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ANALYTIC TORSION, DYNAMICAL ZETA FUNCTIONS, AND THE FRIED CONJECTURE

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We prove the equality of the analytic torsion and the value at zero of a Ruelle dynamical zeta function associated with an acyclic unitarily flat vector bundle on a closed locally symmetric reductive manifold. This solves a conjecture of Fried. This article should be read in conjunction with an earlier paper by Moscovici and Stanton.

1. Introduction 1
2. Characteristic forms and analytic torsion 8
3. Preliminaries on reductive groups 11
4. Orbital integrals and Selberg trace formula 15
5. The solution to Fried conjecture 23
6. Reductive groups G with compact center and $\delta(G)=1$ 27
7. Selberg and Ruelle zeta functions 45
8. A cohomological formula for r_j 56
Acknowledgements 70
Index 71
References 73

1. Introduction

The purpose of this article is to prove the equality of the analytic torsion and the value at zero of a Ruelle dynamical zeta function associated with an acyclic unitarily flat vector bundle on a closed locally symmetric reductive manifold, which completes a gap in the proof given by Moscovici and Stanton [1991] and solves a conjecture of Fried [1987].

Let Z be a smooth closed manifold. Let F be a complex vector bundle equipped with a flat Hermitian metric g^F. Let $H^\bullet(Z,F)$ be the cohomology of sheaf of locally flat sections of F. We assume $H^\bullet(Z,F)=0$.

The Reidemeister torsion, introduced in [Reidemeister 1935], is a positive real number one obtains via the combinatorial complex with values in F associated with a triangulation of Z, which can be shown not to depend on the triangulation.

Let g^{TZ} be a Riemannian metric on TZ. Ray and Singer [1971] constructed the analytic torsion $T(F)$ as a spectral invariant of the Hodge Laplacian associated with g^{TZ} and g^F. They showed that if Z is an

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even-dimensional oriented manifold, then $T(F) = 1$. Moreover, if $\dim Z$ is odd, then $T(F)$ does not depend on the metric data.

Ray and Singer [1971] conjectured an equality between the Reidemeister torsion and the analytic torsion, which was later proved by Cheeger [1979] and Müller [1978]. Using the Witten deformation, Bismut and Zhang [1992] gave an extension of the Cheeger–Müller theorem which is valid for arbitrary flat vector bundles.

From the dynamical side, Milnor [1968b, Section 3] pointed out a remarkable similarity between the Reidemeister torsion and the Weil zeta function. A quantitative description of their relation was formulated by Fried [1986] when Z is a closed oriented hyperbolic manifold. Namely, he showed that the value at zero of the Ruelle dynamical zeta function, constructed using the closed geodesics in Z and the holonomy of F, is equal to $T(F)^2$. Fried [1987, p. 66, Conjecture] suggested that a similar result holds true for general closed locally homogeneous manifolds.

In this article, we prove the Fried conjecture for odd-dimensional[1] closed locally symmetric reductive manifolds. More precisely, we show that the dynamical zeta function is meromorphic on $\mathbb{C}$, holomorphic at 0, and that its value at 0 is equal to $T(F)^2$.

The proof of the above result by Moscovici and Stanton [1991], based on the Selberg trace formula and harmonic analysis on reductive groups, does not seem to be complete. We give the proper argument to make it correct. Our proof is based on the explicit formula given by Bismut [2011, Theorem 6.1.1] for semisimple orbital integrals.

The results contained in this article were announced in [Shen 2016]. See also Ma's talk [2017] at Séminaire Bourbaki for an introduction.

Now, we will describe our results in more detail, and explain the techniques used in their proofs.

1A. ***The analytic torsion.*** Let Z be a smooth closed manifold, and let F be a complex flat vector bundle on Z.

Let g^{TZ} be a Riemannian metric on TZ, and let g^F be a Hermitian metric on F. To g^{TZ} and g^F, we can associate an L^2-metric on $\Omega^\bullet(Z, F)$, the space of differential forms with values in F. Let $\square^Z$ be the Hodge Laplacian acting on $\Omega^\bullet(Z, F)$. By Hodge theory, we have a canonical isomorphism

$$\ker \square^Z \simeq H^\bullet(Z, F). \tag{1-1}$$

Let $(\square^Z)^{-1}$ be the inverse of $\square^Z$ acting on the orthogonal space to $\ker \square^Z$. Let $N^{\Lambda^\bullet(T^*Z)}$ be the number operator of $\Lambda^\bullet(T^*Z)$, i.e., multiplication by i on $\Omega^i(Z, F)$. Let $\mathrm{Tr_s}$ denote the supertrace. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > \frac{1}{2}\dim Z$, set

$$\theta(s) = -\mathrm{Tr_s}[N^{\Lambda^\bullet(T^*Z)}(\square^Z)^{-s}]. \tag{1-2}$$

By [Seeley 1967], $\theta(s)$ has a meromorphic extension to $\mathbb{C}$, which is holomorphic at $s = 0$. The analytic torsion is a positive real number given by

$$T(F) = \exp(\theta'(0)/2). \tag{1-3}$$

[1] The even-dimensional case is trivial.

Equivalently, $T(F)$ is given by the following weighted product of the zeta regularized determinants:

$$T(F) = \prod_{i=1}^{\dim Z} \det(\square^Z|_{\Omega^i(Z,F)})^{(-1)^i i/2}. \tag{1-4}$$

1B. *The dynamical zeta function.* Let us recall the general definition of the formal dynamical zeta function associated to a geodesic flow given in [Fried 1987, Section 5].

