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Analytic Torsion and Dynamical Flow: A Survey on the Fried Conjecture

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Abstract. Given an acyclic and unitarily flat vector bundle on a closed manifold, Fried conjectured an equality between the analytic torsion and the value at zero of the Ruelle zeta function associated to a dynamical flow. In this survey, we review the Fried conjecture for different flows, including the suspension flow, the Morse–Smale flow, the geodesic flow, and the Anosov flow.

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Contents

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Introduction

The purpose of this survey is to review some recent progress of the Fried conjecture, which affirms an equality between the analytic torsion and the value at zero of the Ruelle dynamical zeta function.

In the first two sections of the paper, we describe the Ray–Singer analytic torsion of a flat vector bundle and the Ruelle zeta function of a dynamical flow. The next three sections are devoted to the study on the Fried conjecture for certain flows, including the suspension flow, the Morse–Smale flow, the geodesic flow, and the Anosov flow.

The paper is written in an informal way. We only sketch the proofs, and refer to the original papers when necessary.

We now describe in more details the content of this paper, and give the proper historical perspective to the results described in the paper.

0.1. The combinatorial and analytic torsions

Let Z be a smooth closed manifold. Let F be a complex unitarily flat vector bundle on Z, which amounts to specifying a unitary finite-dimensional representation ρ of the fundamental group $\pi_1(Z)$. Denote by $H^\cdot(Z, F)$ the cohomology of the sheaf of locally constant sections of F. Assume that F is acyclic, i.e., $H^\cdot(Z, F) = 0$.

The Reidemeister combinatorial torsion [Re35, Fr35, dR50] is a positive real number defined with the help of a triangulation on Z. However, it does not depend on the triangulation and becomes a topological invariant. It is the first invariant that can distinguish closed manifolds such as lens spaces which are homotopy equivalent but not homeomorphic.

The analytic torsion was introduced by Ray and Singer [RS71] as an analytic counterpart of the Reidemeister torsion. In order to define the analytic torsion one has to choose a Riemannian metric on Z and a Hermitian metric on F. The analytic torsion is a certain weighted alternating product of regularized determinants of the Hodge Laplacians acting on the space of differential forms with values in F.

The celebrated Cheeger–Müller Theorem [C79, M78] tells us that the Ray–Singer analytic torsion coincides with the Reidemeister combinatorial torsion. Bismut–Zhang [BZ92] and Müller [M93] simultaneously considered generalizations of this result. Müller [M93] extended this result to the case where F is unimodular, i.e., $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(Z)$. Bismut and Zhang [BZ92, Theorem 0.2] generalised the original Cheeger–Müller Theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics. There are also various extensions to the equivariant case by Lott–Rothenberg [LoRo91], Lück [Lü93], and Bismut–Zhang [BZ94], to the family case by Bismut–Goette [BG01] under the assumption of the existence of the fibrewise Morse function, and to manifolds with boundaries by Brüning–Ma [BrMa13].

0.2. Dynamical zeta function

The grandmother of all zeta functions is the Riemann zeta function defined for $\mathrm{Re}\,(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{0.1}$$

Riemann showed that $\zeta(s)$ extends meromorphically to $\mathbf{C}$ with a single pole at $s = 1$ and that there is a functional equation relating $\zeta(s)$ and $\zeta(1-s)$. The Euler product formula asserts that for $\mathrm{Re}\,(s) > 1$,

$$\zeta(s) = \prod_{p:\mathrm{prime}} \left(1 - p^{-s}\right)^{-1}. \tag{0.2}$$

It tells us that $\zeta(s)$ encodes the distribution of the prime numbers. Some statistical properties, like the prime number theorem, can be deduced from the information on the poles and zeros of $\zeta(s)$.

After Riemann's work, several zeta functions with similar properties have been introduced. In particular, Weil [W49] constructed a zeta function using the Frobenius map T defined on an algebraic variety Z over a finite field. It counts the closed orbits of the discrete dynamical system (Z, T). To a smooth closed manifold with a diffeomorphism, similar zeta function was also introduced by Artin and Mazur [ArMaz65].

To a flat vector bundle on the underlying manifold, we can associate a weight to the dynamical systems. In [Ru76a], Ruelle introduced his zeta function for a weighted dynamical system of a diffeomorphism or a flow.

For the geodesic flow on the unit tangent bundle of a Riemann surface of genus $\geqslant 2$, an application of the Selberg trace formula [Sel56] shows that the Ruelle dynamical zeta function has a meromorphic extension to $\mathbf{C}$. Similar methods can be generalized to hyperbolic manifolds by Fried [F86a], to the locally symmetric space of rank 1 by Bunke and Olbrich [BuO95], to the locally symmetric space of higher ranks by Moscovich–Stanton [MoSt91], Shen [Sh18], and Shen–Yu [ShY17].

Independent of the above Selberg theory, which is based on the spectral theory of the Laplacian, a thermodynamical formalism, which is based on the spectral theory of the transfer operator, is used to study the Anosov flow (or more generally Axiom A flow[1]) by Ruelle [Ru76b]. He showed that if the flow itself and the stable and unstable foliations are all analytic, then his zeta function has a meromorphic extension to **C**. Fried [F86c] generalized Ruelle's result by requiring only the flow and the stable foliation to be analytic. An important extension of the above results was given by Rugh [Rug96], for three-dimensional manifolds, and then by Fried [F95], in arbitrary dimensions, but still assuming the analyticity of the flow. The extension of such results to the C^∞ setting was only given very recently by Giulietti, Liverani, and Pollicott [GiLiPo13] and by Dyatlov and Zworski [DyZ16] (See also [FaT17, DyGu16, DyGu18, BWSh20] for related works). This recent progress on the dynamical zeta function is based on the introduction of the anisotropic space. We refer the reader to the book of Baladi [Ba18] for an introduction of these techniques.

0.3. The Fried conjecture

It was Milnor [Mi68a] who observed a remarkable similarity between the Reidemeiter torsion and the Weil zeta function. A precise and quantitative description of their relation was obtained by Fried [F86a, F86b] for the geodesic flow on the unit tangent bundle of a hyperbolic manifold. He showed that the analytic/combinatorial torsion of an acyclic unitarily flat vector bundle on a unit tangent bundle of a closed oriented hyperbolic manifold is equal to the value at zero of the Ruelle dynamical zeta function[2] associated to the geodesic flow. He conjectured later [F87, F95] that similar results hold true for more general flows.

Four kinds of flows will be examined in this survey. As a warm up, we will begin with two simple flows: the suspension flow and Morse–Smale flow. As we will see, the Fried conjecture for the suspension flow is just the Lefschetz fixed point formula, and the Fried conjecture for the Morse–Smale flow is a consequence of the Cheeger–Müller/Bismut–Zhang theorem (see [F87, Theorem 3.1] and [ShY18, Theorem 0.2]). Next, we will consider the geodesic flow on the unit tangent bundle of the locally symmetric spaces. In this case, the Fried conjecture can be deduced formally via the V-invariant of Bismut–Goette [BG04]. Following previous contributions by Moscovici–Stanton [MoSt91], a rigorous proof is given by the author [Sh18] using Bismut's orbital integral formula [B11]. In [ShY17], Shen and Yu made a further generalization to closed locally symmetric orbifolds. Finally, we will study the Anosov flow. If the underlying manifold has dimension 3, it was known by Sánchez-Morgado [SM93, SM96a] that the Fried conjecture holds true

[1] In this case, the meromorphic extension problem is called the Smale conjecture [Sm67].

[2] Here the Ruelle dynamical zeta function should be twisted by the holonomy of the flat vector bundle.

if the flow is transitive and analytic, and if the flat vector bundle satisfies certain holonomy conditions. Using a variation formula, Dang, Guillarmou, Rivière, and Shen [DaGuRiSh20] removed the analyticity assumption in Sánchez-Morgado's result. For general Anosov flow, e.g., the geodesic flow on the unit tangent bundle of a negatively curved manifold, the Fried conjecture is still open. The only known result is that under certain spectral condition on the transfer operator, in [DaGuRiSh20], the authors have shown that the value at zero of the Ruelle dynamical zeta function does not depend on a small perturbation of the Anosov flow.

Due to the limitation of the length of this survey, there are several interesting topics which have not been included. For example, the Fried conjecture for hyperbolic manifolds with cusps [Pa09] or for non-unitarily flat vector bundles [Wo08, M12, M20, Sh20a, Sh20b, Sp20], the relation between the dynamical zeta function with the eta invariant [Mil78, MoSt89, B19] and with the holomorphic torsion [F88, MoSt18]. We refer the reader to the cited references for more information.

0.4. Organization of the paper

This paper is organized as follows. In Section 1, we give the main properties of the Ray–Singer analytic torsion.

In Section 2, we introduce the dynamical zeta function and state the Fried conjecture.

In Sections 3–6, we discuss the Fried conjecture for the suspension flow, the Morse–Smale flow, the geodesic flow, and the Anosov flow.

Throughout the paper, we use the supersymmetric convention. For a matrix A acting on a $\mathbf{Z}$-graded space $E^{\cdot}$, the supertrace of A is defined by

$$\mathrm{Tr}_{\mathrm{s}}^{E^{\cdot}}[A] = \mathrm{Tr}^{E^{\cdot}}[(-1)^N A], \tag{0.3}$$

where N is the number operator on $E^{\cdot}$ which sends $e \in E^q$ to $qe \in E^q$. We use the notation

$$\mathbf{R}_+ = [0, \infty), \quad \mathbf{R}_+^* = (0, \infty), \quad \mathbf{N} = \{0, 1, 2, \ldots\}, \quad \mathbf{N}^* = \{1, 2, 3, \ldots\}. \tag{0.4}$$

For a finite set A, we denote by $|A|$ the cardinality of A. By a closed orbit we mean a non-trivial closed orbit, i.e., its period is positive.

1. The Ray–Singer analytic torsion

The purpose of this section is to recall the definition of the Ray–Singer analytic torsion [RS71] of a Hermitian flat vector bundle on a closed Riemannian manifold.

This section is organized as follows. In Section 1.1, we introduce the flat vector bundle.

In Section 1.2, we recall the definition of the Ray–Singer analytic torsion.

In Section 1.3, we consider a fibration $M \to Z$ of closed Riemannian manifolds. We give a formula relating the analytic torsions of a Hermitian flat vector bundle on M and of its direct image on Z, which is equipped with an L^2 Hermitian metric.

1.1. The flat vector bundle

Let Z be a smooth closed manifold. Let F be a complex flat vector bundle on Z with flat connection ∇^F. Equivalently, F can be obtained via a finite-dimensional complex representation[3] $\rho : \pi_1(Z) \to \mathrm{GL}_r(\mathbf{C})$ of the fundamental group $\pi_1(Z)$ of Z. The flat vector bundle F is called unitarily flat if ρ is unitary.

Let $\left(\Omega^{\cdot}(Z,F), d^Z\right)$ be the de Rham complex of smooth sections of $\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}} F$ on Z. Let $H^{\cdot}(Z,F)$ be the cohomology of the above complex. We say that F is acyclic if $H^{\cdot}(Z,F)=0$. The Euler characteristic number of F is then given by

$$\chi(Z,F) = \sum_{q=0}^{\dim Z} (-1)^q \dim H^q(Z,F). \tag{1.1}$$

When F is the real trivial line bundle $\mathbf{R}$, we use the notation $(\Omega^{\cdot}(Z), d^Z), H^{\cdot}(Z)$ and $\chi(Z)$.

Example 1.1. Take $Z = \mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$. Set $\alpha \in \mathbf{C}$. Let F be the trivial complex line bundle $\mathbf{C}$ equipped with the connection $\nabla^F = d + \alpha dt$. The holonomy ρ is then given by $\rho : n \in \mathbf{Z} \to e^{n\alpha} \in \mathbf{C}^*$. It is easy to see that $H^{\cdot}(Z,F) = 0$ if and only if $\alpha \notin 2i\pi\mathbf{Z}$, and that F is unitarily flat if and only if $\alpha \in i\mathbf{R}$.

Example 1.2. Let $0 \to F_0 \to F_1 \to F_2 \to 0$ be an exact sequence of flat vector bundles on Z. Using the associated long exact sequence

$$\cdots \to H^p(Z,F_0) \to H^p(Z,F_1) \to H^p(Z,F_2) \to \cdots, \tag{1.2}$$

we see that if two of F_i are acyclic, then the third one is also acyclic.

1.2. Hodge Laplacian and analytic torsion

Let g^{TZ} be a Riemannian metric on TZ, and let g^F be a Hermitian metric on F. For $s_1, s_2 \in \Omega^{\cdot}(Z,F)$, put

$$\langle s_1, s_2\rangle_{\Omega^{\cdot}(Z,F)} = \int_{z\in Z} \langle s_1(z), s_2(z)\rangle_{\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}}F} dv_Z, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle_{\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}}F}$ is the metric on $\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}} F$ induced by g^{TZ}, g^F, and dv_Z is the Riemannian volume of (Z, g^{TZ}). Then, $\langle\cdot,\cdot\rangle_{\Omega^{\cdot}(Z,F)}$ defines an L^2-metric on $\Omega^{\cdot}(Z,F)$.

Let $d^{Z,*}$ be the formal adjoint of d^Z with respect to $\langle\,,\rangle_{\Omega^{\cdot}(Z,F)}$. Put

$$\Box^Z = d^Z d^{Z,*} + d^{Z,*} d^Z. \tag{1.4}$$

Then, $\Box^Z$ acts on $\Omega^{\cdot}(Z,F)$ and preserves its degree. It is a formally self-adjoint non-negative second-order elliptic differential operator. Since Z is compact, $\Box^Z$ has a unique self-adjoint extension, which we still denote by $\Box^Z$.

[3]We use the convention that $F = \pi_1(Z)\backslash(X\times\mathbf{C}^r)$, where $\pi_1(Z)$ acts on the left on the universal covering X of Z by the deck transformation, and acts on the left on $\mathbf{C}^r$ by ρ.

Take $0 \leqslant q \leqslant \dim Z$. Let $\square_q^Z$ be the restriction of $\square^Z$ on $\Omega^q(Z,F)$. By the Hodge theory, we have a canonical isomorphism of vector spaces,

$$\ker \square_q^Z \simeq H^q(Z,F). \tag{1.5}$$

Denote by $\left(\square_q^Z\right)^{-1}$ the inverse of $\square_q^Z$ acting on the orthogonal space of $\ker \square_q^Z$ in $\Omega^q(Z,F)$.

Definition 1.3. For $s \in \mathbf{C}$ such that $\mathrm{Re}\,(s) > \dim Z/2$, set

$$\theta_q(s) = \mathrm{Tr}\left[\left(\square_q^Z\right)^{-s}\right]. \tag{1.6}$$

By [Se67] or [BeGeVe04, Proposition 9.35], $\theta_q(s)$ has a meromorphic extension to $\mathbf{C}$ which is holomorphic at $s=0$. The regularized determinant of $\square_q^Z$ is a positive number defined by $\exp(-\theta_q'(0))$. We write

$$\det\left(\square_q^Z\right) = \exp\left(-\theta_q'(0)\right). \tag{1.7}$$

Definition 1.4. The Ray–Singer analytic torsion [RS71] of F is defined by

$$T_F(Z) = \prod_{q=1}^{\dim Z} \det\left(\square_q^Z\right)^{(-1)^q q/2} = \exp\left(\frac{1}{2}\sum_{q=1}^{\dim Z}(-1)^{q-1} q\,\theta_q'(0)\right) \in \mathbf{R}_+^*. \tag{1.8}$$

Let $o(TZ)$ be the orientation line bundle of Z. Let $\overline{F}^*$ be the antidual bundle of F. The following proposition is a consequence of the Poincaré duality.

Proposition 1.5. *The following identity holds,*

$$T_F(Z) = \left(T_{\overline{F}^*\otimes o(TZ)}(Z)\right)^{(-1)^{\dim Z-1}}. \tag{1.9}$$

In particular, if Z has even dimension and is orientable, and if F is unitarily flat, then

$$T_F(Z) = 1. \tag{1.10}$$

The next theorem is a special case of the anomaly formula of Bismut–Zhang [BZ92, Theorem 4.7], which generalizes a result of Ray–Singer [RS71, Theorem 2.1]. Its proof is based on the local index techniques, and consists in calculating the constant term in the asymptotic expansion of $\mathrm{Tr}[A\exp(-t\square^Z)]$ as $t \to 0$, where A is a certain smooth section of $\mathrm{End}(\Lambda^\cdot(T^*Z)\otimes_{\mathbf{R}} F)$.

Theorem 1.6. *Assume $H^\cdot(Z,F)=0$. The following statements hold.*

- *If Z has odd dimension, then $T_F(Z)$ is independent of g^{TZ}, g^F.*
- *If F is unimodular (i.e., $|\det\rho(\gamma)| = 1$ for any $\gamma \in \pi_1(X)$), then $T_F(Z)$ is independent of g^{TZ} and the unimodular metric g^F (i.e., g^F indues a flat metric on the determinant line bundle $\det F = \Lambda^{\max} F$).*

In either of the above situations, the analytic torsion becomes a topological invariant. The celebrated Cheeger–Müller/Bismut–Zhang Theorem [C79, M78, M93, BZ92] compares the Ray–Singer analytic torsion with the Reidemeister combinatorial torsion [Re35, Fr35, dR50]. We refer the reader to the above references for more details.

Example 1.7. We use the notation in Example 1.1. We equip TZ with the standard metric $(dt)^2$ and equip F with the trivial metric. Then,

$$d^Z = dt\frac{\partial}{\partial t} + \alpha dt. \quad d^{Z,*} = -i_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} + \overline{\alpha} i_{\frac{\partial}{\partial t}}, \quad \square^Z = -\left(\frac{\partial}{\partial t} - \overline{\alpha}\right)\left(\frac{\partial}{\partial t} + \alpha\right). \tag{1.11}$$

Using the Fourier transform, we get the spectrum

$$\mathrm{Sp}\,\square_0^Z = \mathrm{Sp}\,\square_1^Z = \left\{\left|\alpha + 2ik\pi\right|^2 : k \in \mathbf{Z}\right\}, \tag{1.12}$$

and

$$T_F(Z) = \begin{cases} \left|2\sinh\left(\frac{\alpha}{2}\right)\right|^{-1} & , \quad \alpha \notin 2i\pi\mathbf{Z}, \\ 1 & , \quad \alpha \in 2i\pi\mathbf{Z}. \end{cases} \tag{1.13}$$

We also have [BZ92, Theorem 0.3].

Theorem 1.8. *We use the notation in Example 1.2. Assume that two of F_i are acyclic. If Z has odd dimension, or if g^{F_0}, g^{F_1}, and g^{F_2} induce a flat metric on $\det F_0 \otimes (\det F_1)^{-1} \otimes \det F_2$, then*

$$T_{F_1}(Z) = T_{F_0}(Z) T_{F_2}(Z). \tag{1.14}$$

Here, for a line bundle L, the notion L^{-1} denotes the dual bundle of L.

1.3. A formula for fibrations

Let $\pi : M \to Z$ be a fibration of closed manifolds with closed fibre Y. Let $TY \subset TM$ be the relative tangent bundle on M of the fibre Y. Let F be a flat vector bundle on M.

Let g^{TM}, g^{TY}, g^{TZ} be the Riemannian metrics on TM, TY, TZ. Let g^F be a Hermitian metric on F. Let $T_F(M)$ be the Ray–Singer analytic torsion with respect to g^{TM} and g^F.