Let (Z, g^{TZ}) be a connected manifold with nonpositive sectional curvature. Let $\Gamma = \pi_1(Z)$ be the fundamental group of Z, and let $[\Gamma]$ be the set of the conjugacy classes of Γ. We identify $[\Gamma]$ with the free homotopy space of Z. For $[\gamma] \in [\Gamma]$, let $B_{[\gamma]}$ be the set of closed geodesics, parametrized by $[0, 1]$, in the class $[\gamma]$. The map $x_\bullet \in B_{[\gamma]} \to (x_0, \dot{x}_0/|\dot{x}_0|)$ induces an identification between $\coprod_{[\gamma]\in[\Gamma]-\{1\}} B_{[\gamma]}$ and the fixed points of the geodesic flow at time $t = 1$ acting on the unit tangent bundle SZ. Then, $B_{[\gamma]}$ is equipped with the induced topology, and is connected and compact. Moreover, all the elements in $B_{[\gamma]}$ have the same length $l_{[\gamma]}$. Also, the Fuller index $\mathrm{ind}_F(B_{[\gamma]}) \in \mathbb{Q}$ is well defined [Fried 1987, Section 4]. Given a finite-dimensional representation ρ of Γ, for $\sigma \in \mathbb{C}$, the formal dynamical zeta function is then defined by

$$R_\rho(\sigma) = \exp\Bigg(\sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)]\, \mathrm{ind}_F(B_{[\gamma]}) e^{-\sigma l_{[\gamma]}} \Bigg). \tag{1-5}$$

Note that our definition is the inverse of the one introduced by Fried [1987, p. 51].

The Fuller index can be made explicit in many case. If $[\gamma] \in [\Gamma]-\{1\}$, the group $\mathbb{S}^1$ acts locally freely on $B_{[\gamma]}$ by rotation. Assume that the $B_{[\gamma]}$ are smooth manifolds. This is the case if (Z, g^{TZ}) has a negative sectional curvature or if Z is locally symmetric. Then $\mathbb{S}^1\backslash B_{[\gamma]}$ is an orbifold. Let $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) \in \mathbb{Q}$ be the orbifold Euler characteristic [Satake 1957]. Denote by

$$m_{[\gamma]} = \left|\ker(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]}))\right| \in \mathbb{N}^* \tag{1-6}$$

the multiplicity of a generic element in $B_{[\gamma]}$. By [Fried 1987, Lemma 5.3], we have

$$\mathrm{ind}_F(B_{[\gamma]}) = \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}. \tag{1-7}$$

By (1-5) and (1-7), the formal dynamical zeta function is then given by

$$R_\rho(\sigma) = \exp\Bigg(\sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} e^{-\sigma l_{[\gamma]}} \Bigg). \tag{1-8}$$

We will say that the formal dynamical zeta function is well defined if $R_\rho(\sigma)$ is holomorphic for $\mathrm{Re}(\sigma) \gg 1$ and extends meromorphically to $\sigma \in \mathbb{C}$.

Observe that if (Z, g^{TZ}) is of negative sectional curvature, then $B_{[\gamma]} \simeq \mathbb{S}^1$ and

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = 1. \tag{1-9}$$

In this case, $R_\rho(\sigma)$ was recently shown to be well defined by Giulietti, Liverani and Pollicott [Giulietti et al. 2013] and Dyatlov and Zworski [2016]. Moreover, Dyatlov and Zworski [2017] showed that if (Z, g^{TZ}) is a negatively curved surface, the order of the zero of $R_\rho(\sigma)$ at $\sigma = 0$ is related to the genus of Z.

1C. ***The Fried conjecture.*** Let us briefly recall the results in [Fried 1986]. Assume Z is an odd-dimensional connected orientable closed hyperbolic manifold. Take $r \in \mathbb{N}$. Let $\rho : \Gamma \to U(r)$ be a unitary representation of the fundamental group Γ. Let F be the unitarily flat vector bundle on Z associated to ρ.

Using the Selberg trace formula, Fried [1986, Theorem 3] showed that there exist explicit constants $C_\rho \in \mathbb{R}^*$ and $r_\rho \in \mathbb{Z}$ such that as $\sigma \to 0$,

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{1-10}$$

Moreover, if $H^\bullet(Z, F) = 0$, then

$$C_\rho = 1, \quad r_\rho = 0, \tag{1-11}$$

so that

$$R_\rho(0) = T(F)^2. \tag{1-12}$$

Fried [1987, p. 66, Conjecture] suggested that the same holds true when Z is a general closed locally homogeneous manifold.

1D. ***The V-invariant.*** In this and in the following subsections, we give a formal proof of (1-12) using the V-invariant of Bismut and Goette [2004].

Let S be a closed manifold equipped with an action of a compact Lie group L, with Lie algebra $\mathfrak{l}$. If $a \in \mathfrak{l}$, let a^S be the corresponding vector field on S. Bismut and Goette [2004] introduced the V-invariant $V_a(S) \in \mathbb{R}$.

Let f be an a^S-invariant Morse–Bott function on S. Let $B_f \subset S$ be the critical submanifold. Since $a^S|_{B_f} \in TB_f$, $V_a(B_f)$ is also well defined. By [Bismut and Goette 2004, Theorem 4.10], $V_a(S)$ and $V_a(B_f)$ are related by a simple formula.

1E. ***Analytic torsion and the V-invariant.*** Let us argue formally. Let LZ be the free loop space of Z equipped with the canonical $\mathbb{S}^1$-action. Write $LZ = \coprod_{[\gamma]\in[\Gamma]} (LZ)_{[\gamma]}$ as a disjoint union of its connected components. Let a be the generator of the Lie algebra of $\mathbb{S}^1$ such that $\exp(a) = 1$. As explained in [Bismut 2005, Equation (0.3)], if F is a unitarily flat vector bundle on Z such that $H^\bullet(Z, F) = 0$, at least formally, we have

$$\log T(F) = -\sum_{[\gamma]\in[\Gamma]} \operatorname{Tr}[\rho(\gamma)] V_a((LZ)_{[\gamma]}). \tag{1-13}$$

Suppose that (Z, g^{TZ}) is an odd-dimensional connected closed manifold of nonpositive sectional curvature, and suppose that the energy functional

$$E : x_\bullet \in LZ \to \frac{1}{2}\int_0^1 |\dot{x}_s|^2\, ds \tag{1-14}$$

on LZ is Morse–Bott. The critical set of E is just $\coprod_{[\gamma]\in[\Gamma]} B_{[\gamma]}$, and all the critical points are local minima. Applying [Bismut and Goette 2004, Theorem 4.10] to the infinite-dimensional manifold $(LZ)_{[\gamma]}$ equipped with the $\mathbb{S}^1$-invariant Morse–Bott functional E, we have the formal identity

$$V_a((LZ)_{[\gamma]}) = V_a(B_{[\gamma]}). \tag{1-15}$$

Since $B_{[1]} \simeq Z$ is formed of the trivial closed geodesics, by the definition of the V-invariant,

$$V_a(B_{[1]}) = 0. \tag{1-16}$$

By [Bismut and Goette 2004, Proposition 4.26], if $[\gamma] \in [\Gamma] - \{1\}$, then

$$V_a(B_{[\gamma]}) = -\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{2m_{[\gamma]}}. \tag{1-17}$$

By (1-13), (1-15)–(1-17), we get a formal identity

$$\log T(F) = \frac{1}{2} \sum_{[\gamma] \in [\Gamma] - \{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{m_{[\gamma]}}, \tag{1-18}$$

which is formally equivalent to (1-12).