Moreover, the restriction $F|_Y$ of F to the fibre Y is still a flat vector bundle. So we can consider the fibrewise Ray–Singer analytic torsion $T_{F|_Y}(Y)$ with respect to g^{TY} and $g^F|_Y$. It is a smooth function on Z.

Also, for $0 \leqslant q \leqslant \dim Y$, the fibrewise cohomology $H^q(Y, F|_Y)$ of Y forms a flat vector bundle on Z (see [BLo95, Section III.f]). Using the Hodge theory (1.5), the L^2 metric on $\Omega^q(Y, F|_Y)$ induces a Hermitian metric $g^{H^q(Y,F|_Y)}$ on $H^q(Y, F|_Y)$. Let $T_{H^q(Y,F|_Y)}(Z)$ be the Ray–Singer analytic torsion with respect to g^{TZ} and $g^{H^p(Y,F|_Y)}$.

The following theorem is a special case of a very general result of Ma [Ma02, Theorem 0.1 and (0.6)], which is a generalization of Dai–Melrose [DMe12] and Lück–Schick–Thielmann [LüScTh98, Theorem 0.2].

Theorem 1.9. *Assume $H^\cdot(M,F)=0$ and $H^\cdot(Z,H^\cdot(Y,F|_Y))=0$. If M has odd dimension or if F is unimodular, then*

$$T_F(M)=\exp\left(\int_Z e\left(TZ,\nabla^{TZ}\right)\log T_{F|_Y}(Y)\right)\prod_{q=0}^{\dim Y}\left(T_{H^q(Y,F|_Y)}(Z)\right)^{(-1)^q}, \quad (1.15)$$

where $e\left(TZ,\nabla^{TZ}\right)\in\Omega^{\dim Z}(Z,o(TZ))$ is the Euler characteristic form on Z with respect to the Levi-Civita connection ∇^{TZ}. In addition, if Z has odd dimension, then

$$T_F(M)=\prod_{q=0}^{\dim Y}\left(T_{H^q(Y,F|_Y)}(Z)\right)^{(-1)^q}. \quad (1.16)$$

2. The Fried conjecture

It was known dating back to Poincaré, Hopf, Lefschetz, etc., that certain topological invariants, like Euler characteristic number, can be expressed with the help of some dynamical objects, like the fixed point set. Fried considered a similar problem relating the Ray–Singer analytic torsion with the closed orbits of certain dynamical systems, which he called the Lefschetz formula for flows.

More precisely, given a diffeomorphism of a manifold, the Lefschetz fixed point formula asserts that the Lefschetz number can be written as a sum of the Lefschetz indices of each fixed point of the diffeomorphism. Formally, the Lefschetz index is a way to count cohomologically the cardinality of the fixed point set.

For a dynamical flow, Fried uses the Fuller index [Fu67] to count cohomologically the cardinality of the set of closed orbits of the flow. However, due to the infinite numbers of closed orbits, the Fuller index of the set of all closed orbits is not well defined and should be regularized. For this Fried relies on the meromorphic extension and the regularity at 0 of the Ruelle dynamical zeta function. The purpose of this section is to formulate the above as a general conjecture, known as the Fried conjecture.

This section is organized as follows. In Section 2.1, we consider a diffeomorphism on a manifold. We recall the Lefschetz fixed point formula.

In Section 2.2, we consider a flow on a manifold. We introduce the Fuller index.

In Section 2.3, we construct the Ruelle dynamical zeta function, and state the Fried conjecture.

2.1. Lefschetz index

Let Z be a closed manifold. Let $T\in\mathrm{Diffeo}(Z)$ be a diffeomorphism of Z. The pull back T^* defines a morphism of the complex $(\Omega^\cdot(Z),d)$. It induces a morphism on the cohomology $H^{\cdot}(Z)$. The Lefschetz number of T is defined by

$$L_T = \mathrm{Tr}_{\mathrm{s}}^{H^{\cdot}(Z)}[T^*] = \sum_{q=0}^{\dim Z} (-1)^q \,\mathrm{Tr}\left[T^*|_{H^q(Z)}\right]. \tag{2.1}$$

Clearly, if T is the identity map, then the Lefschetz number L_T is equal to the Euler characteristic number $\chi(Z)$.

Let

$$Z^T = \{z \in Z : Tz = z\} \tag{2.2}$$

be the set of the fixed points of T. Recall that a fixed point $z \in Z^T$ is called non-degenerate if $\det(1 - D_z T)|_{T_z Z} \neq 0$. For such a fixed point $z \in Z^T$, the Lefschetz index of T at z is defined by

$$\mathrm{ind}_L(T, z) = \mathrm{sgn}\big(\det(1 - D_z T)|_{T_z Z}\big). \tag{2.3}$$

If T has only non-degenerate fixed points, then Z^T is finite, and the Lefschetz fixed point theorem [BeGeVe04, Corollary 6.7] states that

$$L_T = \sum_{z \in Z^T} \mathrm{ind}_L(T, z). \tag{2.4}$$

More generally, if Z^T is not finite, we need the following assumption.

Assumption 2.1. *We assume that*

(1) *the fixed point set Z^T is a disjoint union of finitely many closed connected submanifolds $Z_i \subset Z$, so that*

$$Z^T = \coprod_{i \in I} Z_i; \tag{2.5}$$

(2) *for $z \in Z^T$, the eigenspace of $D_z T : T_z Z \to T_z Z$ associated to the eigenvalue 1 coincides with its characteristic space, and is equal to $T_z Z^T$.*

Let $N_{Z_i/Z} = (TZ|_{Z_i})/TZ_i$ be the normal bundle of Z_i in Z. Our assumption (2) is equivalent to

$$\det(1 - D_z T)|_{N_{Z_i/Z,z}} \neq 0, \qquad \text{for all } i \in I, z \in Z_i. \tag{2.6}$$

Note that for each $i \in I$, the sign of the above determinant does not depend on $z \in Z_i$.

Definition 2.2. The Lefschetz index of the diffeomorphism T at Z_i is defined by

$$\mathrm{ind}_L(T, Z_i) = \mathrm{sgn}\left(\det(1 - D_z T)|_{N_{Z_i/Z,z}}\right) \cdot \chi(Z_i) \in \mathbf{Z}, \tag{2.7}$$

where $z \in Z_i$.

Under Assumption 2.1, we have the Lefschetz fixed point theorem[4]

$$L_T = \sum_{i \in I} \mathrm{ind}_L(T, Z_i). \tag{2.8}$$

Formally, Lefschetz index $\mathrm{ind}_L(T, Z_i)$ counts cohomologically the cardinality of Z_i. Equation (2.8) says that the Lefschetz number can be obtained by counting cohomologically the cardinality of the fixed point set Z^T.

2.2. Fuller index

Let M be a closed manifold. Let $V \in C^\infty(M, TM)$ be a smooth vector field on M, and let $(\phi_t)_{t \in \mathbf{R}}$ be the flow generated by V.

Definition 2.3. We define the periodic set

$$\wp(\phi_\cdot) = \{(x,t) \in M \times \mathbf{R}_+^* : \phi_t(x) = x\}, \tag{2.9}$$

and the length spectrum

$$\ell(\phi_\cdot) = \{t \in \mathbf{R}_+^* : \text{ there is } x \in M \text{ such that } \phi_t(x) = x\}. \tag{2.10}$$

A point $(x,t) \in \wp(\phi_\cdot)$ corresponds to a closed orbit $\{\phi_s(x)\}_{0 \leqslant s \leqslant t}$ starting from x of period t. And $\ell(\phi_\cdot)$ represents all the possible periods. Clearly, if $\ell \in \ell(\phi_\cdot)$, then $k\ell \in \ell(\phi_\cdot)$ for all $k \in \mathbf{N}^*$.

Since our ultimate object "closed orbit" disregards the starting points, we will consider certain quotient space of $\wp(\phi_\cdot)$. Define an $\mathbf{R}$-action on $\wp(\phi_\cdot)$ by

$$s \cdot (x,t) = (\phi_{ts} x, t), \qquad \text{for } s \in \mathbf{R}, (x,t) \in \wp(\phi_\cdot). \tag{2.11}$$

By (2.11), the group $\mathbb{S}^1 = \mathbf{R}/\mathbf{Z}$ acts on $\wp(\phi_\cdot)$. Set

$$\overline{\wp}(\phi_\cdot) = \wp(\phi_\cdot)/\mathbb{S}^1. \tag{2.12}$$

We identify $\overline{\wp}(\phi_\cdot)$ with the space of closed orbits of the flow $\phi_\cdot$.

To count cohomologically the points in the $\overline{\wp}(\phi_\cdot)$, as in Section 2.1, we need an analogue of Assumption 2.1 for flows.

Assumption 2.4. *We assume that*

(1) *the length spectrum* $\ell(\phi_\cdot) \subset (0, \infty)$ *is discrete;*
(2) *for any* $\ell \in \ell(\phi_\cdot)$*, the diffeomorphism* $\phi_\ell \in \mathrm{Diffeo}(M)$ *satisfies Assumption 2.1.*

Recall that $M^{\phi_\ell} \subset M$ is the fixed point set of ϕ_ℓ in M. By (1) of Assumption 2.4, we can write

$$\wp(\phi_\cdot) = \coprod_{\ell \in \ell(\phi_\cdot)} M^{\phi_\ell} \times \{\ell\}. \tag{2.13}$$

[4] When T is an isometry of Z, a proof can be found in [BeGeVe04, Section 6.4]. For general T, the Lefschetz fixed point formula can be proved by adapting the argument given in [BZ92, Theorem 4.20].

To avoid the problem of disconnectedness of M^{ϕ_ℓ}, we write instead

$$\wp(\phi.) = \coprod_{i \in I} M_i \times \{\ell_i\}, \tag{2.14}$$

where M_i is a connected component of $M^{\phi_{\ell_i}}$. Note that not all the ℓ_i are necessarily distinct. Moreover, (1) of Assumption 2.4 implies that the $\mathbb{S}^1$-action on $\wp(\phi.)$ is locally free. Thus, $M_i/\mathbb{S}^1$ is a closed connected orbifold. By (2.12), we have

$$\overline{\wp}(\phi.) = \coprod_{i \in I} M_i/\mathbb{S}^1 \times \{\ell_i\}. \tag{2.15}$$

Denote by $\chi_{\mathrm{orb}}(M_i/\mathbb{S}^1) \in \mathbf{Q}$ the orbifold Euler characteristic number [Sa57] of $M_i/\mathbb{S}^1$. Set

$$m_i = \left|\ker\left(\mathbb{S}^1 \to \mathrm{Diffeo}(M_i)\right)\right| \in \mathbf{N}^* \tag{2.16}$$

to be the generic multiplicity of the closed orbits in $M_i/\mathbb{S}^1 \times \{\ell_i\}$.

By (2) of Assumption 2.4, as in Section 2.1, the sign

$$\mathrm{sgn}\left(\det(1 - D\phi_{\ell_i})|_{N_{M_i/M}}\right) \in \{\pm 1\} \tag{2.17}$$

is well defined for all $i \in I$.

Definition 2.5. The Fuller index of the flow $\phi.$ at $M_i \times \{\ell_i\}$ is defined by

$$\mathrm{ind}_F(\phi., M_i) = \mathrm{sgn}\left(\det(1 - D\phi_{\ell_i})|_{N_{M_i/M}}\right) \cdot \frac{\chi_{\mathrm{orb}}(M_i/\mathbb{S}^1)}{m_i} \in \mathbf{Q}. \tag{2.18}$$

Remark 2.6. The pair $(M_i, \mathbb{S}^1)$ defines a non-effective orbifold. Its "non-effective" Euler characteristic number is given by $\frac{\chi_{\mathrm{orb}}(M_i/\mathbb{S}^1)}{m_i}$. In this way, the Fuller index $\mathrm{ind}_F(\phi., M_i)$ is a strict analogue of the Lefschetz index (2.7).

More generally, let F be a flat vector bundle on M with holonomy ρ. Since M_i is connected, for any $x \in M_i$, the closed orbit $\{\phi_t(x)\}_{0 \leqslant t \leqslant \ell_i}$ lies in the same freely homotopy class of the loops on M. Up to conjugation, the holonomy[5] ρ_i of F associated to the closed orbit $\{\phi_t(x)\}_{0 \leqslant t \leqslant \ell_i}$ does not depend on the choice of $x \in M_i$. We can define the twisted Fuller index by

$$\mathrm{ind}_F(\phi., M_i, F) = \mathrm{ind}_F(\phi., M_i)\, \mathrm{Tr}\left[\rho_i\right]. \tag{2.19}$$

2.3. The Ruelle dynamical zeta function

We assume that the flow satisfies Assumption 2.4. As an analogue of (2.4), Fried raised the question: for what kind of flows the infinite sum

$$\sum_{i \in I} \mathrm{ind}_F(\phi., M_i, F) \tag{2.20}$$

[5] The morphism ρ_i can be obtained by the parallel transport with respect to the flat connection along the closed orbit $\{\phi_t(x)\}_{0 \leqslant t \leqslant \ell_i}$ from $t = \ell_i$ to $t = 0$.

can be regularized, and is equal to some topological invariant. More precisely, Fried conjectured that if F is unitarily flat and acyclic, then the regularized sum (2.20) is equal to $\log T_F(M)$.

To regularize the sum (2.20), Fried used the Ruelle dynamical zeta function [Ru76b, F87].

Definition 2.7. For $\sigma \in \mathbf{C}$, we define formally

$$R_{\phi,\rho}(\sigma) = \exp\left(\sum_{i\in I} \mathrm{ind}_F(\phi_\cdot, M_i, F)e^{-\ell_i\sigma}\right). \tag{2.21}$$

We call $R_{\phi,\rho}$ is well defined if the following properties hold.

1. There is $\sigma_0 > 0$ such that for $\sigma \in \mathbf{C}$ with $\mathrm{Re}\,(\sigma) > \sigma_0$, the sum on the right-hand side of (2.21) converges to a holomorphic function;
2. The holomorphic function $R_{\phi,\rho}(\sigma)$, defined for $\mathrm{Re}\,(\sigma) > \sigma_0$, has a meromorphic extension to $\sigma \in \mathbf{C}$.

Remark 2.8. The above definition of the Ruelle dynamical zeta function is the reciprocal of the one introduced by Fried [F87, Section 5].

The Fried conjecture now can be formulated as follows.

Conjecture 2.9. *For a wide class of flows on a closed manifold M, if F is a unitarily flat vector bundle on M and $H^\cdot(M,F) = 0$, then the Ruelle dynamical zeta function $R_{\phi,\rho}(\sigma)$ is well defined, and is regular at $\sigma = 0$ such that*

$$|R_{\phi,\rho}(0)| = T_F(M). \tag{2.22}$$

Example 2.10. Consider the rotation flow $\phi_\cdot$ on $\mathbb{S}^1$, i.e.,

$$\phi_t(x) = x + t \mod \mathbf{Z}, \qquad \text{for } t \in \mathbf{R}, x \in \mathbf{R}/\mathbf{Z}. \tag{2.23}$$

For the flat vector bundle defined in Example 1.1, the Ruelle dynamical zeta function is given by

$$R_{\phi,\rho}(\sigma) = (1 - e^{-\sigma+\alpha})^{-1}. \tag{2.24}$$

If F is acyclic and unitarily flat (i.e., $\alpha \in i\mathbf{R}$ and $\alpha \notin 2i\pi\mathbf{Z}$), by (1.13) and (2.24), we have

$$|R_{\phi,\rho}(0)| = T_F\left(\mathbb{S}^1\right). \tag{2.25}$$

Example 2.11. Consider now the geodesic flow of $\mathbb{S}^1$. That is a flow $\widetilde{\phi}$ defined on $\mathbb{S}^1 \times \{\pm 1\}$. On $\mathbb{S}^1 \times \{1\}$, $\widetilde{\phi}$ is just the rotation flow considered in Example 2.10, and on the other copy $\mathbb{S}^1 \times \{-1\}$, $\widetilde{\phi}$ is the inverse of the rotation flow. Let $\pi : \mathbb{S}^1 \times \{\pm 1\} \to \mathbb{S}^1$ be the natural projection.

Recall that F is defined in Example 1.1. Let π^*F be the pull back of the flat vector bundle on $\mathbb{S}^1 \times \{\pm 1\}$. The Ruelle dynamical zeta function is then given by

$$R_{\widetilde{\phi},\rho}(\sigma) = (1 - e^{-\sigma+\alpha})^{-1}(1 - e^{-\sigma-\alpha})^{-1}. \tag{2.26}$$

If $\alpha \in \mathbf{C}$ and $\alpha \notin 2i\pi\mathbf{Z}$, we have

$$|R_{\widetilde{\phi},\rho}(0)| = T_{\pi^* F}\left(\mathbb{S}^1 \times \{\pm 1\}\right). \tag{2.27}$$

Example 2.10 shows that the Fried conjecture fails for non-unitarily flat vector bundles, while Example 2.11 shows that for some special flow the Fried conjecture holds even for certain non-unitarily flat vector bundles.

However, even for unitarily flat vector bundles, we can not expect that the Fried conjecture holds for general flows. In fact, a fixed point or a closed orbit is a limit set of dimensions 0 or 1. Wilson [Wi66, Corollary 2] has constructed a smooth flow on any closed manifold M with vanishing Euler characteristic number, whose only limit sets are a finite collection of torus of dimension $\dim M - 2$. In particular, if $\dim M \geqslant 4$ with $\chi(M) = 0$, there exists a flow without fixed points or closed orbits. If $\dim M = 3$, Schweitzer [Sch74, Theorem A] has constructed a C^1-flow without fixed points or closed orbits.

In the following sections, we will discuss several different classes of flows where the Fried conjecture has or might have a positive solution. This includes the suspension flow, the Morse–Smale flow, the geodesic flow, and the Anosov flow.

3. Suspension flow of a diffeomorphism

In this section, we consider the suspension flow of a diffeomorphism. We show that in this case the Fried conjecture is a consequence of the Lefschetz fixed point formula (2.8).

This section is organized as follows. In Section 3.1, we introduce the suspension flow.

In Section 3.2, we describe the flat vector bundle on a suspension, and evaluate its analytic torsion.

In Section 3.3, we give an explicit formula for the Ruelle dynamical zeta function associated to the suspension flow. Then, we show the Fried conjecture.

3.1. The definition of the suspension flow

Let Y be a connected closed manifold. Let $T \in \mathrm{Diffeo}(Y)$ be a diffeomorphism of Y. Let M be the suspension of T. Then $M = \mathbf{R} \times_{\mathbf{Z}} Y$ is the quotient space of $\mathbf{R} \times Y$ by the $\mathbf{Z}$-action given by

$$n \cdot (t, y) = (t - n, T^n y), \qquad \text{for } n \in \mathbf{Z}, (t, y) \in \mathbf{R} \times Y. \tag{3.1}$$

Clearly, $M \to \mathbf{R}/\mathbf{Z}$ is a fibration with fibre Y.

The suspension flow $\phi_\cdot$ on M is defined by

$$\phi_s([t, y]) = [t + s, y], \qquad \text{for } s \in \mathbf{R}, [t, y] \in \mathbf{R} \times_{\mathbf{Z}} Y. \tag{3.2}$$

3.2. The flat vector bundle on the suspension

Let E be a flat vector bundle on Y. Assume that the diffeomorphism T lifts to E. This means that there is a bundle morphism T_E of E such that the diagram

$$\begin{tikzcd} E \arrow[r, "T_E"] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "T"] & Y \end{tikzcd} \tag{3.3}$$

commutes. Let

$$F = \mathbf{R} \times_{\mathbf{Z}} E \tag{3.4}$$

be the suspension of E with respect to T_E. Then F is a flat vector bundle on M.

Proposition 3.1. *All the flat vector bundles on M can be obtained in this way.*

Proof. Using the long exact sequence of the homotopy groups associated to the fibration $M \to \mathbf{R}/\mathbf{Z}$, we get an exact sequence of groups

$$1 \to \pi_1(Y) \to \pi_1(M) \to \mathbf{Z} \to 1. \tag{3.5}$$

The diffeomorphism T defines an isomorphism $T_* : \pi_1(Y) \to \pi_1(Y)$ of groups such that $\pi_1(M)$ is the semidirect product of $\pi_1(Y)$ and $\mathbf{Z}$ with respect to T_*.

Any r-dimensional representation of $\pi_1(M)$ can be constructed by an r-dimensional representation ρ of $\pi_1(Y)$ and an element $A \in \mathrm{GL}_r(\mathbf{C})$ such that

$$A\rho(\gamma) = \rho(T_*\gamma)A, \qquad \text{for all } \gamma \in \pi_1(Y). \tag{3.6}$$

It is easy to check that ρ and A induce our E and T_E. □

As in Section 2.1, the pull back T_E^* defines a morphism on $H^{\cdot}(Y,E)$. Let $H^{\cdot}_T(Y,E)$ be a flat vector bundle on $\mathbf{R}/\mathbf{Z}$ defined by the holonomy

$$\rho : n \in \mathbf{Z} \to (T_E^*)^n . \tag{3.7}$$

This is just the direct image $H^{\cdot}(Y, F|_Y)$ of F described in Section 1.3.

Proposition 3.2. *The flat vector bundle F on M is acyclic if and only if 1 is not an eigenvalue of T_E^* on $H^{\cdot}(Y,E)$.*

Proof. Let $(E_r^{p,q}, d_r)_{r\geqslant 0}$ be the Leray spectral sequence [BoTu82, p. 169] associated to the fibration $M \to \mathbf{R}/\mathbf{Z}$. By [BoTu82, Theorem 14.18], $(E_r^{p,q}, d_r)_{r\geqslant 0}$ converges to $H^{\cdot}(M,F)$, and

$$E_2^{p,q} = H^p\big(\mathbf{R}/\mathbf{Z}, H_T^q(Y,E)\big). \tag{3.8}$$

By (3.8), for $r \geqslant 2$, $p \neq 0,1$, we have $E_r^{p,q} = 0$. Since d_r sends $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, we see that $d_2 = d_3 = \cdots = 0$. So the spectral sequence degenerates at $r = 2$. It implies

$$\dim H^p(M,F) = \dim H^0\big(\mathbf{R}/\mathbf{Z}, H_T^p(Y,E)\big) + \dim H^1\big(\mathbf{R}/\mathbf{Z}, H_T^{p-1}(Y,E)\big). \tag{3.9}$$

By (3.9), $H^{\cdot}(M,F)=0$ if and only if $H^{\cdot}\big(\mathbf{R}/\mathbf{Z},H^{\cdot}_{T}(Y,E)\big)=0$. By Example 1.1, we see that our proposition holds true if T^{*}_{E} is diagonalizable. Thanks to Example 1.2, we can deduce our proposition in full generality. □

If A is a matrix acting on a $\mathbf{Z}$-graded space $E^{\cdot}$ which preserves the degree, then write

$$\det{}_{\mathrm{s}}(A)|_{E^{\cdot}}=\prod_{q\in\mathbf{Z}}\Big(\det(A)|_{E^{q}}\Big)^{(-1)^{q}}. \tag{3.10}$$

Proposition 3.3. *Assume $H^{\cdot}(M,F)=0$. If M has odd dimension or if F is unimodular, then*

$$T_{F}(M)=\Big|\det{}_{\mathrm{s}}\left(T^{*}_{E}\right)|_{H^{\cdot}(Y,E)}\Big|^{1/2}\Big|\det{}_{\mathrm{s}}\left(1-T^{*}_{E}\right)|_{H^{\cdot}(Y,E)}\Big|^{-1}. \tag{3.11}$$

Proof. By Theorem 1.9, we have

$$T_{F}(M)=\prod_{q=0}^{\dim Y}\Big(T_{H^{q}_{T}(Y,E)}\big(\mathbf{R}/\mathbf{Z}\big)\Big)^{(-1)^{q}}. \tag{3.12}$$

Now (3.11) is a consequence of (1.13), (1.14), and (3.12). □

3.3. The Ruelle dynamical zeta function of the suspension flow

Note that if for all $n\in\mathbf{N}^{*}$, the diffeomorphism T^{n} satisfies Assumption 2.1, then the suspension flow $\phi_{\cdot}$ associated to T satisfies Assumption 2.4.

Theorem 3.4. *Assume that for all $n\in\mathbf{N}^{*}$, the diffeomorphism $T^{n}\in\mathrm{Diffeo}(Y)$ satisfies Assumption 2.1. The Ruelle dynamical zeta function is given by*

$$R_{\phi,\rho}(\sigma)=\Big(\det{}_{\mathrm{s}}\left(1-T^{*}_{E}e^{-\sigma}\right)|_{H^{\cdot}(Y,E)}\Big)^{-1}. \tag{3.13}$$

If $H^{\cdot}(M,F)=0$, then $R_{\phi,\rho}$ is regular at $\sigma=0$, so that

$$R_{\phi,\rho}(0)=\Big(\det{}_{\mathrm{s}}\left(1-T^{*}_{E}\right)|_{H^{\cdot}(Y,E)}\Big)^{-1}. \tag{3.14}$$

Proof. For the suspension flow, we have $\ell(\phi_{\cdot})\subset\mathbf{N}^{*}$. The periodic set is given by

$$\wp(\phi_{\cdot})=\coprod_{n=1}^{\infty}M^{\phi_{n}}\times\{n\}. \tag{3.15}$$

Let $\pi:\mathbf{R}\times Y\to M$ be the natural projection. We have

$$M^{\phi_{n}}=\pi\left([0,1]\times Y^{T^{n}}\right). \tag{3.16}$$

We claim that Fuller index (twisted by F) of $\phi_{\cdot}$ at $M^{\phi_{n}}$ is given by

$$\mathrm{ind}_{F}(\phi_{\cdot},M^{\phi_{n}},F)=\frac{1}{n}\mathrm{Tr}_{\mathrm{s}}^{H^{\cdot}(Y,E)}\left[(T^{*}_{E})^{n}\right]. \tag{3.17}$$

Let us show (3.17) in the case where $Y^{T^{n}}$ is discrete. The proof for the general case is similar, and we omit the details. For $p\in\mathbf{N}^{*}$ and $p|n$, let $Y^{T^{n}}_{p}\subset Y^{T^{n}}$ be the subset of Y^{T^n} formed by the points of Y^{T^n} whose prime period is p. Take $y \in Y_p^{T^n}$. By (2.18), the contribution of the closed orbit $\{[t,y]\}_{0\leqslant t\leqslant n}$ to the Fuller index (twisted by F) is given by

$$\frac{\operatorname{sgn}\left(\det\left(1-DT_y^n\right)|_{T_yY}\right)}{n/p}\operatorname{Tr}\left[\tau_{y,n}\right], \tag{3.18}$$

where $\tau_{y,n}$ is the parallel transport of F with respect to the flat connection along the curve $\{[t,y]\}_{0\leqslant t\leqslant n}$ from $t=n$ to $t=0$. We have

$$\operatorname{Tr}\left[\tau_{y,n}\right]=\operatorname{Tr}^{E_y}\left[T_E^{-n}\right]. \tag{3.19}$$

Since $y, Ty, \ldots, T^{p-1}y$ are all in $Y_p^{T^n}$ and correspond to the same closed orbit, we have

$$\begin{aligned}\operatorname{ind}_F(\phi_\cdot, M^{\phi_n}, F) &= \sum_{p|n}\frac{1}{p}\sum_{y\in Y_p^{T^n}}\frac{\operatorname{sgn}\left(\det\left(1-DT_y^n\right)|_{T_yY}\right)}{n/p}\operatorname{Tr}^{E_y}\left[T_E^{-n}\right]\\ &= \frac{1}{n}\sum_{y\in Y^{T^n}}\operatorname{sgn}\left(\det\left(1-DT_y^n\right)|_{T_yY}\right)\operatorname{Tr}^{E_y}\left[T_E^{-n}\right].\end{aligned} \tag{3.20}$$

By a twisted version of (2.4) (see [BeGeVe04, Theorem 6.6]) and (3.20), we get (3.17).

By (3.17), using

$$\log(1-z)=-\sum_{n=1}^{\infty}\frac{z^n}{n}, \tag{3.21}$$

we get (3.13). The rest of our proposition is a consequence of Proposition 3.2 and (3.13). □

Corollary 3.5. *Suppose that for all $n\in\mathbf{N}^*$ the diffeomorphism $T^n\in\operatorname{Diffeo}(Y)$ satisfies Assumption 2.1. Assume that $H^\cdot(M,F)=0$ and that F is unimodular. Then, we have*

$$|R_{\phi,\rho}(0)|=T_F(M). \tag{3.22}$$

Proof. By (3.11) and (3.14), it is enough to show

$$\left|\det{}_{\mathrm{s}}(T_E^*)|_{H^\cdot(Y,E)}\right|=1. \tag{3.23}$$

Recall that $H_T^\cdot(Y,E)$ is a flat vector bundle on $\mathbf{R}/\mathbf{Z}$. Equation (3.23) is equivalent to say that the line bundle $\otimes_{q=0}^{\dim Y}\left(\det H_T^q(Y,E)\right)^{(-1)^q}$ is unitarily flat.

Remark that (see [BLo95, Section I.g]) if (W,∇^W) is a flat vector bundle on a manifold S, for any Hermitian metric g^W on W, the 1-form $\frac{1}{2}\operatorname{Tr}\left[g^{W,-1}\nabla^W g^W\right]$ is closed. Moreover, its class in $H^1(S)$ does not depend on the choice of g^W, and is called the first odd Chern class $c_1^{\mathrm{odd}}(W)$ of W. Clearly, $\det(W)$ is unitarily flat if and only if $c_1^{\mathrm{odd}}(W)=0$.

By above, it is equivalent to show the first odd Chern class of the fibrewise cohomology vanishes,

$$\sum_{q=0}^{\dim Y} (-1)^q c_1^{\mathrm{odd}}(H_T^q(Y,E)) = 0. \tag{3.24}$$

Since F is unimodular, (3.24) is a consequence of [BLo95, Theorem 0.1]. $\square$

4. Morse–Smale flow

The Morse–Smale flow is the simplest structurally stable dynamical system which has only two types of recurrent behaviors: closed orbits and fixed points. In this section, we study the Fried conjecture for the Morse–Smale flow. We show that the Fried conjecture in this case is a consequence of Cheeger–Müller/Bismut–Zhang theorem. The result in this section is originally due to Fried [F87, Section 3] and is extended by Shen–Yu [ShY18].

This section is organized as follows. In Section 4.1, we introduce the Morse–Smale flow.

In Section 4.2, we give an explicit formula for the Ruelle dynamical zeta function, and show the Fried conjecture.

4.1. The definition of Morse–Smale flow

Let M be a closed manifold with a smooth vector field $V \in C^\infty(M,TM)$. Let $(\phi_t)_{t\in\mathbf{R}}$ be the flow on M generated by V.

Definition 4.1. The nonwandering set of $\phi_\cdot$ is defined by

$$\Big\{x \in M:\ \forall \text{ open neighborhood } U \text{ of } x, \forall T>0, \text{ we have } U \cap \bigcup_{t\geqslant T} \phi_t(U) \neq \varnothing\Big\}. \tag{4.1}$$

Definition 4.2. A fixed point $x \in M$ of the flow $\phi_\cdot$ is called hyperbolic if there is a ϕ_t-invariant splitting

$$T_xM = E_x^u \oplus E_x^s, \tag{4.2}$$

and there exist $C>0, \theta>0$ and a Riemannian metric on M such that for $v \in E_x^u$, $v' \in E_x^s$, and $t>0$, we have

$$|\phi_{-t,*}v| \leqslant Ce^{-\theta|t|}\,|v|, \qquad |\phi_{t,*}v'| \leqslant Ce^{-\theta|t|}\,|v'|. \tag{4.3}$$

The index $\mathrm{ind}(x) \in \mathbf{N}$ of x is defined by

$$\mathrm{ind}(x) = \dim E_x^u. \tag{4.4}$$

The unstable and stable manifolds of x are defined by

$$\begin{aligned} W_x^u &= \big\{y \in M:\ \lim_{t\to-\infty} d_M(\phi_t(y),x) = 0\big\},\\ W_x^s &= \big\{y \in M:\ \lim_{t\to+\infty} d_M(\phi_t(y),x) = 0\big\}, \end{aligned} \tag{4.5}$$

where d_M is the Riemannian distance on M.

Definition 4.3. A closed orbit γ of the flow $\phi_\cdot$ is called hyperbolic, if there is a ϕ_t-invariant continuous splitting

$$TM|_\gamma = \mathbf{R}V \oplus E^u_\gamma \oplus E^s_\gamma \tag{4.6}$$

of vector bundles over γ such that (4.3) holds. The index $\mathrm{ind}(\gamma) \in \mathbf{N}$ of γ is defined by

$$\mathrm{ind}(\gamma) = \dim E^u_\gamma. \tag{4.7}$$

The unstable and stable manifolds of γ are defined by

$$\begin{aligned} W^u_\gamma &= \bigcup_{x\in\gamma} \Big\{ y \in M : \lim_{t\to-\infty} d_M(\phi_t(y), \phi_t(x)) = 0 \Big\}, \\ W^s_\gamma &= \bigcup_{x\in\gamma} \Big\{ y \in M : \lim_{t\to+\infty} d_M(\phi_t(y), \phi_t(x)) = 0 \Big\}. \end{aligned} \tag{4.8}$$

Denote by A the set of fixed points and by B the set of prime closed orbits.

Definition 4.4. A vector field V or a flow $\phi_\cdot$ is called Morse–Smale (see [PalMe82, Definition, p.118]) if

- the sets A and B are finite and contain only hyperbolic elements;
- the nonwandering set of $\phi_\cdot$ is equal to $A \cup \bigcup_{\gamma\in B} \gamma$;
- the stable manifold of any element in $A \coprod B$ intersects transversally with the unstable manifolds of any element in $A \coprod B$.

In the rest of this section, we assume that V is a Morse–Smale vector field.

4.2. The Ruelle dynamical zeta function of the Morse–Smale flow

For $\gamma \in B$, denote by $\ell_\gamma \in \mathbf{R}^*_+$ its period. A closed orbit $\gamma \in B$ is called untwist if E^u_γ is orientable along γ, and is called twist otherwise. Put

$$\Delta(\gamma) = \begin{cases} 1, & \text{if } \gamma \text{ is untwist,} \\ -1, & \text{if } \gamma \text{ is twist.} \end{cases} \tag{4.9}$$

Recall that F is a flat vector bundle on M with holonomy ρ. For $\gamma \in B$, denote by $\rho(\gamma)$ the holonomy along γ, which is also the parallel transport with respect to the flat connection along γ^{-1} (cf. footnote 3). Clearly, $\rho(\gamma)$ is well defined up to a conjugation.

Proposition 4.5. *The Ruelle dynamical zeta function is given by*

$$R_{\phi,\rho}(\sigma) = \prod_{\gamma\in B} \det\left(1 - \Delta(\gamma)\rho(\gamma)e^{-\sigma\ell_\gamma}\right)^{(-1)^{1+\mathrm{ind}(\gamma)}}. \tag{4.10}$$

Proof. Without loss of generality, we assume that there is only one prime closed orbit γ. For $x \in \gamma$, $k \in \mathbf{N}^*$, let us calculate the sign of $\det(1 - D\phi_{k\ell_\gamma})|_{T_xM/\mathbf{R}V(x)}$. By (4.6), we have

$$\det(1 - D\phi_{k\ell_\gamma})|_{T_xM/\mathbf{R}V(x)} = \det\left(1 - (D\phi_{\ell_\gamma})^k\right)|_{E^s_{\gamma,x}} \det\left(1 - (D\phi_{\ell_\gamma})^k\right)|_{E^u_{\gamma,x}}. \tag{4.11}$$

Note that the non-real eigenvalues α and $\overline{\alpha}$ of $(D\phi_{\ell_\gamma})^k$ come in pair. Note also that by property of E^s, all the absolute value of the real eigenvalues of $(D\phi_{\ell_\gamma})^k$ on $E^s_{\gamma,x}$ is bounded by 1. So the sign of (4.11) comes from the positive eigenvalues of $D\phi_{\ell_\gamma}|_{E^u_{\gamma,x}}$. More precisely, we have

$$\begin{aligned}\operatorname{sgn}\left(\det\left(1-(D\phi_{\ell_\gamma})^k\right)|_{T_xM/\mathbf{R}V(x)}\right) &= \operatorname{sgn}\det\left(-\left(D\phi_{\ell_\gamma}\right)^k|_{E^u_{\gamma,x}}\right)\\ &= (-1)^{\operatorname{ind}(\gamma)}\Delta(\gamma)^k.\end{aligned} \tag{4.12}$$

Equation (4.10) is a consequence of (2.21), (3.21), and (4.12). □

The following theorem is due to Fried [F87, Theorem 3.1] if F is unitarily flat, and can be extended to unimodular vector bundle [ShY18, Proposition 2.12].

Theorem 4.6. *Assume that F is a unimodular flat vector bundle, and that $\phi_\cdot$ does not have fixed points. If for all $\gamma \in B$, $\Delta(\gamma)$ is not an eigenvalue of $\rho(\gamma)$, then we have $H^\cdot(M,F)=0$, and $R_{\phi,\rho}$ is regular at $\sigma=0$, so that*

$$|R_{\phi,\rho}(0)| = \prod_{\gamma\in B}\left|\det\left(1-\Delta(\gamma)\rho(\gamma)\right)\right|^{(-1)^{1+\operatorname{ind}(\gamma)}} = T_F(M). \tag{4.13}$$

Proof. Following [Fra82, Definition 9.10], let

$$\varnothing = M^0 \subset M^1 \subset \cdots \subset M^N = M \tag{4.14}$$

be a Smale filtration on M associated to the flow $\phi_\cdot$. Note that each $M^p \subset M$ is a submanifold with boundary, and can be constructed by the sublevel set of a smooth Lyapunov function. Also, we have

- on each ∂M^p, V points toward the inside of M^p;
- there is only one prime closed orbit γ in $M^{p+1}\backslash M^p$ and

$$\gamma = \bigcap_{t\in\mathbf{R}} \phi_t\left(M^{p+1}\backslash M^p\right). \tag{4.15}$$

Note that M^{p+1} can be obtained from M^p by attaching a round $\operatorname{ind}(\gamma)$-handle,

$$M^{p+1} = M^p \cup_{\mathbb{S}^1\times\mathbb{S}^{\operatorname{ind}(\gamma)}\times\mathbb{D}^{m-\operatorname{ind}(\gamma)-1}} \mathbb{S}^1\times\mathbb{D}^{\operatorname{ind}(\gamma)}\times\mathbb{D}^{m-\operatorname{ind}(\gamma)-1}, \tag{4.16}$$

where $m=\dim M$. By [Fra82, Theorem 9.11] (see also [SM96b, Section 2]),

$$H^q\left(M^{p+1},M^p,F\right) = H^{q-\operatorname{ind}(\gamma)}\left(\gamma, o(E^u_\gamma)\otimes_{\mathbf{R}} F|_\gamma\right), \tag{4.17}$$

where $o(E^u_\gamma)$ is the orientation line bundle of E^u_γ along the closed orbit γ.