1F. *The main result of the article.* Let G be a linear connected real reductive group [Knapp 1986, p. 3], and let θ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ. Let $\mathfrak{k}$ and $\mathfrak{g}$ be the Lie algebras of K and G, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition. Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action of G and θ. Assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Set $X = G/K$. Then B induces a Riemannian metric g^{TX} on the tangent bundle $TX = G \times_K \mathfrak{p}$ such that X is of nonpositive sectional curvature.

Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G. Set $Z = \Gamma \backslash X$. Then Z is a closed locally symmetric manifold with $\pi_1(Z) = \Gamma$. Recall that $\rho : \Gamma \to \mathrm{U}(r)$ is a unitary representation of Γ, and that F is the unitarily flat vector bundle on Z associated with ρ. The main result of this article gives the solution of the Fried conjecture for Z. In particular, this conjecture is valid for all the closed locally symmetric spaces of noncompact type.

Theorem 1.1. *Assume* $\dim Z$ *is odd. The dynamical zeta function* $R_\rho(\sigma)$ *is holomorphic for* $\mathrm{Re}(\sigma) \gg 1$ *and extends meromorphically to* $\sigma \in \mathbb{C}$. *Moreover, there exist explicit constants* $C_\rho \in \mathbb{R}^*$ *and* $r_\rho \in \mathbb{Z}$, *see* (7-75), *such that, when* $\sigma \to 0$,

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho + 1}). \tag{1-19}$$

If $H^\bullet(Z, F) = 0$, *then*

$$C_\rho = 1, \quad r_\rho = 0, \tag{1-20}$$

so that

$$R_\rho(0) = T(F)^2. \tag{1-21}$$

Let $\delta(G)$ be the nonnegative integer defined by the difference between the complex ranks of G and K. Since $\dim Z$ is odd, $\delta(G)$ is odd. For $\delta(G) \neq 1$, Theorem 1.1 is originally due to Moscovici and Stanton [1991] and was recovered by Bismut [2011]. Indeed, it was proved in [Moscovici and Stanton 1991, Corollary 2.2, Remark 3.7] or [Bismut 2011, Theorem 7.9.3] that $T(F) = 1$ and $\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]}) = 0$ for all $[\gamma] \in [\Gamma] - \{1\}$.

Remark that both of the above two proofs use the Selberg trace formula. However, in the evaluation of the geometric side of the Selberg trace formula and of orbital integrals, Moscovici and Stanton relied on Harish-Chandra's Plancherel theory, while Bismut used his explicit formula [2011, Theorem 6.1.1] obtained via the hypoelliptic Laplacian.

Our proof of Theorem 1.1 relies on Bismut's formula.

1G. ***Our results on*** $\boldsymbol{R_\rho(\sigma)}$**.** Assume that $\delta(G)=1$. To show that $R_\rho(\sigma)$ extends as a meromorphic function on $\mathbb{C}$ when Z is hyperbolic, Fried [1986] showed that $R_\rho(\sigma)$ is an alternating product of certain Selberg zeta functions. Moscovici and Stanton's idea was to introduce the more general Selberg zeta functions and to get a similar formula for $R_\rho(\sigma)$.

Let us recall some facts about reductive group G with $\delta(G)=1$. In this case, there exists a unique (up to conjugation) standard parabolic subgroup $Q\subset G$ with Langlands decomposition $Q=M_QA_QN_Q$ such that $\dim A_Q=1$. Let $\mathfrak{m},\mathfrak{b},\mathfrak{n}$ be the Lie algebras of M_Q,A_Q,N_Q. Let $\alpha\in\mathfrak{b}^*$ be such that, for $a\in\mathfrak{b}$, $\operatorname{ad}(a)$ acts on $\mathfrak{n}$ as a scalar $\langle\alpha,a\rangle\in\mathbb{R}$ (see Proposition 6.3). Let M be the connected component of identity of M_Q. Then M is a connected reductive group with maximal compact subgroup $K_M=M\cap K$ and with Cartan decomposition $\mathfrak{m}=\mathfrak{p}_\mathfrak{m}\oplus\mathfrak{k}_\mathfrak{m}$. We have the identity of real K_M-representations

$$\mathfrak{p}\simeq\mathfrak{p}_\mathfrak{m}\oplus\mathfrak{b}\oplus\mathfrak{n}. \tag{1-22}$$

An observation due to Moscovici and Stanton is that $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})\neq 0$ only if γ can be conjugated by an element of G into A_QK_M. For $\sigma\in\mathbb{C}$, we define the formal Selberg zeta function by

$$Z_j(\sigma)=\exp\left(-\sum_{[\gamma]\in[\Gamma]-\{1\}}\operatorname{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}\frac{\operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}[\operatorname{Ad}(k^{-1})]}{\left|\det(1-\operatorname{Ad}(e^ak^{-1}))|_{\mathfrak{n}\oplus\theta\mathfrak{n}}\right|^{\frac{1}{2}}}e^{-\sigma l_{[\gamma]}}\right), \tag{1-23}$$

where $a\in\mathfrak{b}$, $k\in K_M$ are such that γ can be conjugated to e^ak^{-1}. We remark that $l_{[\gamma]}=|a|$. To show the meromorphicity of $Z_j(\sigma)$, Moscovici and Stanton tried to identify $Z_j(\sigma)$ with the geometric side of the zeta regularized determinant of the resolvent of some elliptic operator acting on some vector bundle on Z. However, the vector bundle used in [Moscovici and Stanton 1991], whose construction involves the adjoint representation of K_M on $\Lambda^i(\mathfrak{p}_\mathfrak{m}^*)\otimes\Lambda^i(\mathfrak{n}^*)$, does not live on Z, but only on $\Gamma\backslash G/K_M$.

We complete this gap by showing that such an object exists as a virtual vector bundle on Z in the sense of K-theory. More precisely, let $\mathrm{RO}(K),\mathrm{RO}(K_M)$ be the real representation rings of K and K_M. We can verify that the restriction $\mathrm{RO}(K)\to\mathrm{RO}(K_M)$ is injective. Note that $\mathfrak{p}_\mathfrak{m},\mathfrak{n}\in\mathrm{RO}(K_M)$. In Section 6C, using the classification theory of real simple Lie algebras, we show $\mathfrak{p}_\mathfrak{m},\mathfrak{n}$ are in the image of $\mathrm{RO}(K)$. For $0\leqslant j\leqslant\dim\mathfrak{n}$, let $E_j=E_j^+-E_j^-\in\mathrm{RO}(K)$ such that the following identity in $\mathrm{RO}(K_M)$ holds:

$$\left(\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}(-1)^i\Lambda^i(\mathfrak{p}_\mathfrak{m}^*)\right)\otimes\Lambda^j(\mathfrak{n}^*)=E_j|_{K_M}. \tag{1-24}$$

Let $\mathcal{E}_j=G\times_K E_j$ be a $\mathbb{Z}_2$-graded vector bundle on X. It descends to a $\mathbb{Z}_2$-graded vector bundle $\mathcal{F}_j$ on Z. Let C_j be a Casimir operator of G action on $C^\infty(Z,\mathcal{F}_j\otimes_\mathbb{R}F)$. In Theorem 7.6, we show that there are $\sigma_j \in \mathbb{R}$ and an odd polynomial P_j such that if $\mathrm{Re}(\sigma) \gg 1$, $Z_j(\sigma)$ is holomorphic and

$$Z_j(\sigma) = \det_{\mathrm{gr}}(C_j + \sigma_j + \sigma^2)\exp(r\,\mathrm{vol}(Z)P_j(\sigma)), \tag{1-25}$$

where $\det_{\mathrm{gr}}$ is the zeta regularized $\mathbb{Z}_2$-graded determinant. In particular, $Z_j(\sigma)$ extends meromorphically to $\mathbb{C}$.

By a direct calculation of linear algebra, we have

$$R_\rho(\sigma) = \prod_{j=0}^{\dim \mathfrak{n}} Z_j\big(\sigma + \big(j - \tfrac{1}{2}\dim \mathfrak{n}\big)|\alpha|\big)^{(-1)^{j-1}}, \tag{1-26}$$

from which we get the meromorphic extension of $R_\rho(\sigma)$. Note that the meromorphic function

$$T(\sigma) = \prod_{i=1}^{\dim Z} \det(\sigma + \square^Z|_{\Omega^i(Z,F)})^{(-1)^i i} \tag{1-27}$$

has a Laurent expansion near $\sigma = 0$,

$$T(\sigma) = T(F)^2\sigma^{\chi'(X,F)} + \mathcal{O}(\sigma^{\chi'(X,F)+1}), \tag{1-28}$$

where $\chi'(X,F)$ is the derived Euler number; see (2-8). Note also that the Hodge Laplacian $\square^Z$ coincides with the Casimir operator acting on $\Omega^\bullet(Z,F)$. The Laurent expansion (1-19) can be deduced from (1-25)–(1-28) and the identity in $\mathrm{RO}(K)$,

$$\sum_{i=1}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{\dim \mathfrak{n}} (-1)^j E_j. \tag{1-29}$$

1H. *Proof of* (1-20). To understand how the acyclicity of F is reflected in the function $R_\rho(\sigma)$, we need some deep results of representation theory. Let $\hat{p} : \Gamma\backslash G \to Z$ be the natural projection. The enveloping algebra of $U(\mathfrak{g})$ acts on $C^\infty(\Gamma\backslash G, \hat{p}^*F)$. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let $V^\infty \subset C^\infty(\Gamma\backslash G, \hat{p}^*F)$ be the subspace of $C^\infty(\Gamma\backslash G, \hat{p}^*F)$ on which the action of $\mathcal{Z}(\mathfrak{g})$ vanishes, and let V be the closure of V^∞ in $L^2(\Gamma\backslash G, \hat{p}^*F)$. Then V is a unitary representation of G. The compactness of $\Gamma\backslash G$ implies that V is a finite sum of irreducible unitary representations of G. By standard arguments [Borel and Wallach 2000, Chapter VII, Theorem 3.2, Corollary 3.4], the cohomology $H^\bullet(Z,F)$ is canonically isomorphic to the $(\mathfrak{g},K)$-cohomology $H^\bullet(\mathfrak{g},K;V)$ of V.

In [Vogan and Zuckerman 1984; Vogan 1984], the authors classified all irreducible unitary representations with nonzero $(\mathfrak{g},K)$-cohomology. On the other hand, Salamanca-Riba [1999] showed that any irreducible unitary representation with vanishing $\mathcal{Z}(\mathfrak{g})$-action is in the class specified by Vogan and Zuckerman, which means that it possesses nonzero $(\mathfrak{g},K)$-cohomology.

By the above considerations, the acyclicity of F is equivalent to $V = 0$. This is essentially the algebraic ingredient in the proof of (1-20). Indeed, in Corollary 8.18, we give a formula for the constants C_ρ and r_ρ, obtained by Hecht–Schmid formula [1983] with the help of the $\mathfrak{n}$-homology of V.

1I. ***The organization of the article.*** This article is organized as follows. In Section 2, we recall the definitions of certain characteristic forms and of the analytic torsion.

In Section 3, we introduce the reductive groups and the fundamental rank $\delta(G)$ of G.

In Section 4, we introduce the symmetric space. We recall basic principles for the Selberg trace formula, and we state formulas by Bismut [2011, Theorem 6.1.1] for semisimple orbital integrals. We recall the proof, given in Theorem 7.9.1 of the same paper, of a vanishing result of the analytic torsion $T(F)$ in the case $\delta(G) \neq 1$, which is originally due to Moscovici and Stanton [1991, Corollary 2.2].