By Examples 1.1 and 1.2, we see that $\Delta(\gamma)$ is not an eigenvalue of $\rho(\gamma)$ if and only if

$$H^\cdot\left(M^{p+1},M^p,F\right)=0. \tag{4.18}$$

Using the long exact sequence

$$\cdots\to H^q(M^p,M^{p-1},F)\to H^q(M^p,F)\to H^q(M^{p-1},F)\to\cdots, \tag{4.19}$$

we see that

$$H^{\cdot}(M,F) = H^{\cdot}(M^N,F) \simeq H^{\cdot}\left(M^{N-1},F\right) \simeq \cdots \simeq H^{\cdot}\left(M^0,F\right) = 0. \tag{4.20}$$

The first equality of (4.13) is a trivial consequence of (4.10). Let us show the second equality of (4.13).

We claim that

$$T_F(M) = \prod_{\gamma \in B} \left(T_{o(E^u_\gamma)\otimes_{\mathbf{R}} F|_\gamma}(\gamma)\right)^{(-1)^{\mathrm{ind}(\gamma)}}. \tag{4.21}$$

Indeed, if F is unitary, using (4.16), (4.19), the Cheeger–Müller theorem [C79, M78], we can deduce (4.21) from the excision property of the Reidemeister torsion. If F is only unimodular, using Bismut–Zhang's theorem [BZ92, Theorem 0.2] and a property of Milnor torsion, (4.21) still holds true (see [ShY18, Proposition 2.13]).

By (1.13), (1.14), and (4.21), we get the second equality of (4.13). □

It is natural to ask if a similar result holds when the Morse–Smale flow has both fixed points and closed orbits. However, Sánchez-Morgado [SM96b] has shown that the heteroclinic orbits have a non-trivial contribution in the torsion invariant, and in this way he constructed a counterexample to the Fried conjecture on Seifert manifolds.

In [ShY18], Shen and Yu constructed a Milnor metric, which indeed contains the heteroclinic contributions, and obtained a proof for a modified version of the Fried conjecture for generally Morse–Smale flow with or without fixed points and for arbitrary flat vector bundle with arbitrary Hermitian metric. The formulation and the proof are based on [BZ92], using the determinant line of the cohomology. We refer the reader to [ShY18] for more details.

Let us mention that there is another interpretation of the Ruelle dynamical zeta function provided by Dang–Rivière [DaRi20c]. See also [DaRi19, DaRi20a, DaRi20b, DaRi21] for related works.

5. Geodesic flow

In this section, we discuss the Fried conjecture in the most interesting case the geodesic flow on the unit tangent bundle of a non-positively curved Riemannian manifold of dimension $\geqslant 3$. (The case of surface will be treated in Section 6.4.) In this case, the Fried conjecture can be proved formally via Bismut–Goette's V-invariant [BG04]. However, this argument remains non-rigorous due to the involved infinite-dimensional loop space. For locally symmetric spaces, we give the rigorous argument [F86a, MoSt91, Sh18] which is based on the Selberg trace formula and the Bismut orbital integral formula [B11].

This section is organized as follows. In Section 5.1, we study the flat vector bundle defined on the unit tangent bundle.

In Section 5.2, we study the geodesic flow on the unit tangent bundle of a non-positively curved Riemannian manifold.

In Section 5.3, we sketch the formal proof of the Fried conjecture via Bismut–Goette's V-invariant.

In Section 5.4, we introduce the reductive group, symmetric space, and the locally symmetric space. We state the main Theorem 5.9 of this section.

Sections 5.5–5.8 are devoted to sketching the proof of Theorem 5.9.

In Section 5.5, we recall the Selberg trace formula and the Bismut orbital integral formula, which are our main analytic tools.

In Section 5.6, we give a new proof of Moscovici–Stanton's vanishing theorem [MoSt91], which is due to Bismut [B11].

In Section 5.7, we sketch the proof of the Fried conjecture for the real reductive group of **R**-rank one, which is originally due to Fried [F86a].

In Section 5.8, we sketch the proof of the Fried conjecture for general reductive groups [Sh18].

5.1. Flat vector bundle on the unit tangent bundle

Let (Z, g^{TZ}) be a compact Riemannian manifold of dimension m. Let

$$M = SZ = \{(x,v) \in TZ : |v| = 1\} \tag{5.1}$$

be the unit tangent bundle. Then $\pi : M \to Z$ is a fibration of sphere $\mathbb{S}^{m-1}$.

Proposition 5.1. *Assume* $\dim Z \geqslant 3$. *Then all the flat vector bundles on M are the pull back of a flat vector bundle F on Z. Moreover, F is unitary* (*resp. unimodular*) *if and only if π^*F is unitary* (*resp. unimodular*).

Proof. Consider the long exact sequence of homotopy groups

$$\cdots \to \pi_1(\mathbb{S}^{m-1}) \to \pi_1(M) \to \pi_1(Z) \to \pi_0(\mathbb{S}^{m-1}) \to \cdots \tag{5.2}$$

associated to the fibration $M \to Z$. Since $m \geqslant 3$, $\pi_1(\mathbb{S}^{m-1}) = \pi_0(\mathbb{S}^{m-1}) = 1$. By (5.2), we have the isomorphism of the groups

$$\pi_1(M) \simeq \pi_1(Z), \tag{5.3}$$

from which we deduce our proposition. □

Proposition 5.2. *Assume that Z is orientable and that* $\dim Z \geqslant 3$. *Then π^*F is acyclic on M if and only if F is acyclic on Z. In additional, if* $\dim Z$ *is odd, then*[6]

$$T_{\pi^*F}(M) = T_F(Z)^2. \tag{5.4}$$

Proof. Let

$$\begin{aligned} \cdots \to H^{p-m}(Z,F) \to H^p(Z,F) \to H^p(M,\pi^*F) \to H^{p-m+1}(Z,F) \\ \to H^{p+1}(Z,F) \to \cdots \end{aligned} \tag{5.5}$$

[6] Since both of M and Z have odd dimensions, by Theorem 1.6, we do not need to specify the metric data to define the analytic torsion.

be the Gysin exact sequence [BoTu82, Proposition 14.33] associated to the fibration $M \to Z$ of orientable $(m-1)$-spheres. Comparing the degrees in (5.5), for $0 \leqslant p \leqslant m-2$ and $m+1 \leqslant q \leqslant 2m-1$, we have

$$H^p(M, \pi^* F) = H^p(Z, F), \qquad H^q(M, \pi^* F) = H^{q-m+1}(Z, F). \tag{5.6}$$

By (5.6), using $m \geqslant 3$, we get the first statement of our proposition. By (1.16), we get (5.4). □

Remark 5.3. Assume that $\dim Z \geqslant 3$ is odd. If we do not assume that Z is orientable, a detailed analysis on the Leray spectral sequence tells us that $\pi^* F$ is acyclic on M if and only if F and $o(TZ) \otimes_{\mathbf{R}} F$ are acyclic on Z. In this case, we have

$$T_{\pi^* F}(M) = T_F(Z) \cdot T_{o(TZ)\otimes_{\mathbf{R}} F}(Z). \tag{5.7}$$

Let us consider the geodesic flow on a negatively curved orientable surface (Z, g^{TZ}) of genus $g \geqslant 2$, which will be used in Section 6.4. The long exact sequence of the homotopy groups (5.2) is simply

$$1 \to \mathbf{Z} \to \pi_1(M) \to \pi_1(Z) \to 1. \tag{5.8}$$

Let $a_0 \in \pi_1(M)$ be a generator of $\mathbf{Z} \subset \pi_1(M)$. Let F be a flat vector bundle of rank r on M with holonomy ρ. Then $\rho(a_0) \in \mathrm{GL}_r(\mathbf{C})$.

Corollary 5.4. *Let us assume that Z is a negatively curved orientable surface. Then $H^\cdot(M, F) = 0$ if and only if 1 is not an eigenvalue of $\rho(a_0)$. In this case, we have*

$$T_F(M) = \Big| \det\big(\rho(a_0)\big)\Big|^{(1-g)} \cdot \Big| \det\big(1 - \rho(a_0)\big)\Big|^{2(g-1)}. \tag{5.9}$$

Proof. We use the Leray spectral sequence associated to the fibration $M \to Z$ as in the proof of Propositions 3.2. By [BoTu82, Theorem 14.18], $(E_r^{p,q}, d_r)_{r\geqslant 0}$ converges to $H^\cdot(M, F)$, and

$$E_2^{p,q} = H^p\big(Z, H^q(Y, F|_Y)\big). \tag{5.10}$$

Since the fibre Y is a circle, for $r \geqslant 2$ and $q \neq 0$ or 1, we see that $E_r^{p,q} = 0$. Since d_r sends $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$. We get $d_3 = d_4 = \cdots = 0$, and d_2 vanishes except

$$d_2 : E_2^{0,1} \to E_2^{2,0}. \tag{5.11}$$

Indeed, by [BoTu82, p. 178], up to a sign, $d_2|_{E_2^{0,1}}$ is the multiplication by the Euler characteristic class of Z, which is an isomorphism since $g \geqslant 2$. So the spectral sequence degenerates at $r = 3$ such that

$$\begin{aligned} H^0(M,F) &= H^0(Z, H^0(Y, F|_Y)), \quad & H^1(M,F) &= H^1(Z, H^0(Y, F|_Y)), \\ H^2(M,F) &= H^0(Z, H^1(Y, F|_Y)), \quad & H^3(M,F) &= H^2(Z, H^1(Y, F|_Y)). \end{aligned} \tag{5.12}$$

Since Y is a circle and since Z is orientable, we have an isomorphism of vector bundles on Z,

$$H^1(Y, F|_Y) \simeq H^0(Y, F|_Y). \tag{5.13}$$

By (5.12) and (5.13), we see that F is acyclic on M if and only if $H^0(Y, F|_Y)$ is acyclic on Z. By the Gauss–Bonnet theorem,

$$\chi(Z, H^0(Y, F|_Y)) = 2\,\mathrm{rk}[H^0(Y, F|_Y)](1-g), \tag{5.14}$$

where $\mathrm{rk}[H^0(Y, F|_Y)]$ denotes the rank of the vector bundle $H^0(Y, F|_Y)$. By (5.14), we see that $H^0(Y, F|_Y)$ is acyclic on Z if and only if $H^0(Y, F|_Y) = 0$. The latter means exactly that 1 is not an eigenvalue of $\rho(a_0)$. By (1.13), (1.14), and (1.15), we get (5.9). □

Remark 5.5. Equation (5.9) is obtained in [F86b] by a purely topological method under the assumption that F is unitarily flat.

5.2. Geodesic flow of the non-positively curved manifolds

Let (Z, g^{TZ}) be a Riemannian manifold with non-positive sectional curvature. Let $(\phi_t)_{t\in\mathbf{R}}$ be the geodesic flow on the unit tangent bundle $M = SZ$. Let $V \in C^\infty(M, TM)$ be the generator of $\phi_\cdot$.

Let $LZ = C^\infty(\mathbb{S}^1, Z)$ be the free loop space of Z. It is equipped with a canonical $\mathbb{S}^1$-action given by

$$s \cdot x_\cdot = x_{\cdot+s}, \qquad \text{for } s \in \mathbb{S}^1, x_\cdot \in LZ. \tag{5.15}$$

Denote by Γ the fundamental group of Z. Let $[\Gamma]$ be the set of conjugacy classes of Γ. We identify $[\Gamma]$ with the set of freely homotopy classes of loops on Z. For $[\gamma] \in [\Gamma]$, let $(LZ)_{[\gamma]} \subset LZ$ be the subset of LZ formed by all the loops on Z with the freely homotopy class $[\gamma]$. Write

$$LZ = \coprod_{[\gamma]\in[\Gamma]} (LZ)_{[\gamma]}. \tag{5.16}$$

Let E be the $\mathbb{S}^1$-invariant energy functional on LZ defined by

$$E : x_\cdot \in LZ \to \frac{1}{2}\int_0^1 |\dot{x}_s|^2 ds. \tag{5.17}$$

Then E is a convex function whose critical points are closed geodesics. Let $B_{[\gamma]}$ be the critical points set of E in $(LZ)_{[\gamma]}$. Since (Z, g^{TZ}) has non-positive sectional curvature,

$$B_{[1]} \simeq Z. \tag{5.18}$$

Let us give a more explicit description of $B_{[\gamma]}$. Take an element γ in $[\gamma]$. Up to evident isomorphisms, our objects constructed below do not depend on the choice of γ in $[\gamma]$. Let X be the universal cover of Z. Let g^{TX} be the induced Riemannian metric on X by g^{TZ}. Let $d_X(\cdot,\cdot)$ be the Riemannian distance on X. Since Z and also X have non-positive sectional curvature, the displacement function $x \in X \to d_X(x, \gamma x)$ is convex. Let $\ell_{[\gamma]} \in \mathbf{R}_+$ be the minimal value[7] of the displacement function on X and let $X(\gamma) \subset X$ be its minimal set. Since Z is compact and since the displacement function is convex, $X(\gamma)$ is a non-empty

[7] As the notation indicates, $\ell_{[\gamma]}$ does not depend on the choice of $\gamma \in [\gamma]$.

convex set (see [BaGrSch85, Section 6], [Ma17, Proposition 3.9]). Let $\Gamma(\gamma)$ be the centralizer of γ in Γ. Then $\Gamma(\gamma)$ acts on $X(\gamma)$. We have the identification

$$x. \in B_{[\gamma]} \simeq [\widetilde{x}_0] \in \Gamma(\gamma)\backslash X(\gamma), \tag{5.19}$$

where $\widetilde{x}.$ is the lifting of $x.$ in X. By the convexity of $X(\gamma)$, we see that $B_{[\gamma]}$ is connected. Recall that $M = SZ$. We can also identify $B_{[\gamma]}$ with a connected component of $M^{\phi_{\ell_{[\gamma]}}}$ by

$$x. \in B_{[\gamma]} \to (x_0, \dot{x}_0/\ell_{[\gamma]}) \in M^{\phi_{\ell_{[\gamma]}}}. \tag{5.20}$$

Since $M^{\phi_{\ell_{[\gamma]}}}$ is compact, we see that $B_{[\gamma]}$ is compact. By (5.20), equations (2.14) and (2.15) become

$$\wp(\phi.) = \coprod_{[\gamma]\in[\Gamma]-\{1\}} B_{[\gamma]} \times \{\ell_{[\gamma]}\}, \quad \overline{\wp}(\phi.) = \coprod_{[\gamma]\in[\Gamma]-\{1\}} B_{[\gamma]}/\mathbb{S}^1 \times \{\ell_{[\gamma]}\}. \tag{5.21}$$

Proposition 5.6. *Assume that (Z, g^{TZ}) has non-positive sectional curvature. The following statements hold.*

(1) *Condition (1) of Assumption 2.4 holds.*
(2) *Condition (2) of Assumption 2.4 holds if Z is a negatively curved manifold or if Z is a locally symmetric space.*

Proof. For Condition (1), it is enough to show that for $r \geqslant 0$ the set

$$\{[\gamma] \in [\Gamma] : \ell_{[\gamma]} \leqslant r\} \tag{5.22}$$

is finite. Fix a point $x \in X$ in the universal cover X. Fix a fundamental domain $F_Z \subset X$ of Z in X. There is $z_0 \in F_Z$ and $\gamma_1 \in \Gamma$ such that

$$\ell_{[\gamma]} = d_X(\gamma_1 z_0, \gamma\gamma_1 z_0). \tag{5.23}$$

By (5.23) and by the triangular inequality, we have

$$d_X(\gamma_1 x, \gamma\gamma_1 x) \leqslant \ell_{[\gamma]} + 2 \max_{z\in F_Z} d(x, z). \tag{5.24}$$

Using (5.24), we get

$$|\{[\gamma] \in [\Gamma] : \ell_{[\gamma]} \leqslant r\}| \leqslant |\{\gamma \in \Gamma : d_X(x, \gamma x) \leqslant r + 2 \max_{z\in F_Z} d(x, z)\}|. \tag{5.25}$$

It is known by [Mi68b, Remark p.1, Lemma 2] that the set on the right-hand side of (5.25) is finite[8].

The statement (2) is clear if Z is a negatively curved manifold. When Z is a locally symmetric space, the statement (2) is a consequence of [B11, (3.3.17), (3.5.10)]. □

5.3. The V-invariant and a formal proof of the Fried conjecture

Let us give a formal proof of (2.22) using the V-invariant of Bismut–Goette [BG04].

[8] Indeed, it is smaller than Ce^{Cr} for certain $C > 0$.

5.3.1. The V-invariant. Let S be a closed manifold equipped with an action of a compact Lie group L, with Lie algebra $\mathfrak{l}$. If $a \in \mathfrak{l}$, let a^S be the corresponding vector field on S. Bismut–Goette [BG04] introduced the V-invariant $V_a(S) \in \mathbf{R}$.

Let f be an a^S-invariant Morse–Bott function on S. Let $B_f \subset S$ be the critical submanifold. Since $a^S|_{B_f} \in TB_f$, $V_a(B_f)$ is also well defined. By [BG04, Theorem 4.10], $V_a(S)$ and $V_a(B_f)$ are related by a simple formula.

5.3.2. A formal proof of the Fried conjecture. Let us argue formally as in [Sh18, Section 1E]. Let a be the generator of the Lie algebra of $\mathbb{S}^1$ such that $\exp(a) = 1$.

As explained in [B05, Equation (0.3)], if F is a unitarily flat vector bundle on Z such that $H^{\cdot}(Z, F) = 0$, at least formally, we have

$$\log T_F(Z) = -\sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}[\rho(\gamma)] V_a\big((LZ)_{[\gamma]}\big). \tag{5.26}$$

Let (Z, g^{TZ}) be an odd-dimensional oriented Riemannian manifold with non-positive sectional curvature. Assume that the energy functional is Morse–Bott[9]. Applying [BG04, Theorem 4.10] to the infinite-dimensional manifold $(LZ)_{[\gamma]}$ with the $\mathbb{S}^1$-invariant Morse–Bott functional E, we have the formal identity

$$V_a\big((LZ)_{[\gamma]}\big) = V_a\big(B_{[\gamma]}\big). \tag{5.27}$$

Since $B_{[1]} \simeq Z$ is formed by the trivial geodesics, by the definition of the V-invariant,

$$V_a(B_{[1]}) = 0. \tag{5.28}$$

By [BG04, Proposition 4.26], if $[\gamma] \in [\Gamma] - \{1\}$, then

$$V_a(B_{[\gamma]}) = -\frac{\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1)}{2m_{[\gamma]}}. \tag{5.29}$$

By Proposition 5.2 and by (5.26)–(5.29), we get a formal identity

$$\log T_{\pi^*F}(M) = 2\log T_F(Z) = \sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1)}{m_{[\gamma]}}, \tag{5.30}$$

which is formally just (2.22)[10].