In Section 5, we introduce the dynamical zeta function $R_\rho(\sigma)$, and we state Theorem 1.1 as Theorem 5.5. We prove Theorem 1.1 when $\delta(G) \neq 1$ or when G has noncompact center.

Sections 6–8 are devoted to establishing Theorem 1.1 when G has compact center and when $\delta(G) = 1$.

In Section 6, we introduce geometric objects associated with such reductive groups G.

In Section 7, we introduce Selberg zeta functions, and we prove that $R_\rho(\sigma)$ extends meromorphically, and we establish (1-19).

Finally, in Section 8, after recalling some constructions and results of representation theory, we prove that (1-20) holds.

Throughout the paper, we use the superconnection formalism of [Quillen 1985] and [Berline et al. 2004, Section 1.3]. If A is a $\mathbb{Z}_2$-graded algebra and if $a, b \in A$, the supercommutator $[a, b]$ is given by

$$[a,b] = ab - (-1)^{\deg a \deg b} ba. \tag{1-30}$$

If B is another $\mathbb{Z}_2$-graded algebra, we denote by $A \widehat{\otimes} B$ the super tensor product algebra of A and B. If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector space, the algebra $\mathrm{End}(E)$ is $\mathbb{Z}_2$-graded. If $\tau = \pm 1$ on $E^\pm$ and if $a \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr_s}[a]$ is defined by

$$\mathrm{Tr_s}[a] = \mathrm{Tr}[\tau a]. \tag{1-31}$$

We make the convention that $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$.

2. Characteristic forms and analytic torsion

The purpose of this section is to recall some basic constructions and properties of characteristic forms and the analytic torsion.

This section is organized as follows. In Section 2A, we recall the construction of the Euler form, the $\widehat{A}$-form and the Chern character form.

In Section 2B, we introduce the regularized determinant.

Finally, in Section 2C, we recall the definition of the analytic torsion of flat vector bundles.

2A. ***Characteristic forms.*** If V is a real or complex vector space of dimension n, we denote by V^* the dual space and by $\Lambda^\bullet(V) = \sum_{i=0}^n \Lambda^i(V)$ its exterior algebra. Let Z be a smooth manifold. If V is a vector bundle on Z, we denote by $\Omega^\bullet(Z, V)$ the space of smooth differential forms with values in V. When $V = \mathbb{R}$, we write $\Omega^\bullet(Z)$ instead.

Let E be a real Euclidean vector bundle of rank m with a metric connection ∇^E. Let $R^E = \nabla^{E,2}$ be the curvature of ∇^E. It is a 2-form with values in antisymmetric endomorphisms of E.

If A is an antisymmetric matrix, denote by $\mathrm{Pf}[A]$ the Pfaffian [Bismut and Zhang 1992, Equation (3.3)] of A. Then $\mathrm{Pf}[A]$ is a polynomial function of A which is a square root of $\det[A]$. Let $o(E)$ be the orientation line of E. The Euler form $e(E,\nabla^E)$ of (E,∇^E) is given by

$$e(E,\nabla^E) = \mathrm{Pf}\left[\frac{R^E}{2\pi}\right] \in \Omega^m(Z, o(E)). \tag{2-1}$$

If m is odd, then $e(E,\nabla^E) = 0$.

For $x \in \mathbb{C}$, set

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)}. \tag{2-2}$$

The form $\widehat{A}(E,\nabla^E)$ of (E,∇^E) is given by

$$\widehat{A}(E,\nabla^E) = \left[\det\left(\widehat{A}\left(-\frac{R^E}{2i\pi}\right)\right)\right]^{\frac{1}{2}} \in \Omega^\bullet(Z). \tag{2-3}$$

If E' is a complex Hermitian vector bundle equipped with a metric connection $\nabla^{E'}$ with curvature $R^{E'}$, the Chern character form $\mathrm{ch}(E',\nabla^{E'})$ of $(E',\nabla^{E'})$ is given by

$$\mathrm{ch}(E',\nabla^{E'}) = \mathrm{Tr}\left[\exp\left(-\frac{R^{E'}}{2i\pi}\right)\right] \in \Omega^\bullet(Z). \tag{2-4}$$

The differential forms $e(E,\nabla^E)$, $\widehat{A}(E,\nabla^E)$ and $\mathrm{ch}(E',\nabla^{E'})$ are closed. They are the Chern–Weil representatives of the Euler class of E, the $\widehat{A}$-genus of E and the Chern character of E'.

2B. ***Regularized determinant.*** Let (Z, g^{TZ}) be a smooth closed Riemannian manifold of dimension m. Let (E, g^E) be a Hermitian vector bundle on Z. The metrics g^{TZ}, g^E induce an L^2-metric on $C^\infty(Z,E)$.

Let P be a second-order elliptic differential operator acting on $C^\infty(Z,E)$. Suppose that P is formally self-adjoint and nonnegative. Let P^{-1} be the inverse of P acting on the orthogonal space to $\ker(P)$. For $\mathrm{Re}(s) > m/2$, set

$$\theta_P(s) = -\mathrm{Tr}[(P^{-1})^s]. \tag{2-5}$$

By [Seeley 1967] or [Berline et al. 2004, Proposition 9.35], $\theta(s)$ has a meromorphic extension to $s \in \mathbb{C}$ which is holomorphic at $s = 0$. The regularized determinant of P is defined as

$$\det(P) = \exp(\theta_P'(0)). \tag{2-6}$$

Assume now that P is formally self-adjoint and bounded from below. Denote by $\mathrm{Sp}(P)$ the spectrum of P. For $\lambda \in \mathrm{Sp}(P)$, set

$$m_P(\lambda) = \dim_{\mathbb{C}} \ker(P - \lambda) \tag{2-7}$$

to be its multiplicity. If $\sigma \in \mathbb{R}$ is such that $P + \sigma > 0$, then $\det(P+\sigma)$ is defined by (2-6). Voros [1987] has shown that the function $\sigma \to \det(P+\sigma)$, defined for $\sigma \gg 1$, extends holomorphically to $\mathbb{C}$ with zeros at $\sigma = -\lambda$ of the order $m_P(\lambda)$, where $\lambda \in \mathrm{Sp}(P)$.

2C. *Analytic torsion.* Let Z be a smooth connected closed manifold of dimension m with fundamental group Γ. Let F be a complex flat vector bundle on Z of rank r. Equivalently, F can be obtained via a complex representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$.

Let $H^\bullet(Z,F) = \bigoplus_{i=0}^m H^i(Z,F)$ be the cohomology of the sheaf of locally flat sections of F. We define the Euler number and the derived Euler number by

$$\chi(Z,F) = \sum_{i=0}^m (-1)^i \dim_{\mathbb{C}} H^i(Z,F), \quad \chi'(Z,F) = \sum_{i=1}^m (-1)^i i \dim_{\mathbb{C}} H^i(Z,F). \tag{2-8}$$

Let $(\Omega^\bullet(Z,F), d^Z)$ be the de Rham complex of smooth sections of $\Lambda^\bullet(T^*Z) \otimes_{\mathbb{R}} F$ on Z. We have the canonical isomorphism of vector spaces

$$H^\bullet(\Omega^\bullet(Z,F), d^Z) \simeq H^\bullet(Z,F). \tag{2-9}$$

In the sequel, we will also consider the trivial line bundle $\mathbb{R}$. We denote simply by $H^\bullet(Z)$ and $\chi(Z)$ the corresponding objects. Note that, in this case, the complex dimension in (2-8) should be replaced by the real dimension.

Let g^{TZ} be a Riemannian metric on TZ, and let g^F be a Hermitian metric on F. They induce an L^2-metric $\langle \cdot , \cdot \rangle_{\Omega^\bullet(Z,F)}$ on $\Omega^\bullet(Z,F)$. Let $d^{Z,*}$ be the formal adjoint of d^Z with respect to $\langle \cdot , \cdot \rangle_{\Omega^\bullet(Z,F)}$. Put

$$D^Z = d^Z + d^{Z,*}, \quad \square^Z = D^{Z,2} = [d^Z, d^{Z,*}]. \tag{2-10}$$

Then, $\square^Z$ is a formally self-adjoint nonnegative second-order elliptic operator acting on $\Omega^\bullet(Z,F)$. By Hodge theory, we have the canonical isomorphism of vector spaces

$$\ker \square^Z \simeq H^\bullet(Z,F). \tag{2-11}$$

Definition 2.1. The analytic torsion of F is a positive real number defined by

$$T(F, g^{TZ}, g^F) = \prod_{i=1}^m \det(\square^Z|_{\Omega^i(Z,F)})^{(-1)^i i/2}. \tag{2-12}$$

Recall that the flat vector bundle F carries a flat metric g^F if and only if the holonomy representation ρ factors through $\mathrm{U}(r)$. In this case, F is said to be unitarily flat. If Z is an even-dimensional orientable manifold and if F is unitarily flat with a flat metric g^F, by Poincaré duality, $T(F, g^{TZ}, g^F) = 1$. If $\dim Z$ is odd and if $H^\bullet(Z,F) = 0$, by [Bismut and Zhang 1992, Theorem 4.7], then $T(F, g^{TZ}, g^F)$ does not depend on g^{TZ} or g^F. In the sequel, we write instead

$$T(F) = T(F, g^{TZ}, g^F). \tag{2-13}$$

By Section 2B,

$$T(\sigma) = \prod_{i=1}^{\dim Z} \det(\sigma + \square^Z|_{\Omega^i(Z,F)})^{(-1)^i i} \tag{2-14}$$

is meromorphic on $\mathbb{C}$. When $\sigma \to 0$, we have

$$T(\sigma) = T(F)^2 \sigma^{\chi'(Z,F)} + \mathcal{O}(\sigma^{\chi'(Z,F)+1}). \tag{2-15}$$

3. Preliminaries on reductive groups

The purpose of this section is to recall some basic facts about reductive groups.

This section is organized as follows. In Section 3A, we introduce the reductive group G.

In Section 3B, we introduce the semisimple elements of G, and we recall some related constructions.

In Section 3C, we recall some properties of Cartan subgroups of G. We introduce a nonnegative integer $\delta(G)$, which has paramount importance in the whole article. We also recall Weyl's integral formula on reductive groups.

Finally, in Section 3D, we introduce the regular elements of G.

3A. ***The reductive group.*** Let G be a linear connected real reductive group [Knapp 1986, p. 3]; that is, G is a closed connected group of real matrices that is stable under transpose. Let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K. The Cartan involution θ acts naturally as a Lie algebra automorphism of $\mathfrak{g}$. Then $\mathfrak{k}$ is the eigenspace of θ associated with the eigenvalue 1. Let $\mathfrak{p}$ be the eigenspace with the eigenvalue -1, so that

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{3-1}$$

Then we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{3-2}$$

Put

$$m = \dim \mathfrak{p}, \quad n = \dim \mathfrak{k}. \tag{3-3}$$

By [Knapp 1986, Proposition 1.2], we have the diffeomorphism

$$(Y, k) \in \mathfrak{p} \times K \to e^Y k \in G. \tag{3-4}$$

Let B be a real-valued nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action Ad of G on $\mathfrak{g}$, and also under θ. Then (3-1) is an orthogonal splitting of $\mathfrak{g}$ with respect to B. We assume B to be positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. The form $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ defines an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$ such that the splitting (3-1) is still orthogonal. We denote by $|\cdot|$ the corresponding norm.