5.4. Reductive group, globally and locally symmetric space

We give a rigorous argument in the case of locally symmetric space of reductive type. Let us recall some preliminaries that are needed.

[9] Since this part is completely formal, we will not give the precise definition of the Morse–Bott function on the loop space LZ. Here we can understand it by the fact that the energy functional is convex and its critical points form a smooth manifold.

[10] We need also to show that the sign that appeared in the Fuller index is positive when $\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1) \neq 0$. Indeed, it follows from the fact that Z has an odd dimension and is orientable, and the fact that $\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1) = 0$ if $\dim B_{[\gamma]}$ is even. We omit the details. For negatively curved manifolds, a proof can be found in [GiLiPo13, Appendix B].

5.4.1. Reductive group. Let G be a linear connected real reductive group [K86, p. 3], and let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution. This means that G is a closed connected group of real matrices that is stable under transpose, and θ is the composition of transpose and inverse of matrices. Let K be the maximal compact subgroup of G which is the fixed point set of θ.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K. The Cartan involution θ acts naturally as a Lie algebra automorphism of $\mathfrak{g}$. Then $\mathfrak{k}$ is the eigenspace of θ associated with the eigenvalue 1. Let $\mathfrak{p}$ be the eigenspace with the eigenvalue -1, so that

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{5.31}$$

We have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{5.32}$$

By [K86, Proposition 1.2], we have the diffeomorphism

$$(Y, k) \in \mathfrak{p} \times K \to e^Y k \in G. \tag{5.33}$$

Let B be a real-valued non-degenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action Ad of G on $\mathfrak{g}$, and also under θ. Then (5.31) is an orthogonal splitting of $\mathfrak{g}$ with respect to B. We assume B to be positive on $\mathfrak{p}$, and negative on $\mathfrak{k}$. The form $\langle \cdot, , \cdot \rangle = -B(\cdot, \theta \cdot)$ defines an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$ such that the splitting (5.31) is still orthogonal. We denote by $|\cdot|$ the corresponding norm.

The real (resp. complex) rank of G, denoted by $\mathrm{rk}_{\mathbf{R}}\, G$ (resp. $\mathrm{rk}_{\mathbf{C}}\, G$), is defined by the dimension of the maximal abelian subspace of $\mathfrak{p}$ (resp. the Cartan subalgebra of $\mathfrak{g}$). The fundamental rank of G is defined by

$$\delta(G) = \mathrm{rk}_{\mathbf{C}}\, G - \mathrm{rk}_{\mathbf{C}}\, K \in \mathbf{N}. \tag{5.34}$$

Clearly, $\delta(\mathfrak{g}) \in \mathbf{N}$ is also well defined. The following proposition is a consequence of the classification theory of real simple Lie algebras (see [B11, Remark 7.9.2]).

Theorem 5.7. *The only simple Lie algebras*[11] $\mathfrak{g}$ *with* $\delta(\mathfrak{g}) = 1$ *are given by*

$$\mathfrak{sl}_3(\mathbf{R}) \ \textit{or} \ \mathfrak{so}(p, q) \ \textit{with} \ pq > 1 \ \textit{odd}. \tag{5.35}$$

In Table 1, we list some reductive groups G, the maximal compact subgroups K, the dimensions $\mathfrak{p}$, and the fundamental ranks $\delta(G)$.

5.4.2. Semisimple elements. If $\gamma \in G$, we denote by $Z(\gamma) \subset G$ the centralizer of γ in G, and by $\mathfrak{z}(\gamma) \subset \mathfrak{g}$ its Lie algebra. If $a \in \mathfrak{g}$, let $Z(a) \subset G$ be the stabilizer of a in G, and let $\mathfrak{z}(a) \subset \mathfrak{g}$ be its Lie algebra.

An element $\gamma \in G$ is said to be semisimple if γ can be conjugate to $e^a k^{-1}$ such that

$$a \in \mathfrak{p}, \qquad k \in K, \qquad \mathrm{Ad}(k) a = a. \tag{5.36}$$

[11] We have also $\delta(\mathbf{R}) = 1$. But the abelian Lie algebra $\mathbf{R}$ is not considered as simple Lie algebra.

TABLE 1. Examples of reductive Lie group.

G	$\mathbf{R}$	$\mathrm{GL}_n^+(\mathbf{R})$	$\mathrm{SL}_n(\mathbf{R})$	$\mathrm{SO}^0(p,q)$
K	$\{0\}$	$\mathrm{SO}(n)$	$\mathrm{SO}(n)$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$
$\dim\mathfrak{p}$	1	$\frac{n(1+n)}{2}$	$\frac{n(1+n)}{2}-1$	pq
$\delta(G)$	1	$n-\left[\frac{n}{2}\right]$	$n-1-\left[\frac{n}{2}\right]$	$\left[\frac{p+q}{2}\right]-\left[\frac{p}{2}\right]-\left[\frac{q}{2}\right]$

Let $\gamma = e^a k^{-1}$ be such that (5.36) holds. By [B11, (3.3.4), (3.3.6)], we have

$$Z(\gamma) = Z(a) \cap Z(k), \qquad \mathfrak{z}(\gamma) = \mathfrak{z}(a) \cap \mathfrak{z}(k). \tag{5.37}$$

Set

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \qquad \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}, \qquad K(\gamma) = Z(\gamma) \cap K. \tag{5.38}$$

From (5.37) and (5.38), we get

$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{5.39}$$

By [K02, Proposition 7.25], $Z(\gamma)$ is a reductive subgroup (not necessarily connected) of G with maximal compact subgroup $K(\gamma)$, and with Cartan decomposition (5.39).

5.4.3. The symmetric space. Set $X = G/K$ to be the associated symmetric space. Then

$$p : G \to X = G/K \tag{5.40}$$

is a K-principle bundle.

Let τ be a finite-dimensional orthogonal representation of K on the real Euclidean space E_τ. Then $\mathcal{E}_\tau = G \times_K E_\tau$ is a real Euclidean vector bundle on X. The space of smooth sections $C^\infty(X, \mathcal{E}_\tau)$ on X can be identified with the subspace $C^\infty(G, E_\tau)^K$ of K-invariant smooth E_τ-valued functions on G.

By adjoint action, K acts isometrically on $\mathfrak{p}$. Using the above construction, the tangent bundle of X is given by

$$TX = G \times_K \mathfrak{p}. \tag{5.41}$$

The bilinear form $B|_{\mathfrak{p}}$ induces a G-invariant Riemannian metric g^{TX} on X. It is well known that (X, g^{TX}) is a Riemannian manifold of non-positive sectional curvature. For $x, y \in X$, we denote by $d_X(x, y)$ the Riemannian distance on X.

5.4.4. The locally symmetric space. Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G. Take $Z = \Gamma\backslash X = \Gamma\backslash G/K$. Then Z is a connected closed orientable Riemannian locally symmetric manifold with non-positive sectional curvature. Since X is contractible, $\pi_1(Z) = \Gamma$ and X is the universal cover of Z.

By [Sel60, Lemmas 1], Γ contains the identity element and non-elliptic semisimple elements. Recall that $\Gamma(\gamma)$ is the centralizer of γ in Γ defined before (5.19). By [Sel60, Lemma 2], $\Gamma(\gamma)$ is cocompact in $Z(\gamma)$.

The following proposition is [DuKV79, Proposition 5.15], relating the geometric object considered in Section 5.2 and the group theoretic object considered in this section.

Proposition 5.8. *For $[\gamma] \in [\Gamma] - \{1\}$, we have the following identifications,*

$$\ell_{[\gamma]} = |a|, \qquad X(\gamma) \simeq Z(\gamma)/K(\gamma), \qquad B_{[\gamma]} \simeq \Gamma(\gamma)\backslash Z(\gamma)/K(\gamma). \tag{5.42}$$

Let us state the main theorem of [Sh18, Theorem 1.1], which generalizes [F86a, Theorem 3] in the case where $\mathrm{rk}_{\mathbf{R}}[G] = 1$ and [MoSt91, Corollary 2.2] in the case where $\delta(G) \neq 1$.

Theorem 5.9. *Assume that $Z = \Gamma\backslash G/K$ and $\dim Z \geqslant 3$ is odd. Let F be a unitarily flat vector bundle on Z. The following statements hold.*

(1) *The dynamical zeta function $R_{\phi,\rho}(\sigma)$ is well defined in the sense of Definition 2.7.*

(2) *There exist explicit constants $C_\rho \in \mathbf{R}^*$ and $r_\rho \in \mathbf{Z}$ (see (5.99)) such that, when $\sigma \to 0$,*

$$R_{\phi,\rho}(\sigma) = C_\rho T_F(Z)^2 \sigma^{r_\rho} + \mathcal{O}\left(\sigma^{r_\rho+1}\right). \tag{5.43}$$

(3) *If $H^{\cdot}(Z,F) = 0$, then*

$$C_\rho = 1, \qquad\qquad r_\rho = 0, \tag{5.44}$$

so that

$$R_{\phi,\rho}(0) = T_F(Z)^2 = T_{\pi^*(F)}(M). \tag{5.45}$$

We sketch the proof of Theorem 5.9 in Sections 5.5–5.8.

Remark 5.10. If we do not require Γ to be torsion free, then $Z = \Gamma\backslash G/K$ is an orbifold. In [ShY17], we show that the above theorem still holds true.

Remark 5.11. Using Hirzebruch proportionality [H66, Theorem 22.3.1] and Bott's formula [Bo65, p. 175], we can show that if $\delta(G) \geqslant 2$, then for all $[\gamma] \in [\Gamma] - \{1\}$,

$$\chi(B_{[\gamma]}/\mathbb{S}^1) = 0. \tag{5.46}$$

So $R_{\phi,\rho}(\sigma) \equiv 1$.

Remark 5.12. We do not need to study the case where $Z = \Gamma\backslash G/K$ with $\dim Z \geqslant 4$ is even. Recall that $\delta(G)$ and $\dim Z$ have the same parity. If $\delta(G) \geqslant 2$, by (1.10) and Remark 5.11, we have $R_{\phi,\rho}(0) = T_F(Z) = 1$. If $\delta(G) = 0$, there are no acyclic flat vector bundles on Z. Indeed, as in (5.46), by Hirzebruch proportionality [H66, Theorem 22.3.1] and Bott's formula [Bo65, p. 175], we can deduce that

$$(-1)^{\frac{1}{2}\dim Z}\chi(Z) > 0. \tag{5.47}$$

Using the Gauss–Bonnet–Chern Theorem, by (5.47), we see that

$$\chi(Z,F) = \mathrm{rk}[F]\chi(Z) \neq 0, \tag{5.48}$$

which implies $H^{\cdot}(Z,F) \neq 0$.

Remark 5.13. By Remark 5.11, if $\dim Z$ is odd and $\delta(G) \geqslant 3$, to show the Fried conjecture, it is enough to show $T_F(Z) = 1$, which is known as the Moscovici–Stanton vanishing theorem (see Section 5.6).

5.5. The Selberg trace formula

In this section, we recall the Selberg trace formula. We introduce the Casimir operator and its heat kernel, orbital integral, and an explicit orbital integral formula [B11, Theorem 6.1.1] for the heat operator of the Casimir.

5.5.1. Casimir operator. Let $\mathscr{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. We identify $\mathscr{U}(\mathfrak{g})$ with the algebra of left-invariant differential operators on G. Let $C^{\mathfrak{g}} \in \mathscr{U}(\mathfrak{g})$ be the Casimir element of $(\mathfrak{g}, B)$. If $\{e_1, \ldots, e_{\dim \mathfrak{p}}\}$ is an orthonormal basis of $\mathfrak{p}$, and if $\{f_1, \ldots, f_{\dim \mathfrak{k}}\}$ is an orthonormal basis of $\mathfrak{k}$, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{\dim \mathfrak{p}} e_i^2 + \sum_{i=1}^{\dim \mathfrak{k}} f_i^2. \tag{5.49}$$

It is well known that $C^{\mathfrak{g}}$ is in the center of $\mathscr{U}(\mathfrak{g})$.

We define $C^{\mathfrak{k}}$ in the same way. Let τ be a finite-dimensional representation of K on E_τ. We denote by $C^{\mathfrak{k}, E_\tau} \in \mathrm{End}(E_\tau)$ the corresponding Casimir operator acting on E_τ, so that

$$C^{\mathfrak{k}, E_\tau} = \sum_{i=1}^{\dim \mathfrak{k}} \tau(f_i)^2. \tag{5.50}$$

Let $C^{\mathfrak{g}, X, \tau}$ be the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$. Then $C^{\mathfrak{g}, X, \tau}$ is a formally self-adjoint second-order elliptic differential operator which is bounded from below. If $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$, then $C^\infty(X, \mathcal{E}_\tau) = \Omega^\cdot(X)$. In this case, we write $C^{\mathfrak{g}, X} = C^{\mathfrak{g}, X, \tau}$. By [B11, Proposition 7.8.1], $C^{\mathfrak{g}, X}$ coincides with the Hodge Laplacian acting on $\Omega^\cdot(X)$.

We denote by $\widehat{p} : \Gamma\backslash G \to Z$ and $\pi : X \to Z$ the natural projections, so that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "p"] & \Gamma\backslash G \arrow[d, "\widehat{p}"] \\ X \arrow[r, "\pi"] & Z \end{tikzcd} \tag{5.51}$$

commutes. Recall that the group Γ acts isometrically on the left on X. This action lifts to all the homogeneous Euclidean vector bundles $\mathcal{E}_\tau$ constructed in Subsection 5.4.3. It descends to a Euclidean vector bundle $\mathcal{F}_\tau = \Gamma\backslash\mathcal{E}_\tau$ on Z. Let F be the unitarily flat vector bundle of rank r on Z. Let $\rho : \Gamma \to \mathrm{U}(r)$ be the holonomy of F. Let $C^{\mathfrak{g}, Z, \tau, \rho}$ be the Casimir element of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{R}} F)$. As before, when $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$, we write $C^{\mathfrak{g}, Z, \rho} = C^{\mathfrak{g}, Z, \tau, \rho}$. Then,

$$\square^Z = C^{\mathfrak{g}, Z, \rho}. \tag{5.52}$$

5.5.2. The heat kernel of $C^{\mathfrak{g},X,\tau}$. We fix a finite-dimensional real orthogonal representation (τ, E_τ) of K.

Let $p_t^{X,\tau}(x,x')$ be the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to the Riemannian volume dv_X on X. The Gaussian estimate on the heat kernel tells us that for $t>0$, there exist $c>0$ and $C>0$ such that for $x,x'\in X$,

$$\left|p_t^{X,\tau}(x,x')\right| \leqslant C\exp\left(-c\, d_X^2(x,x')\right). \tag{5.53}$$

Set

$$p_t^{X,\tau}(g) = p_t^{X,\tau}(p1,pg). \tag{5.54}$$

It is a smooth $\mathrm{End}(E_\tau)$-valued function on G. For $g\in G$ and $k,k'\in K$, we have

$$p_t^{X,\tau}(kgk') = \tau(k)p_t^{X,\tau}(g)\tau(k'). \tag{5.55}$$

We can recover $p_t^{X,\tau}(x,x')$ by

$$p_t^{X,\tau}(x,x') = p_t^{X,\tau}(g^{-1}g'), \tag{5.56}$$

where $g,g'\in G$ are such that $pg=x, pg'=x'$.

In the sequel, we do not distinguish $p_t^{X,\tau}(x,x')$ and $p_t^{X,\tau}(g)$. We refer to both of them as the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$.

5.5.3. The Selberg trace formula. We will write the trace of $\exp(-tC^{\mathfrak{g},Z,\tau,\rho}/2)$ with the help of the heat kernel $p_t^{X,\tau}(g)$ on X. Let dv_K be the Haar measure on K such that the volume of K is equal to 1. By (5.33), dv_K and $B|_{\mathfrak{p}}$ induce a Haar measure dv_G on G. Let $dv_{Z(\gamma)}$ be the Haar measure on $Z(\gamma)$ constructed in the same way. Let $dv_{Z(\gamma)\backslash G}$ be the right invariant measure on $Z(\gamma)\backslash G$ such that

$$dv_G = dv_{Z(\gamma)}dv_{Z(\gamma)\backslash G}. \tag{5.57}$$

It is known (see [B11, (3.4.36)]) that for $s\gg 1$,

$$\int_{g\in Z(\gamma)\backslash G} e^{-sd_X(p\gamma g,pg)}dv_{Z(\gamma)\backslash G} < \infty. \tag{5.58}$$

Definition 5.14. Let $\gamma\in G$ be semisimple. We define the orbital integral of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ by

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right] = \int_{g\in Z(\gamma)\backslash G}\mathrm{Tr}^{E_\tau}\left[p_t^{X,\tau}(g^{-1}\gamma g)\right]dv_{Z(\gamma)\backslash G}. \tag{5.59}$$

By (5.53) and (5.58), we see that the integration in (5.59) is well defined.

Theorem 5.15. *There exist $c>0$, $C>0$ such that for $t>0$, we have*

$$\sum_{[\gamma]\in[\Gamma]-\{1\}}\mathrm{vol}\left(B_{[\gamma]}\right)\left|\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]\right| \leqslant C\exp\left(-\frac{c}{t}+Ct\right). \tag{5.60}$$

For $t>0$, the following identity holds,

$$\operatorname{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2\right)\right]=\sum_{[\gamma]\in[\Gamma]}\operatorname{vol}\left(B_{[\gamma]}\right)\operatorname{Tr}[\rho(\gamma)]\operatorname{Tr}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right]. \tag{5.61}$$

Proof. The estimate in (5.60) and the convergence of (5.61) are consequences of the following fact: there exist $c>0$, $C>0$ such that for $t>0$ and $x\in X$, we have

$$\sum_{\gamma\in\Gamma-\{1\}}\left|p_t^{X,\tau}(x,\gamma x)\right|\leqslant C\exp\left(-\frac{c}{t}+Ct\right). \tag{5.62}$$

The proof of (5.62) can be found in [Sh18, Proposition 4.8] (see also [MaMari15, Theorem 4]), where we use an estimate similar to the one in (5.25).