Let $Z_G \subset G$ be the center of G with Lie algebra $\mathfrak{z}_\mathfrak{g} \subset \mathfrak{g}$. Set

$$\mathfrak{z}_\mathfrak{p} = \mathfrak{z}_\mathfrak{g} \cap \mathfrak{p}, \quad \mathfrak{z}_\mathfrak{k} = \mathfrak{z}_\mathfrak{g} \cap \mathfrak{k}. \tag{3-5}$$

By [Knapp 1986, Corollary 1.3], Z_G is reductive. As in (3-1) and (3-4), we have the Cartan decomposition

$$\mathfrak{z}_\mathfrak{g} = \mathfrak{z}_\mathfrak{p} \oplus \mathfrak{z}_\mathfrak{k}, \quad Z_G = \exp(\mathfrak{z}_\mathfrak{p})(Z_G \cap K). \tag{3-6}$$

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ be the compact form of $\mathfrak{g}$. Let $G_\mathbb{C}$ and U be the connected group of complex matrices associated with the Lie algebras $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{u}$. By [Knapp 1986, Propositions 5.3 and 5.6], if G has compact center, i.e., its center Z_G is compact, then $G_\mathbb{C}$ is a linear connected complex reductive group with maximal compact subgroup U.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element. If $e_1, \dots, e_m$ is an orthonormal basis of $\mathfrak{p}$, and if $e_{m+1}, \dots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{n+m} e_i^2. \tag{3-7}$$

Classically, $C^{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$.

We define $C^{\mathfrak{k}}$ similarly. Let τ be a finite-dimensional representation of K on V. We denote by $C^{\mathfrak{k},V}$ or $C^{\mathfrak{k},\tau} \in \mathrm{End}(V)$ the corresponding Casimir operator acting on V, so that

$$C^{\mathfrak{k},V} = C^{\mathfrak{k},\tau} = \sum_{i=m+1}^{m+n} \tau(e_i)^2. \tag{3-8}$$

3B. *Semisimple elements.* If $\gamma \in G$, we denote by $Z(\gamma) \subset G$ the centralizer of γ in G, and by $\mathfrak{z}(\gamma) \subset \mathfrak{g}$ its Lie algebra. If $a \in \mathfrak{g}$, let $Z(a) \subset G$ be the stabilizer of a in G, and let $\mathfrak{z}(a) \subset \mathfrak{g}$ be its Lie algebra.

An element $\gamma \in G$ is said to be semisimple if γ can be conjugated to $e^a k^{-1}$ such that

$$a \in \mathfrak{p}, \quad k \in K, \quad \mathrm{Ad}(k)a = a. \tag{3-9}$$

Let $\gamma = e^a k^{-1}$ be such that (3-9) holds. By [Bismut 2011, Equations (3.3.4), (3.3.6)], we have

$$Z(\gamma) = Z(a) \cap Z(k), \quad \mathfrak{z}(\gamma) = \mathfrak{z}(a) \cap \mathfrak{z}(k). \tag{3-10}$$

Set

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \quad \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}. \tag{3-11}$$

From (3-10) and (3-11), we get

$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{3-12}$$

By [Knapp 2002, Proposition 7.25], $Z(\gamma)$ is a reductive subgroup of G with maximal compact subgroup $K(\gamma) = Z(\gamma) \cap K$, and with Cartan decomposition (3-12). Let $Z^0(\gamma)$ be the connected component of the identity in $Z(\gamma)$. Then $Z^0(\gamma)$ is a reductive subgroup of G, with maximal compact subgroup $Z^0(\gamma) \cap K$. Also, $Z^0(\gamma) \cap K$ coincides with $K^0(\gamma)$, the connected component of the identity in $K(\gamma)$.

An element $\gamma \in G$ is said to be elliptic if γ is conjugated to an element of K. Let $\gamma \in G$ be semisimple and nonelliptic. Up to conjugation, we can assume $\gamma = e^a k^{-1}$ such that (3-9) holds and that $a \neq 0$. By (3-10), $a \in \mathfrak{p}(\gamma)$. Let $\mathfrak{z}^{a,\perp}(\gamma)$, $\mathfrak{p}^{a,\perp}(\gamma)$ be respectively the orthogonal spaces to a in $\mathfrak{z}(\gamma)$, $\mathfrak{p}(\gamma)$, so that

$$\mathfrak{z}^{a,\perp}(\gamma) = \mathfrak{p}^{a,\perp}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{3-13}$$

Moreover, $\mathfrak{z}^{a,\perp}(\gamma)$ is a Lie algebra. Let $Z^{a,\perp,0}(\gamma)$ be the connected subgroup of $Z^0(\gamma)$ that is associated with the Lie algebra $\mathfrak{z}^{a,\perp}(\gamma)$. By [Bismut 2011, Equation (3.3.11)], $Z^{a,\perp,0}(\gamma)$ is reductive with maximal compact subgroup $K^0(\gamma)$ with Cartan decomposition (3-13), and

$$Z^0(\gamma) = \mathbb{R} \times Z^{a,\perp,0}(\gamma), \tag{3-14}$$

so that e^{ta} maps into $t|a|$.

3C. *Cartan subgroups.* A Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$. A Cartan subgroup of G is the centralizer of a Cartan subalgebra.

By [Knapp 1986, Theorem 5.22], there is only a finite number of nonconjugate (via K) θ-stable Cartan subalgebras $\mathfrak{h}_1, \dots, \mathfrak{h}_{l_0}$. Let $H_1, \dots, H_{l_0}$ be the corresponding Cartan subgroups. Clearly, the Lie algebra of H_i is $\mathfrak{h}_i$. Set

$$\mathfrak{h}_{i\mathfrak{p}} = \mathfrak{h}_i \cap \mathfrak{p}, \quad \mathfrak{h}_{i\mathfrak{k}} = \mathfrak{h}_i \cap \mathfrak{k}. \tag{3-15}$$

We call $\dim \mathfrak{h}_{i\mathfrak{p}}$ the noncompact dimension of $\mathfrak{h}_i$. By [Knapp 1986, Theorem 5.22(c); 2002, Proposition 7.25], H_i is an abelian reductive group with maximal compact subgroup $H_i \cap K$, and with Cartan decomposition

$$\mathfrak{h}_i = \mathfrak{h}_{i\mathfrak{p}} \oplus \mathfrak{h}_{i\mathfrak{k}}, \quad H_i = \exp(\mathfrak{h}_{i\mathfrak{p}})(H_i \cap K). \tag{3-16}$$

Note that in general, H_i is not necessarily connected.