Let us prove (5.61). We disregard the convergence problem. Let $p_t^{Z,\tau,\rho}(z,z')$ be the smooth kernel of $\exp(-tC^{\mathfrak{g},Z,\tau,\rho}/2)$ with respect to the Riemannian volume on Z. For $x\in X$ and $g\in G$ such that $pg=x$, we have

$$p_t^{Z,\tau,\rho}(\pi x,\pi x)=\sum_{\gamma\in\Gamma}\rho(\gamma)p_t^{X,\tau}\left(g^{-1}\gamma g\right). \tag{5.63}$$

Recall that $F_Z\subset X$ is the fundamental domain of Z in X. Then, $p^{-1}(F_Z)\subset G$ is a fundamental domain of $\Gamma\backslash G$ in G. Since $\operatorname{vol}(K)=1$, by (5.63), we have

$$\operatorname{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2\right)\right]=\int_{g\in p^{-1}(F_Z)}\sum_{\gamma\in\Gamma}\operatorname{Tr}\left[\rho(\gamma)\right]\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]dv_G. \tag{5.64}$$

Take $\gamma\in\Gamma$. Recall that $[\gamma]\in[\Gamma]$ is the conjugacy class of γ in Γ. We claim that

$$\int_{g\in p^{-1}(F_Z)}\sum_{\gamma'\in[\gamma]}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]dv_G=\operatorname{vol}(\Gamma(\gamma)\backslash Z(\gamma))\operatorname{Tr}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right]. \tag{5.65}$$

Indeed, since $\gamma'\to(\gamma')^{-1}\gamma\gamma'$ induces an identification of $\Gamma(\gamma)\backslash\Gamma\simeq[\gamma]$, we have

$$\sum_{\gamma'\in[\gamma]}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma' g\right)\right]=\sum_{\gamma'\in\Gamma(\gamma)\backslash\Gamma}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}(\gamma')^{-1}\gamma\gamma' g\right)\right]. \tag{5.66}$$

By (5.66), and by changing variable, we have

$$\int_{g\in p^{-1}(F_Z)}\sum_{\gamma'\in[\gamma]}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma' g\right)\right]dv_G \tag{5.67}$$
$$=\int_{\bigcup_{\gamma'\in\Gamma(\gamma)\backslash\Gamma}\gamma' p^{-1}(F_Z)}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]dv_G=\int_{\Gamma(\gamma)\backslash G}\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]dv_{\Gamma(\gamma)\backslash G},$$

where in the last equality we use the fact that $\bigcup_{\gamma'\in\Gamma(\gamma)\backslash\Gamma}\gamma' p^{-1}(F_Z)$ is a fundamental domain of $\Gamma(\gamma)\backslash G$ in G. Using the fibration $\Gamma(\gamma)\backslash G\to Z(\gamma)\backslash G$, and using the fact that the integrand $\operatorname{Tr}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]$ is constant on the fibre, by (5.59) and (5.67), we get (5.65).

By Proposition 5.8, (5.64), and (5.65), using $\mathrm{vol}(K(\gamma)) = 1$, we get (5.61). $\square$

5.5.4. Bismut orbital integral formula. Let us recall an explicit orbital integral formula for $\mathrm{Tr}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right]$ obtained by Bismut [B11, Theorem 6.1.1].

Let $\gamma = e^a k^{-1} \in G$ be semisimple as in (5.36). Set

$$\mathfrak{p}_0 = \ker(\mathrm{ad}(a)) \cap \mathfrak{p}, \qquad \mathfrak{k}_0 = \ker(\mathrm{ad}(a)) \cap \mathfrak{k}, \qquad \mathfrak{z}_0 = \ker(\mathrm{ad}(a)) \cap \mathfrak{z}. \tag{5.68}$$

Let $\mathfrak{p}_0^\perp \subset \mathfrak{p}$, $\mathfrak{k}_0^\perp \subset \mathfrak{k}$, $\mathfrak{z}_0^\perp \subset \mathfrak{z}$ be the orthogonal spaces of $\mathfrak{p}_0, \mathfrak{k}_0, \mathfrak{z}_0$ with respect to B. Let $\mathfrak{p}_0^\perp(\gamma) \subset \mathfrak{p}_0$, $\mathfrak{k}_0^\perp(\gamma) \subset \mathfrak{k}_0$, $\mathfrak{z}_0^\perp(\gamma) \subset \mathfrak{z}_0$ be the orthogonal spaces of $\mathfrak{p}(\gamma), \mathfrak{k}(\gamma), \mathfrak{z}(\gamma)$ with respect to B. Then

$$\mathfrak{z}_0^\perp = \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp, \qquad \mathfrak{z}_0^\perp(\gamma) = \mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma). \tag{5.69}$$

In particular, we have the orthogonal decompositions with respect to B,

$$\mathfrak{p} = \underbrace{\mathfrak{p}(\gamma) \oplus \mathfrak{p}_0^\perp(\gamma)}_{\mathfrak{p}_0} \oplus \mathfrak{p}_0^\perp, \quad \mathfrak{k} = \underbrace{\mathfrak{k}(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma)}_{\mathfrak{k}_0} \oplus \mathfrak{k}_0^\perp, \quad \mathfrak{g} = \underbrace{\mathfrak{z}(\gamma) \oplus \mathfrak{z}_0^\perp(\gamma)}_{\mathfrak{z}_0} \oplus \mathfrak{z}_0^\perp. \tag{5.70}$$

Let us introduce the J-function [B11, Section 5.5] of Bismut. In [B11, Section 5.5], it has been shown that the analytic function defined for $Y \in \mathfrak{k}(\gamma)$ by

$$\frac{1}{\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{p}_0^\perp(\gamma)}} \tag{5.71}$$

has a natural square root, which is still analytic in $Y \in \mathfrak{k}(\gamma)$. For a Hermitian matrix H, define

$$\widehat{A}(H) = \det{}^{1/2}\left(\frac{H/2}{\sinh(H/2)}\right). \tag{5.72}$$

The square root in (5.72) is the positive square root of a positive real number.

Definition 5.16. Let J_γ be the analytic function on $\mathfrak{k}(\gamma)$ defined for $Y \in \mathfrak{k}(\gamma)$ by

$$\begin{aligned} J_\gamma(Y) = {} & \frac{1}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{p}(\gamma)}\big)}{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{k}(\gamma)}\big)} \\ & \left[\frac{1}{\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}} \frac{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}. \end{aligned} \tag{5.73}$$

Remark 5.17. There is $C_\gamma > 0$ such that for all $Y \in \mathfrak{k}(\gamma)$, we have

$$|J_\gamma(Y)| \leqslant C_\gamma e^{C_\gamma |Y|}. \tag{5.74}$$

Recall that $C^{\mathfrak{k},\mathfrak{p}}$ and $C^{\mathfrak{k},\mathfrak{k}}$ are defined as in (5.50) associated with the adjoint actions of K on $\mathfrak{p}$ and $\mathfrak{k}$. Write

$$c_{\mathfrak{g}} = -\frac{1}{8}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] - \frac{1}{24}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right] \in \mathbf{R}_+. \tag{5.75}$$

For $Y \in \mathfrak{k}(\gamma)$, let dY be the Lebesgue measure on $\mathfrak{k}(\gamma)$ induced by $-B$. The main result of [B11, Theorem 6.1.1] is the following.

Theorem 5.18. *For $t>0$, we have*

$$\operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]=\frac{1}{(2\pi t)^{\dim\mathfrak{z}(\gamma)/2}}\exp\left(-\frac{|a|^2}{2t}-\frac{c_{\mathfrak{g}}}{2}t\right)\\ \int_{Y\in\mathfrak{k}(\gamma)}J_\gamma(Y)\operatorname{Tr}^{E_\tau}\left[\tau\left(k^{-1}\right)\exp(-i\tau(Y))\right]\exp\left(-|Y|^2/2t\right)dY. \tag{5.76}$$

5.6. Moscovici–Stanton's vanishing theorem

In this section, we show Theorem 5.9 in the case of $\delta(G)\neq 1$. By Remark 5.13, it is enough to show $T_F(Z)=1$.

Let $\mathfrak{t}\subset\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. Set

$$\mathfrak{b}=\{Y\in\mathfrak{p}:[Y,\mathfrak{t}]=0\}. \tag{5.77}$$

Then $\mathfrak{h}=\mathfrak{b}\oplus\mathfrak{t}$ is a fundamental Cartan subalgebra of $\mathfrak{g}$.

Let $\gamma=e^ak^{-1}\in G$ be a semisimple element such that (5.36) holds. Let $\mathfrak{t}(\gamma)\subset\mathfrak{k}(\gamma)$ be a Cartan subalgebra of $\mathfrak{k}(\gamma)$. Set

$$\mathfrak{b}(\gamma)=\{Y\in\mathfrak{p}:[Y,\mathfrak{t}(\gamma)]=0,\operatorname{Ad}(k)Y=Y\}. \tag{5.78}$$

By (5.77) and (5.78), we have

$$\dim\mathfrak{b}(\gamma)\geqslant\dim\mathfrak{b}=\delta(G). \tag{5.79}$$

The following theorem is [B11, Theorem 7.9.1] (see also [Sh18, Theorem 4.12]).

Theorem 5.19. *Let $\gamma\in G$ be semisimple such that* $\dim\mathfrak{b}(\gamma)\geqslant 2$. *For $Y\in\mathfrak{k}(\gamma)$, we have*

$$\operatorname{Tr}_{\mathrm{s}}^{\Lambda^\cdot(\mathfrak{p}^*)}\left[N^{\Lambda^\cdot(\mathfrak{p}^*)}\operatorname{Ad}(k^{-1})\exp(-i\operatorname{ad}(Y))\right]=0. \tag{5.80}$$

In particular, for $t>0$, we have

$$\operatorname{Tr}_{\mathrm{s}}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)}\exp\left(-tC^{\mathfrak{g},X}/2\right)\right]=0. \tag{5.81}$$

Proof. Since the term on the left-hand side of (5.80) is $\operatorname{Ad}\left(K(\gamma)\right)$-invariant, it is enough to show (5.80) for $Y\in\mathfrak{t}(\gamma)$. If $Y\in\mathfrak{t}(\gamma)$, we have

$$\begin{aligned}&\operatorname{Tr}_{\mathrm{s}}^{\Lambda^\cdot(\mathfrak{p}^*)}\left[N^{\Lambda^\cdot(\mathfrak{p}^*)}\operatorname{Ad}(k^{-1})\exp(-i\operatorname{ad}(Y))\right]\\&=\frac{\partial}{\partial b}\Big|_{b=0}\det\left(1-e^b\operatorname{Ad}(k)\exp(i\operatorname{ad}(Y))\right)|_{\mathfrak{p}}.\end{aligned} \tag{5.82}$$

Since $\dim\mathfrak{b}(\gamma)\geqslant 2$, by (5.82), we get (5.80) for $Y\in\mathfrak{t}(\gamma)$. By (5.76) and (5.80), we get (5.81). □

In this way, Bismut [B11, Theorem 7.9.3] recovered [MoSt91, Corollary 2.2].

Corollary 5.20. *Let F be a unitarily flat vector bundle on Z. Assume that* $\dim Z$ *is odd and $\delta(G) \neq 1$. Then for any $t > 0$, we have*

$$\mathrm{Tr_s}\left[N^{\Lambda^{\cdot}(T^*Z)} \exp\left(-t\square^Z/2\right)\right] = 0. \tag{5.83}$$

In particular,

$$T_F(Z) = 1. \tag{5.84}$$

Proof. Since $\dim Z$ and $\delta(G)$ have the same parity, and since $\dim Z$ is odd and $\delta(G) \neq 1$, we get $\delta(G) \geqslant 3$. By (5.79), $\dim \mathfrak{b}(\gamma) \geqslant 3$. Thus, (5.83) is a consequence of (5.52), (5.61), and (5.81). □

5.7. The case $G = \mathrm{SO}^0(p, 1)$ with p odd

In this section, we assume $G = \mathrm{SO}^0(p, 1)$ with p odd. Then G is a semisimple Lie group of $\mathbf{R}$-rank 1, and $X = G/K$ is hyperbolic space of dimension p. The result of this section is due to Fried [F86a].

It is easy to show that up to a sign there exists $\alpha \in \mathfrak{b}^*$ such that for $a \in \mathfrak{b}$, $\mathrm{ad}(a)$ acting on $\mathfrak{g}$ has three eigenvalues $0, \pm\langle\alpha, a\rangle$. We have an orthogonal splitting with respect to B,

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}, \tag{5.85}$$

where $\mathfrak{b} \oplus \mathfrak{m}$ (resp. $\mathfrak{n}$, resp. $\overline{\mathfrak{n}}$) is the eigenspace of $\mathrm{ad}(a)$ for the eigenvalue 0 (resp. $\langle\alpha, a\rangle$, resp. $-\langle\alpha, a\rangle$). Let $\mathcal{M} \subset G$ be the connected subgroup of G associated to $\mathfrak{m}$. Then $\mathcal{M}$ is compact and isomorphic to $\mathrm{SO}(p-1)$. Moreover, the action of $\mathcal{M}$ on $\mathfrak{n}$ is just the $\mathrm{SO}(p-1)$-action on $\mathbf{R}^{p-1}$.

One feature of the group $\mathrm{SO}^0(p, 1)$ with p odd is that any non-elliptic semisimple element of γ can be conjugate to $e^a k^{-1}$ such that $a \in \mathfrak{b}, a \neq 0$ and $k \in \mathcal{M}$. Moreover, the subspaces $\mathfrak{z}_0$ and $\mathfrak{z}_0^\perp$ introduced in (5.68) and (5.69) do not depend on the choice of non-elliptic semisimple element of γ, and is given by

$$\mathfrak{z}_0 = \mathfrak{b} \oplus \mathfrak{m}, \qquad \mathfrak{z}_0^\perp = \mathfrak{n} \oplus \overline{\mathfrak{n}}. \tag{5.86}$$

Definition 5.21. For $\sigma \in \mathbf{C}$, we define a formal Selberg zeta function associated to a representation η of $\mathcal{M}$ and to a representation ρ of Γ by

$$Z_{\eta,\rho}(\sigma) = \exp\left(-\sum_{\substack{[\gamma]\in[\Gamma]-\{1\}\\ \gamma\sim e^a k^{-1}}} \mathrm{Tr}[\rho(\gamma)] \frac{\mathrm{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{e^{-\sigma\ell_{[\gamma]}}}{m_{[\gamma]}}\right). \tag{5.87}$$

The formal Selberg zeta function is said to be well defined if the same conditions as in Definition 2.7 hold.

For $\gamma = e^a k^{-1}$ such that $a \in \mathfrak{b}, a \neq 0$ and $k \in \mathcal{M}$, we have

$$\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2} = \sum_{j=0}^{\dim\mathfrak{n}} (-1)^j \,\mathrm{Tr}^{\Lambda^j(\mathfrak{n}^*)}\left[\mathrm{Ad}\left(k^{-1}\right)\right] e^{\left(\frac{1}{2}\dim\mathfrak{n}-j\right)|\alpha||a|}. \tag{5.88}$$

If we consider the Selberg zeta function associated to $\eta_j = \Lambda^j(\mathfrak{n}^*)$, using (5.88) and $\ell_{[\gamma]} = |a|$, we have

$$R_{\phi,\rho}(\sigma) = \prod_{j=0}^{\dim \mathfrak{n}} Z_{\eta_j,\rho}\left(\sigma + \left(j - \frac{\dim \mathfrak{n}}{2}\right)|\alpha|\right)^{(-1)^{j-1}}. \tag{5.89}$$

To show the meromorphic extension of $R_{\phi,\rho}$, it is enough to consider the meromorphic extension of $Z_{\eta_j,\rho}$.

There are two remarkable properties of η_j. First, since η_j is just the representation of $\mathrm{SO}(p-1)$ on $\Lambda^j(\mathbf{R}^{p-1})$, the Casimir of $\mathfrak{m}$ acts on η_j is a scalar. Second, η_j has a lift as a virtual representation of K. More precisely, let $\mathrm{RO}(K), \mathrm{RO}(\mathcal{M})$ be the real representation rings of K and $\mathcal{M}$. The restriction induces a morphism of rings

$$\iota : \mathrm{RO}(K) \to \mathrm{RO}(\mathcal{M}). \tag{5.90}$$

Since K and $\mathcal{M}$ have the same complex rank, ι is an injection. The second property is that η_j has a lift $\widehat{\eta}_j = \widehat{\eta}_j^+ - \widehat{\eta}_j^- \in \mathrm{RO}(K)$, where

$$\widehat{\eta}_j^+ = \Lambda^j(\mathfrak{p}^*) + \Lambda^{j-2}(\mathfrak{p}^*) + \cdots, \qquad \widehat{\eta}_j^- = \Lambda^{j-1}(\mathfrak{p}^*) + \Lambda^{j-3}(\mathfrak{p}^*) + \cdots. \tag{5.91}$$

For a virtual representation $\widehat{\eta} = \widehat{\eta}^+ - \widehat{\eta}^- \in \mathrm{RO}(K)$, where $\widehat{\eta}^+$, $\widehat{\eta}^-$ are two representations of K, we use the notation

$$\det{}_{\mathrm{s}}\left(\sigma + C^{\mathfrak{g},Z,\widehat{\eta},\rho}\right) = \frac{\det\left(\sigma + C^{\mathfrak{g},Z,\widehat{\eta}^+,\rho}\right)}{\det\left(\sigma + C^{\mathfrak{g},Z,\widehat{\eta}^-,\rho}\right)}. \tag{5.92}$$

Recall that $c_{\mathfrak{g}} \in \mathbf{R}_+$ is defined in (5.75). We define $c_{\mathfrak{m}} \in \mathbf{R}_+$ in the same way.

Proposition 5.22. *Assume that η has a lift $\widehat{\eta} \in \mathrm{RO}(K)$ and that the Casimir of $\mathfrak{m}$ on E_η is a scalar $C^{\mathfrak{m},\eta} \in \mathbf{R}_-$. Then $Z_{\eta,\rho}(\sigma)$ is well defined such that*

$$Z_{\eta,\rho}(\sigma) = e^{P_\eta(\sigma)} \det{}_{\mathrm{s}}\left(\sigma^2 - c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta} + C^{\mathfrak{g},Z,\widehat{\eta},\rho}\right). \tag{5.93}$$

where $P_\eta(\sigma)$ is an odd polynomial of σ.

Proof. By a general theorem of elliptic operators [Vo87], we know that the right-hand side of (5.93) has a meromorphic extension to $\mathbf{C}$. So it is enough to show (5.93) for $\sigma \in \mathbf{R}$ and $\sigma \gg 1$. In this case, using the Mellin transform, we have

$$\begin{aligned} &\det{}_{\mathrm{s}}\left(\sigma^2 - c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta} + C^{\mathfrak{g},Z,\widehat{\eta},\rho}\right) \\ &= \exp\left(-\frac{\partial}{\partial s}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr_s}\left[\exp\left(-t\left(\sigma^2 - c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta} + C^{\mathfrak{g},Z,\widehat{\eta},\rho}\right)\right)\right] t^{s-1} dt\right). \end{aligned} \tag{5.94}$$

We use the Selberg trace formula to evaluate the second line of (5.94). For the group of $\mathbf{R}$-rank 1, Bismut's orbital integral formula has a very simple form

[B11, Theorem 8.2.1]. For $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$, $a \neq 0$, and $k \in \mathcal{M}$, we have

$$\begin{aligned} &\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] \qquad (5.95)\\ &= \frac{1}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|a|^2}{2t} + \frac{-c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta}}{2}t\right) \mathrm{Tr}\left[\eta(k^{-1})\right], \end{aligned}$$

and if $\gamma = 1$, we have

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{1}{\sqrt{t}} Q_\eta\left(\frac{1}{t}\right) \exp\left(\frac{-c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta}}{2}t\right). \qquad (5.96)$$

where Q_η is some explicit polynomial. Using Theorem 5.15, (5.94)–(5.96), we deduce (5.93) for $\sigma \in \mathbf{R}$ and $\sigma \gg 1$. □

Let us give an explicit formula for the constant appearing in Theorem 5.9. For $0 \leqslant j \leqslant \dim \mathfrak{n}$, set

$$r_j = \dim\ker C^{\mathfrak{g},Z,\widehat{\eta}_j^+,\rho} - \dim\ker C^{\mathfrak{g},Z,\widehat{\eta}_j^-,\rho}. \qquad (5.97)$$

By (5.52) and (5.91), we have

$$r_j = b_j(Z,F) - b_{j-1}(Z,F) + b_{j-2}(Z,F) - \cdots, \qquad (5.98)$$

where $b_j(Z,F) = \dim H^j(Z,F)$ is the Betti number. Clearly, $r_j = r_{\dim \mathfrak{n} - j}$.

Definition 5.23. Set

$$C_\rho = \prod_{j=0}^{\frac{\dim \mathfrak{n}}{2}-1} \left(-4\left(\frac{\dim \mathfrak{n}}{2} - j\right)^2 |\alpha|^2\right)^{(-1)^{j-1} r_j}, \quad r_\rho = 2\sum_{j=0}^{\frac{\dim \mathfrak{n}}{2}} (-1)^{j-1} r_j. \qquad (5.99)$$

Let us complete the proof of Theorem 5.9 in the case $G = \mathrm{SO}^0(\mathrm{p},1)$ with p odd.

Proof of (5.43) *and* (5.44). Simple calculation shows that

$$c_{\mathfrak{g}} - c_{\mathfrak{m}} + C^{\mathfrak{m},\eta_j} = \left|j - \frac{\dim \mathfrak{n}}{2}\right|^2 |\alpha|^2. \qquad (5.100)$$

By (5.89), (5.93), and (5.100), using the fact that P_η is odd, we see that as $\sigma \to 0$,

$$R_{\phi,\rho}(\sigma) = \left(1 + \mathcal{O}(\sigma)\right) \prod_{j=0}^{\dim \mathfrak{n}} \det\nolimits_{\mathrm{s}}\left(\sigma^2 + 2\sigma\Big(j - \frac{\dim \mathfrak{n}}{2}\Big)|\alpha| + C^{\mathfrak{g},Z,\widehat{\eta}_j,\rho}\right)^{(-1)^{j-1}}. \qquad (5.101)$$

On the other hand, by (1.8), (5.52), and (5.91), we have

$$\begin{aligned} \prod_{j=0}^{\dim \mathfrak{n}} \det\nolimits_{\mathrm{s}}\left(\sigma + C^{\mathfrak{g},Z,\widehat{\eta}_j,\rho}\right)^{(-1)^{j-1}} &= \prod_{q=1}^{\dim Z} \det\left(\sigma + \square_q^Z\right)^{(-1)^q q} \qquad (5.102)\\ &= \left(1 + \mathcal{O}(\sigma)\right)\sigma^{\sum_{q=1}^{\dim Z}(-1)^q q b_q(Z,F)} T_F(Z)^2. \end{aligned}$$

Comparing (5.101) and (5.102), by (5.98), we get (5.43).

By (5.98), if $H^{\cdot}(Z,F) = 0$, then (5.44) follows. □

5.8. The reductive group G with $\delta(G) = 1$

In this section, we assume that G is a real reductive group with $\delta(G) = 1$. From Table 1, we see three examples $\mathbf{R}$, $\mathrm{SL}_3(\mathbf{R})$, and $\mathrm{SO}^0(p, q)$ with $pq > 1$ odd. The result in this section is from [Sh18].

5.8.1. Meromorphic extension. We generalize the construction in Section 5.7 to the case in this section. By [Sh18, Proposition 6.2], we can define $\mathfrak{m}, \mathfrak{n}, \overline{\mathfrak{n}}$ is the same way. So the splitting (5.85) still holds true, and $\dim \mathfrak{n}$ is still even. However, the group $\mathcal{M}$ is not necessarily compact. It is a reductive group with maximal compact subgroup $K \cap \mathcal{M}$ and with Cartan decomposition

$$\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{5.103}$$

One of difference with the $\mathbf{R}$-rank 1 case is that a general non-elliptic semisimple element can not always be conjugate to $e^a k^{-1}$ with $a \in \mathfrak{b}$, $a \neq 0$, and $k \in \mathcal{M} \cap K$. In [Sh18, Proposition 4.1], we observe that this happens if and only if

$$\dim \mathfrak{b}(\gamma) = 1. \tag{5.104}$$

For such an element γ, we still have (5.86). Moreover, if $\dim \mathfrak{b}(\gamma) \geqslant 2$, as in Remark 5.13, we have

$$\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1) = 0. \tag{5.105}$$

This gives the motivation to introduce the following Selberg zeta function.

Definition 5.24. For $\sigma \in \mathbf{C}$, we define a formal Selberg zeta function associated to a representation η of $\mathcal{M}$ and to a representation ρ of Γ by

$$Z_{\eta,\rho}(\sigma) = \exp\left(- \sum_{\substack{[\gamma]\in[\Gamma]-\{1\} \\ \gamma\sim e^a k^{-1}}} \mathrm{Tr}[\rho(\gamma)] \frac{\mathrm{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\chi_{\mathrm{orb}}(B_{[\gamma]}/\mathbb{S}^1)}{m_{[\gamma]}} e^{-\sigma \ell_{[\gamma]}} \right). \tag{5.106}$$

Thanks to (5.105), the semisimple element γ satisfying $\dim \mathfrak{b}(\gamma) \geqslant 2$ does not have a contribution to the Selberg zeta function. Take $\eta_j = \Lambda^j(\mathfrak{n}^*)$ as before. Equation (5.89) still holds true. So we can reduce our problem to showing the meromorphic extension of $Z_{\eta_j,\rho}$.

Note that in this case, the Casimir operator of $\mathfrak{m}$ still acts as a constant on η_j. For technical reasons, we note also that the compact dual of $\mathcal{M}$ acts on $\eta_j \otimes_{\mathbf{R}} \mathbf{C}$. Also, the lifting property still holds but in a more complicated form. Consider the diagram

$$\begin{tikzcd} & \mathrm{RO}(\mathcal{M}) \arrow[d, "\iota_{\mathcal{M}}"] \\ \mathrm{RO}(K) \arrow[r, "\iota_K"] & \mathrm{RO}(K \cap \mathcal{M}), \end{tikzcd} \tag{5.107}$$

where the maps are induced by restriction. In [Sh18, Theorem 6.11], we show that $\iota_{\mathcal{M}}(\eta_j)$, $\Lambda^j(\mathfrak{p}_{\mathfrak{m}}^*) \in \mathrm{RO}(K \cap \mathcal{M})$ have unique lifts in $\mathrm{RO}(K)$.

Proposition 5.25. *Suppose that (η, E_η) is a finite-dimensional real representation of $\mathcal{M}$ such that*

- *the Casimir of $\mathfrak{m}$ acting on E_η is a scalar $C^{\mathfrak{m},\eta} \in \mathbf{R}_-$;*
- *the compact dual of $\mathcal{M}$ acts on $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$;*
- *the restriction $\iota_{\mathcal{M}}(\eta)$ to $K \cap \mathcal{M}$ has a lift in $\mathrm{RO}(K)$.*

Then $Z_{\eta,\rho}(\sigma)$ is well defined such that

$$Z_{\eta,\rho}(\sigma) = e^{P_\eta(\sigma)} \det\nolimits_{\mathrm{s}} \left(\sigma^2 - c_{\mathfrak{g}} + c_{\mathfrak{m}} - C^{\mathfrak{m},\eta} + C^{\mathfrak{g},Z,\widehat{\eta},\rho} \right), \tag{5.108}$$

where $P_\eta(\sigma)$ is an odd polynomial of σ, and $\widehat{\eta} \in \mathrm{RO}(K)$ is a virtual representation of K such that

$$\iota_K\left(\widehat{\eta}\right) = \Lambda^{\cdot}\left(\mathfrak{p}_{\mathfrak{m}}^*\right) \otimes \iota_{\mathcal{M}}\left(\eta\right) \in \mathrm{RO}(K \cap \mathcal{M}). \tag{5.109}$$

Proof. The proof of our proposition is similar to the one of Proposition 5.22, except that the evaluation of the orbital integral is more complicated. Note that when $\dim \mathfrak{b}(\gamma) \geqslant 2$, we have

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = 0. \tag{5.110}$$

This is a refinement of Theorem 5.19. □

Now a method similar to the one given in Section 5.7 shows (5.43) with the constants C_ρ and r_ρ defined by the same formula as in (5.99).

5.8.2. Regularity at $\boldsymbol{\sigma = 0}$. The proof of (5.44) is much more difficult since we do not have a relation between r_j and $b_j(Z,F)$ as in (5.98). We rely on some deep results from the representation theory. Here we only sketch the main steps.

Let $\widehat{G}_u$ be the unitary dual of G. For a unitary representation $V_\pi \in \widehat{G}_u$, denote by $V_{\pi,K}$ the associated Harish-Chandra $(\mathfrak{g}, K)$-module, which is formed by K-finite elements. The center of the complexified enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbf{C}})$ acts on $V_{\pi,K}$ as scalars, which is called the infinitesimal character and is denoted by χ_π. Clearly, for $a \in \mathbf{C}$, we have

$$\chi_\pi(a) = a. \tag{5.111}$$

We call χ_π is trivial if χ_π coincides with the infinitesimal character of the trivial representation of G.

Recall that $\widehat{p} : \Gamma\backslash G \to Z$ is the natural projection. The group G acts unitarily on the right on $L^2(\Gamma\backslash G, \widehat{p}^*F)$. By [GeGraPS69, p.23, Theorem], $L^2(\Gamma\backslash G, \widehat{p}^*F)$ decomposes into a discrete Hilbert direct sum with finite multiplicity of unitary representations of G. We can write

$$L^2\left(\Gamma\backslash G, \widehat{p}^*F\right) = \bigoplus_{\pi \in \widehat{G}_u}^{\mathrm{Hil}} n_\rho(\pi) V_\pi, \tag{5.112}$$

with $n_\rho(\pi) < \infty$.

Recall that τ is a real finite-dimensional orthogonal representation of K on the real Euclidean space E_τ, and that $C^{\mathfrak{g},Z,\tau,\rho}$ is the Casimir element of G acting on $C^\infty(Z,\mathcal{F}_\tau\otimes_{\mathbf{C}} F)$. By (5.112), we have

$$\ker C^{\mathfrak{g},Z,\tau,\rho}=\bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\big(V_{\pi,K}\otimes_{\mathbf{R}} E_\tau\big)^K, \tag{5.113}$$

By properties of elliptic operators, the sum on right-hand side of (5.113) is finite. We will apply (5.113) in the case $\tau=\Lambda^j(\mathfrak{p}^*)$ and also $\tau=\widehat{\eta}_j$.

<u>The case $\tau=\Lambda^j(\mathfrak{p}^*)$.</u> By (5.52), (5.113), and by the Hodge theory (1.5), we have

$$H^{\cdot}(Z,F)=\bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\big(V_{\pi,K}\otimes_{\mathbf{R}}\Lambda^{\cdot}(\mathfrak{p}^*)\big)^K. \tag{5.114}$$

The Hodge theory for Lie algebras [BorW00, Proposition II.3.1] tells us that the right-hand side of (5.114) has a cohomological interpretation,

$$H^{\cdot}(Z,F)=\bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)H^{\cdot}(\mathfrak{g},K;V_{\pi,K}), \tag{5.115}$$

where $H^{\cdot}(\mathfrak{g},K;V_{\pi,K})$ is the $(\mathfrak{g},K)$-cohomology of the Harish-Chandra $(\mathfrak{g},K)$-module $V_{\pi,K}$. By the following property of $(\mathfrak{g},K)$-cohomology [VZu84, V84, SR99] (see also [Sh18, Theorem 8.9]), for $\pi\in\widehat{G}_u$,

$$\chi_\pi \text{ is trivial } \iff H^{\cdot}(\mathfrak{g},K;V_{\pi,K})\neq 0, \tag{5.116}$$

we see that the sum in (5.115) can be reduced to $\pi\in\widehat{G}_u$ with trivial infinitesimal character,

$$H^{\cdot}(Z,F)=\bigoplus_{\pi\in\widehat{G}_u,\chi_\pi \text{ trivial}} n_\rho(\pi)H^{\cdot}(\mathfrak{g},K;V_{\pi,K}), \tag{5.117}$$

and each summand does not vanish except for $n_\rho(\pi)=0$.

<u>The case $\tau=\widehat{\eta}_j$.</u> By (5.113), we have

$$r_j=\sum_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\left(\dim\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}_j^+\big)^K-\dim\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}_j^-\big)^K\right). \tag{5.118}$$

As (5.115), in [Sh18, Theorem 8.14, Corollary 8.15], we give a cohomology interpretation of the right-hand side of (5.118),

$$\begin{aligned} r_j=\frac{1}{\chi(K/K\cap\mathcal{M})}&\sum_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\\ &\times\sum_{i=0}^{\dim\mathfrak{p}_{\mathfrak{m}}}\sum_{j=0}^{\dim\mathfrak{n}}(-1)^{i+j}\dim H^i\big(\mathfrak{m},K\cap\mathcal{M};H_j(\mathfrak{n},V_{\pi,K})\otimes_{\mathbf{R}} E_\eta\big), \end{aligned} \tag{5.119}$$

where $H_j(\mathfrak{n}, V_{\pi,K})$ is the $\mathfrak{n}$-homology of $V_{\pi,K}$ and is a $(\mathfrak{m}, K \cap \mathcal{M})$-module. Moreover, as (5.117), [Sh18, Proposition 8.17, Corollary 8.18] implies that the first sum in (5.119) can be reduced to $\pi \in \widehat{G}_u$ with trivial infinitesimal character, i.e.,

$$\begin{aligned} r_j = \frac{1}{\chi(K/K \cap \mathcal{M})} \sum_{\pi \in \widehat{G}_u, \chi_\pi \text{ trivial}} n_\rho(\pi) \\ \times \sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} \sum_{j=0}^{\dim \mathfrak{n}} (-1)^{i+j} \dim H^i\big(\mathfrak{m}, K \cap \mathcal{M}; H_j(\mathfrak{n}, V_{\pi,K}) \otimes_{\mathbf{R}} E_\eta\big). \end{aligned} \tag{5.120}$$

Equations (5.117) and (5.120) can be considered as an analogue of (5.98). Now we prove (5.44).

The proof of (5.44). If $H^\cdot(Z, F) = 0$, by (5.117), we see that if χ_π is trivial, then $n_\rho(\pi) = 0$. By (5.120), we see that $r_j = 0$ for all j. By (5.99), we complete the proof of (5.44). □

6. Anosov flow

The purpose of this section is to study the Fried conjecture for the Anosov flow. This section is organized as follows. In Section 6.1, we introduce the Anosov flow.

In Section 6.2, we explain the meromorphic extension of the Ruelle dynamical zeta function [GiLiPo13, DyZ16].

In Section 6.3, we explain a proof that under certain resonance conditions the value at zero of the Ruelle dynamical zeta function does not depend on a small perturbation of the Anosov flow.

In Section 6.4, we study the Anosov flow on 3-manifolds.

6.1. Closed orbits of the Anosov flow

Let M be a closed manifold with a smooth vector field $V \in C^\infty(M, TM)$. Let $(\phi_t)_{t\in\mathbf{R}}$ be the flow on M generated by V.

Definition 6.1. A flow $\phi_\cdot$ is called Anosov if there is a ϕ_t-invariant continuous splitting[12]

$$TM = \mathbf{R}V \oplus E^u \oplus E^s \tag{6.1}$$

of C^0-vector bundles on M and there exist $C > 0, \theta > 0$ and a Riemannian metric on M such that for $v \in E^u_x$, $v' \in E^s_x$, and $t > 0$, we have

$$|\phi_{-t,*}v| \leqslant Ce^{-\theta|t|}\,|v|\,, \qquad |\phi_{t,*}v'| \leqslant Ce^{-\theta|t|}\,|v'|\,. \tag{6.2}$$

[12]This requires that $\mathbf{R}V$ is a line bundle on M. It implies $V(x) \neq 0$ for all $x \in M$, and so the Euler characteristic number $\chi(M)$ vanishes.

In this section, we assume $\phi.$ is Anosov. It is well known that the set of closed orbits $\overline{\wp}(\phi.)$ defined in (2.12) is discrete (see [Mar04] or [DyZ16, Appendix A]), so that Assumption 2.4 holds. More precisely, for $\gamma \in \overline{\wp}(\phi.)$, let $\ell_\gamma \in \mathbf{R}_+^*, m_\gamma \in \mathbf{N}^*$ be the period and the multiplicity of γ. Then (2.14) becomes

$$\wp(\phi.) = \coprod_{\gamma \in \overline{\wp}(\phi.)} \mathbb{S}^1 \times \{\ell_\gamma\}. \tag{6.3}$$

Moreover, by [Mar04, Theorem 1.1, p.78] or [DyZ16, Lemma 2.2] there is $C > 0$ such that for $r \geqslant 0$, we have

$$|\{\gamma \in \overline{\wp}(\phi.) : \ell_\gamma \leqslant r\}| \leqslant Ce^{Cr}. \tag{6.4}$$

Thanks to (6.4), the Ruelle dynamical zeta function is well defined for $\sigma \in \mathbf{C}$ with $\mathrm{Re}\,(\sigma) \gg 1$. Recall that for a prime closed orbit γ, $\Delta(\gamma) \in \{\pm 1\}$ is defined in (4.9). Proceeding as in the proof of Proposition 4.5, for $\sigma \in \mathbf{C}$ with $\mathrm{Re}\,(\sigma) \gg 1$, we have

$$R_{\phi,\rho}(\sigma) = \left(\prod_{\gamma:\mathrm{prime}} \det\left(1 - \Delta(\gamma)\rho(\gamma)e^{-\sigma\ell_\gamma}\right)\right)^{(-1)^{\mathrm{rk}[E^u]+1}}, \tag{6.5}$$

where $\mathrm{rk}\,[E^u]$ denotes the rank of the C^0-vector bundle E^u. We note the similarity between (0.2) and (6.5).

Example 6.2. If (Z, g^{TZ}) is a negatively curved manifold, then the geodesic flow on the unit tangent bundle $M = SZ$ is Anosov [A67]. By (6.5), for $\sigma \in \mathbf{C}$ with $\mathrm{Re}\,(\sigma) \gg 1$,

$$R_{\phi,\rho}(\sigma) = \left(\prod_{\gamma:\mathrm{prime}} \det\left(1 - \Delta(\gamma)\rho(\gamma)e^{-\sigma\ell_\gamma}\right)\right)^{(-1)^{\dim Z}}. \tag{6.6}$$

In addition, if Z is orientable, for all prime closed orbits γ, we have $\Delta(\gamma) = 1$ (see [GiLiPo13, Lemma B.1]).

6.2. The meromorphic extension

The proofs of the meromorphic extension of the Ruelle dynamical zeta function for the Anosov flow given in [GiLiPo13] and [DyZ16][13] are based on a spectral interpretation of $R_{\phi,\rho}$.

Let us begin with establishing a relation between $R_{\phi,\rho}$ and the Lie derivation L_V along V acting on $\Omega^{\cdot}(M,F)$. For $t > 0$, write

$$e^{tL_V} : u \in \Omega^{\cdot}(M,F) \to \phi_t^* u \in \Omega^{\cdot}(M,F). \tag{6.7}$$

[13] Unfortunately, there is a sign conflict between the convention used in these two papers and Fried's paper [F87]. In [GiLiPo13, (2.2)] and [DyZ16, (1.1),(B.1)], the sign in the Fuller index (2.18) is defined using $D\phi_{-\ell_\gamma}$. As we adopt Fried's convention, the statements in this section are slightly different with [GiLiPo13, DyZ16].

The Schwartz kernel $e^{tL_V}(x,y)$ of e^{tL_V} is a current on $\mathbf{R}_+^* \times M \times M$ with coefficients in

$$\mathbf{C} \boxtimes (\Lambda^\cdot(T^*M) \otimes_{\mathbf{R}} F) \boxtimes \Big((\Lambda^\cdot(T^*M) \otimes_{\mathbf{R}} F)^* \otimes_{\mathbf{R}} |\det(T^*M)| \Big), \tag{6.8}$$

where $|\det(T^*M)| = \Lambda^{\dim M}(T^*M) \otimes o(TM)$ is the density bundle on M. By (6.1), its wave front set is disjoint from the conormal bundle of the submanifold $\{(t,x,x) \in \mathbf{R}_+^* \times M \times M\} \subset \mathbf{R}_+^* \times M \times M$. So, the restriction on the diagonal $e^{tL_V}(x,x)$ is a well-defined current on $\mathbf{R}_+^* \times M$ with coefficients in

$$\mathbf{C} \boxtimes \Big(\mathrm{End}\left(F \otimes_{\mathbf{R}} \Lambda^\cdot(T^*M)\right) \otimes_{\mathbf{R}} |\det(T^*M)| \Big). \tag{6.9}$$

Definition 6.3. The flat trace of e^{tL_V} is a distribution on $\mathbf{R}_+^*$ defined by

$$\mathrm{Tr}^\flat\left[e^{tL_V}\right] = \int_M \mathrm{Tr}\left[e^{tL_V}(x,x)\right]. \tag{6.10}$$

Note that the flat trace is not a classical trace in the sense of the trace of a trace class operator. However, we can still show that the flat trace of the commutator of e^{tL_V} with a differential operator vanishes.

Let us give an explicit formula for $\mathrm{Tr}^\flat\left[e^{tL_V}\right]$, which is known as the Atiyah–Bott–Guillemin trace formula [Gui77]. Let $\gamma \in \overline{\wp}(\phi_\cdot)$ be a closed orbit. For $x \in \gamma$, we have a morphism

$$D\phi_{\ell_\gamma}(x) : T_xM \to T_xM. \tag{6.11}$$

Up to conjugation it does not depend on the choice of $x \in \gamma$, and is denoted by $D\phi_{\ell_\gamma}|_{T_\gamma M}$. It acts naturally on any tensor of $T_\gamma M$. Acting on $T_\gamma M/\mathbf{R}V$, it is just the linearized Poincaré return map.

For $0 \leqslant q \leqslant \dim M$, denote by $L_{V,q}$ the restriction of L_V on $\Omega^q(M,F)$. Let $N^{\Lambda^\cdot(T^*M)}$ be the number operator on $\Lambda^\cdot(T^*M)$, which sends $s \in \Omega^q(M,F)$ to $qs \in \Omega^q(M,F)$.

Proposition 6.4. *For $0 \leqslant q \leqslant \dim M$, the following identity holds,*

$$\mathrm{Tr}^\flat\left[e^{tL_{V,q}}\right] = \sum_{\gamma \in \overline{\wp}(\phi_\cdot)} \frac{\mathrm{Tr}^{\Lambda^q(T_\gamma^*M)}\left[\left(D\phi_{\ell_\gamma}\right)^{\mathrm{tr}}|_{\Lambda^q(T_\gamma^*M)}\right]}{|\det(1-D\phi_{\ell_\gamma})|_{T_\gamma M/\mathbf{R}V}|} \mathrm{Tr}\left[\rho(\gamma)\right] \frac{\ell_\gamma}{m_\gamma} \delta_{\ell_\gamma}(t). \tag{6.12}$$

In particular, we have

$$\mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)} e^{tL_V}\right] = -\sum_{\gamma \in \overline{\wp}(\phi_\cdot)} \mathrm{sgn}\big(\det(1-D\phi_{\ell_\gamma})|_{T_\gamma M/\mathbf{R}V}\big) \,\mathrm{Tr}\left[\rho(\gamma)\right] \frac{\ell_\gamma}{m_\gamma} \delta_{\ell_\gamma}(t). \tag{6.13}$$

Proof. The Schwartz kernel of $e^{tL_{V,q}}$ is given by

$$e^{tL_{V,q}}(x,y) = \left\{ \tau_0^t \otimes (D\phi_t)^{\mathrm{tr}}|_{\Lambda^q\left(T^*_{\phi_t(x)}M\right)} \right\} \cdot \delta_{\phi_t(x)}(y), \tag{6.14}$$

where $\tau_0^t \in \mathrm{Hom}(F_{\phi_t(x)}, F_x)$ is the parallel transport with respect to ∇^F along the curve $(\phi_s(x))_{0\leqslant s\leqslant t}$ from t to 0. So the restriction of the distribution to the diagonal $e^{tL_V}(x,x)$ is supported on the periodic set $\wp(\phi) \subset \mathbf{R}_+^* \times M$ and is given by[14]

$$\mathrm{Tr}\left[e^{tL_{V,q}}(x,x)\right] = \sum_{\gamma\in\overline{\wp}(\phi.)} \mathrm{Tr}\left[\rho(\gamma)\right] \frac{\mathrm{Tr}^{\Lambda^q(T_\gamma^* M)}\left[\left(D\phi_{\ell_\gamma}\right)^{\mathrm{tr}}|_{\Lambda^q(T_\gamma^* M)}\right]}{|\det(1-D\phi_{\ell_\gamma})|_{T_\gamma M/\mathbf{R}V}|} \delta_{\ell_\gamma}(t)\otimes\delta_{\gamma^\sharp}(x), \tag{6.15}$$

where $\delta_{\gamma^\sharp}(x)$ is the current of integration on the prime closed orbit $\gamma^\sharp$ associated to γ defined by

$$s\in C_c^\infty(M) \to \int_0^{\ell_\gamma/m_\gamma} s\big(\phi_t(x)\big)dt, \tag{6.16}$$

where x is any point on γ. By (6.15), we get (6.12). □

By (2.21), (6.4), and (6.13), for $\mathrm{Re}\,(\sigma) \gg 1$, we have

$$\log R_{\phi,\rho}(\sigma) = -\int_0^\infty \mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)}\exp\big(-t\,(\sigma - L_V)\big)\right]\frac{dt}{t}. \tag{6.17}$$

So formally, $R_{\phi,\rho}(\sigma)$ is a certain flat regularized determinant

$$\prod_{q=1}^{\dim M} \det{}^\flat\,(\sigma - L_{V,q})^{(-1)^q q}. \tag{6.18}$$

Note the similarity between (1.8) and (6.18).

For $\mathrm{Re}\,(\sigma)\gg 1$, we write

$$\begin{aligned}\frac{\partial}{\partial\sigma}\log R_{\phi,\rho}(\sigma) &= \int_0^\infty \mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)}\exp\big(-t\,(\sigma-L_V)\big)\right]dt\\ &= \int_\delta^\infty \mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)}\exp\big(-t\,(\sigma-L_V)\big)\right]dt,\end{aligned} \tag{6.19}$$

where $\delta > 0$ is some positive number smaller than the minimum of the length spectrum. In the second identity of (6.19), we use the fact that the support of the distribution $\mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)}\exp\left(-t\,(\sigma-L_V)\right)\right]$ is away from 0.

To show the meromorphic extension of $R_{\phi,\rho}$, it is enough to show that $\frac{\partial}{\partial\sigma}\log R_{\phi,\rho}$ has a meromorphic extension to $\mathbf{C}$ with simple poles and integer residues. By (6.19), we write formally

$$\frac{\partial}{\partial\sigma}\log R_{\phi,\rho}(\sigma) = \mathrm{Tr}_{\mathrm{s}}^\flat\left[N^{\Lambda^\cdot(T^*M)}e^{-\delta(\sigma-L_V)}\,(\sigma-L_V)^{-1}\right]. \tag{6.20}$$

An important step is to give a proper sense of the operators on the right-hand side of (6.20) and to show its flat trace exists and has a meromorphic extension. We refer the reader to [DyZ16] for more details. Here we just state a weak version

[14] This can be obtained by proceeding as in the proof of Lefschetz fixed point formula. We refer the reader to [DyZ16, Appendix B] for more details.

of [DyZ16, Propositions 3.1–3.3], and explain the reason for which we need to introduce the small $\delta > 0$.

Let

$$T^*M = (\mathbf{R}V)^* \oplus E_u^* \oplus E_s^* \tag{6.21}$$

be the dual of the splitting (6.1). Let $\Delta \subset T^*(M\times M)$ be the diagonal of $T^*(M\times M)$. Set

$$\Omega^- = \left\{ \left(\phi_t(x), \left((D\phi_t)_x^{\mathrm{tr}} \right)^{-1} \cdot \xi, x, \xi \right) \in T^*(M\times M) : \langle V(x), \xi\rangle = 0, t \leqslant 0 \right\}, \tag{6.22}$$

where $\cdot^{\mathrm{tr}}$ denotes the transpose of a matrix. Denote by WF' the wave front set of an operator (see [DyZ16, Appendix C.2]).

Theorem 6.5. *The operator*

$$(\sigma - L_V)^{-1} : \Omega^{\cdot}(M,F) \to \mathcal{D}'(M, \Lambda^{\cdot}(T^*M)\otimes_{\mathbf{R}} F) \tag{6.23}$$

defines a meromorphic family on $\mathbf{C}$. *If it is holomorphic at* σ_0, *then*

$$\mathrm{WF}'\,(\sigma_0 - L_V)^{-1} \subset \Delta \cup \Omega^- \cup (E_s^* \times E_u^*). \tag{6.24}$$

From (6.24), we see that $(\sigma_0 - L_V)^{-1}$ does not necessarily have a well-defined flat trace. But $e^{-\delta(\sigma - L_V)}\,(\sigma - L_V)^{-1}$ does.

6.3. The $R_{\phi,\rho}(0)$ as a topological invariant

The poles of the meromorphic family in (6.23) are called Ruelle–Pollicott resonances. The set of Ruelle–Pollicott resonances is denoted by $\mathrm{Res}_\rho(V)$. If $0 \notin \mathrm{Res}_\rho(V)$, then $R_{\phi,\rho}(\sigma)$ is regular at $\sigma = 0$.[15]

Set

$$\mathscr{V}_\rho(M) = \{V \in C^\infty(M,TM) : V \text{ is Anosov such that } 0 \notin \mathrm{Res}_\rho V\}. \tag{6.25}$$

Thanks to the stability of the Anosov flow [A67] and of its resonance [ButLi07, ButLi13], $\mathscr{V}_\rho(M)$ forms an open subset in $C^\infty(M,TM)$. The following theorem [DaGuRiSh20] tells us the value at zero of the Ruelle dynamical zeta function does not depend on a small perturbation of the flow.

For $V \in \mathscr{V}_\rho(M)$, denote by $\phi_\cdot^V$ the corresponding flow.

Theorem 6.6. *For any flat vector bundle* F, *the map*

$$V \in \mathscr{V}_\rho(M) \to R_{\phi^V,\rho}(0) \in \mathbf{C}^* \tag{6.26}$$

is locally constant.

[15] Due to the cancellation from the supertrace, the converse is not correct.

Proof. Let $(V_b)_{b\in\mathbf{R}}$ be a smooth family of vector fields in $\mathscr{V}_\rho(M)$. Let $R_{b,\rho}$ be the corresponding family of the Ruelle dynamical zeta functions. Take a smooth family $\alpha_b \in \Omega^1(M)$ such that $\alpha_b(V_b) = 1$. Write $\dot{V}_b = \frac{\partial}{\partial b} V_b$.

We claim that for $\sigma \in \mathbf{C}$ with $\mathrm{Re}\,(\sigma) \gg 1$, we have

$$\frac{\partial}{\partial b} \log R_{b,\rho}(\sigma) = -\sigma \int_\delta^\infty \mathrm{Tr}_\mathrm{s}^\flat \left[\alpha_b i_{\dot{V}_b} \exp\big(- t\,(\sigma - L_{V_b})\,\big)\right] dt. \tag{6.27}$$

In [DaGuRiSh20], the proof of (6.27) is obtained by variation of the periods of closed orbits. Here, we give a proof via supersymmetry (cf. [RS71, Theorem 2.1]). We argue formally. The argument can be made rigorous easily. By (6.17), for $\mathrm{Re}\,(\sigma) \gg 1$, we have

$$\frac{\partial}{\partial b} \log R_{b,\rho}(\sigma) = -\int_\delta^\infty \mathrm{Tr}_\mathrm{s}^\flat \left[N^{\Lambda^\cdot (T^*M)} L_{\dot{V}_b} \exp\big(- t\,(\sigma - L_{V_b})\,\big)\right] dt. \tag{6.28}$$

Using the Cartan identity[16] $L_{\dot{V}_b} = \left[d, i_{\dot{V}_b}\right]$, the fact that d commutes with L_{V_b}, and fact that the supertrace vanishes on the supercommutator, we have identities of distributions on $\mathbf{R}_+^*$,

$$\begin{aligned}\mathrm{Tr}_\mathrm{s}^\flat \left[N^{\Lambda^\cdot (T^*M)} L_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right] &= \mathrm{Tr}_\mathrm{s}^\flat \left[N^{\Lambda^\cdot (T^*M)} \left[d, i_{\dot{V}_b}\right] \exp\left(tL_{V_b}\right)\right] \\ &= \mathrm{Tr}_\mathrm{s}^\flat \left[\left[N^{\Lambda^\cdot (T^*M)}, d\right] i_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right] \\ &= \mathrm{Tr}_\mathrm{s}^\flat \left[d i_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right].\end{aligned} \tag{6.29}$$

Since $\alpha_b(V_b) = 1$, we have $i_{\dot{V}_b} = \left[i_{V_b}, \alpha_b i_{\dot{V}_b}\right]$. By (6.29), and proceeding as before we have

$$\begin{aligned}\mathrm{Tr}_\mathrm{s}^\flat \left[N^{\Lambda^\cdot (T^*M)} L_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right] &= \mathrm{Tr}_\mathrm{s}^\flat \left[d\left[i_{V_b}, \alpha_b i_{\dot{V}_b}\right] \exp\left(tL_{V_b}\right)\right] \\ &= \mathrm{Tr}_\mathrm{s}^\flat \left[\left[d, i_{V_b}\right] \alpha_b i_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right] \\ &= \mathrm{Tr}_\mathrm{s}^\flat \left[L_{V_b} \alpha_b i_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right] \\ &= \frac{\partial}{\partial t} \mathrm{Tr}_\mathrm{s}^\flat \left[\alpha_b i_{\dot{V}_b} \exp\left(tL_{V_b}\right)\right].\end{aligned} \tag{6.30}$$

By (6.28) and (6.30), we get (6.27).

By (6.27), using the method as in [DyZ16], we can show that for $\sigma \in \mathbf{C}$, $\mathrm{Re}\,(\sigma) \gg 1$,

$$\frac{\partial}{\partial b} \log R_{b,\rho}(\sigma) = -\,\sigma \mathrm{Tr}_\mathrm{s}^\flat \left[\alpha_b i_{\dot{V}_b} e^{-\delta(\sigma - L_{V_b})} \left(\sigma - L_{V_b}\right)^{-1}\right]. \tag{6.31}$$

Note that thanks to (6.24), the above flat trace is well defined. Moreover, the function $\sigma \to \mathrm{Tr}_\mathrm{s}^\flat \left[\alpha_b i_{\dot{V}_b} e^{-\delta(\sigma - L_{V_b})} \left(\sigma - L_{V_b}\right)^{-1}\right]$ has a meromorphic extension to $\mathbf{C}$, and is regular at $\sigma = 0$ since $V_b \in \mathscr{V}_\rho(M)$.

---

[16] Here $[a, b] = ab - (-1)^{\deg a \deg b} ba$ denotes the supercommutator of a and b (see [BeGeVe04, Section 1.3]).

To complete our proof, it remains to show that for a fixed σ near 0, the function $b \to R_{b,\rho}(\sigma)$ is C^1 and (6.31) holds near 0. This is somehow technical and we refer the reader to [DaGuRiSh20] for a detailed proof. □

6.4. Anosov flow on 3-manifold

Let us restrict ourself to an orientable 3-manifold, where we have a partial solution for the Fried conjecture.

The following proposition [DaGuRiSh20, Proposition 7.3, Lemma 7.4] gives a characterization of the acyclicity of a unitarily flat vector bundle via resonance. Its proof uses [DyZ17, Lemma 2.3] in an essential way.

Proposition 6.7. *Let F be a unitarily flat vector bundle on a closed orientable 3-manifold. For any volume preserving Anosov flow, F is acyclic if and only if 0 is not a resonance.*

Recall the following theorem due to Sánchez-Morgado [SM96a], whose proof is based on the Markov partition [Rat69] and Rugh's technique [Rug96].

Theorem 6.8. *Let F be an acyclic unitarily flat vector bundle with holonomy ρ on a closed orientable analytic 3-manifold M. If $\phi.$ is a transitive analytic Anosov flow, and if there is a prime closed orbit γ such that 1 and $\Delta(\gamma)$ are not eigenvalues of $\rho(\gamma)$, then $R_{\phi,\rho}(\sigma)$ is regular at 0 and*

$$|R_{\phi,\rho}(0)| = T_F(M). \tag{6.32}$$

Note that any smooth manifold has a unique compatible analytic structure, and that any volume preserving Anosov flow is transitive. Since we can always approximate a smooth Anosov flow by an analytic one, and since we can approximate a flat vector bundle by the one with specified holonomy condition in the above theorem provided $H^1(M) \neq 0$, using Theorem 6.6, in [DaGuRiSh20, Section 7.2], we deduce the following Theorem [DaGuRiSh20, Theorem 1].

Theorem 6.9. *Let F be an acyclic unitarily flat vector bundle with holonomy ρ on a closed orientable 3-manifold with $H^1(M) \neq 0$. For any flow $\phi.$ which is a volume preserving Anosov flow or a flow nearby[17], we have*

$$|R_{\phi,\rho}(0)| = T_F(M). \tag{6.33}$$

Let us return to the case of the geodesic flow on the unit tangent bundle $M = SZ$ of a negatively curved orientable surface (Z, g^{TZ}). Recall that $a_0 \in \pi_1(M)$ is defined after (5.8). By Corollary 5.4 and Theorem 6.9, we get:

Corollary 6.10. *Let F be an acyclic unitarily flat vector bundle on the unit tangent bundle of a negatively curved orientable surface (Z, g^{TZ}). Then,*

$$|R_{\phi,\rho}(0)| = T_F(M) = \left| \det\left(1 - \rho(a_0)\right) \right|^{-\chi(Z)}. \tag{6.34}$$

The above corollary can be considered as a complementary of Dyatlov–Zworski's result [DyZ17], where ρ is assumed to be trivial.

[17] It is still an Anosov flow by the stability of Anosov flows [A67].

Theorem 6.11. *Assume that (Z, g^{TZ}) is a negatively curved orientable surface. There is $C \in \mathbf{R}^*$ such that as $\sigma \to 0$, we have*

$$R_{\phi,\mathrm{trivial}}(\sigma) = C\sigma^{-\chi(Z)}\big(1 + \mathcal{O}(\sigma)\big). \tag{6.35}$$

The above two results are generalizations of Fried's results [F86b, Corollaries 1 and 2] for hyperbolic surfaces.

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