Let $W(H_i, G)$ be the Weyl group. If $N_K(\mathfrak{h}_i) \subset K$ and $Z_K(\mathfrak{h}_i) \subset K$ are the normalizer and centralizer of $\mathfrak{h}_i$ in K, then

$$W(H_i, G) = N_K(\mathfrak{h}_i)/Z_K(\mathfrak{h}_i). \tag{3-17}$$

Throughout, we fix a maximal torus T of K. Let $\mathfrak{t} \subset \mathfrak{k}$ be the Lie algebra of T. Set

$$\mathfrak{b} = \{Y \in \mathfrak{p} : [Y, \mathfrak{t}] = 0\}. \tag{3-18}$$

By (3-5) and (3-18), we have

$$\mathfrak{z}_{\mathfrak{p}} \subset \mathfrak{b}. \tag{3-19}$$

Put

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}. \tag{3-20}$$

By [Knapp 1986, Theorem 5.22(b)], $\mathfrak{h}$ is the θ-stable Cartan subalgebra of $\mathfrak{g}$ with minimal noncompact dimension. Also, every θ-stable Cartan subalgebra with minimal noncompact dimension is conjugated to $\mathfrak{h}$ by an element of K. Moreover, the corresponding Cartan subgroup $H \subset G$ of G is connected, so that

$$H = \exp(\mathfrak{b})T. \tag{3-21}$$

We may assume that $\mathfrak{h}_1 = \mathfrak{h}$ and $H_1 = H$.

Note that the complexification $\mathfrak{h}_{i\mathbb{C}} = \mathfrak{h}_i \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. All the $\mathfrak{h}_{i\mathbb{C}}$ are conjugated by inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$. Their common complex dimension $\dim_{\mathbb{C}} \mathfrak{h}_{i\mathbb{C}}$ is called the complex rank $\mathrm{rk}_{\mathbb{C}}(G)$ of G.

Definition 3.1. Put

$$\delta(G) = \mathrm{rk}_{\mathbb{C}}(G) - \mathrm{rk}_{\mathbb{C}}(K) \in \mathbb{N}. \tag{3-22}$$

By (3-18) and (3-22), we have

$$\delta(G) = \dim \mathfrak{b}. \tag{3-23}$$

Note that $m - \delta(G)$ is even. We will see that $\delta(G)$ plays an important role in our article.

Remark 3.2. If $\mathfrak{g}$ is a real reductive Lie algebra, then $\delta(\mathfrak{g}) \in \mathbb{N}$ can be defined in the same way as in (3-23). Since $\mathfrak{g}$ is reductive, by [Knapp 2002, Corollary 1.56], we have

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}], \tag{3-24}$$

where $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra. By (3-6) and (3-24), we have

$$\delta(\mathfrak{g}) = \dim \mathfrak{z}_{\mathfrak{p}} + \delta([\mathfrak{g}, \mathfrak{g}]). \tag{3-25}$$

Proposition 3.3. *The element $\gamma \in G$ is semisimple if and only if γ can be conjugated into $\bigcup_{i=1}^{l_0} H_i$. In this case,*

$$\delta(G) \leqslant \delta(Z^0(\gamma)). \tag{3-26}$$

The two sides of (3-26) are equal if and only if γ can be conjugated into H.

Proof. If $\gamma \in H_i$, by the Cartan decomposition (3-16), there exist $a \in \mathfrak{h}_{i\mathfrak{p}}$ and $k \in K \cap H_i$ such that $\gamma = e^a k^{-1}$. Since H_i is the centralizer of $\mathfrak{h}_i$, we have $\mathrm{Ad}(\gamma)a = a$. Therefore, $\mathrm{Ad}(k)a = a$, so that γ is semisimple.

Assume that $\gamma \in G$ is semisimple and is such that (3-9) holds. We claim that

$$\mathrm{rk}_{\mathbb{C}}(G) = \mathrm{rk}_{\mathbb{C}}(Z^0(\gamma)). \tag{3-27}$$

Indeed, let $\mathfrak{h}' \subset \mathfrak{g}$ be a θ-invariant Cartan subalgebra of $\mathfrak{g}$ containing a. Then, $\mathfrak{h}' \subset \mathfrak{z}(a)$. It implies

$$\mathrm{rk}_{\mathbb{C}}(G) = \mathrm{rk}_{\mathbb{C}}(Z^0(a)). \tag{3-28}$$

By choosing a maximal torus T containing k, by (3-20), we have $\mathfrak{h} \subset \mathfrak{z}(k)$. Then

$$\mathrm{rk}_{\mathbb{C}}(G) = \mathrm{rk}_{\mathbb{C}}(Z^0(k)). \tag{3-29}$$

If we replace G by $Z^0(a)$ in (3-29), by (3-10), we get

$$\mathrm{rk}_{\mathbb{C}}(Z^0(a)) = \mathrm{rk}_{\mathbb{C}}(Z^0(\gamma)). \tag{3-30}$$

By (3-28) and (3-30), we get (3-27).

Let $\mathfrak{h}(\gamma) \subset \mathfrak{z}(\gamma)$ be the θ-invariant Cartan subalgebra defined as in (3-20) when G is replaced by $Z^0(\gamma)$. By (3-27), $\mathfrak{h}(\gamma)$ is also a Cartan subalgebra of $\mathfrak{g}$. Moreover, γ is an element of the Cartan subgroup of G associated to $\mathfrak{h}(\gamma)$. In particular, γ can be conjugated into some H_i.

By the minimality of noncompact dimension of $\mathfrak{h}$, we have

$$\delta(G) = \dim \mathfrak{h} \cap \mathfrak{p} \leqslant \dim \mathfrak{h}(\gamma) \cap \mathfrak{p} = \delta(Z^0(\gamma)), \tag{3-31}$$

which completes the proof of (3-26).

It is obvious that if γ can be conjugated into H, the equality in (3-31) holds. If the equality holds in (3-31), by the uniqueness of the Cartan subalgebra with minimal noncompact dimension, there is $k' \in K$ such that

$$\mathrm{Ad}(k')\mathfrak{h}(\gamma) = \mathfrak{h}, \tag{3-32}$$

which implies that $k'\gamma k'^{,-1} \in H$. $\square$

Now we recall the Weyl integral formula on G, which will be used in Section 8. Let dv_{H_i} and $dv_{H_i\backslash G}$ be respectively the Riemannian volumes on H_i and $H_i\backslash G$ induced by $-B(\cdot,\theta\cdot)$. By [Knapp 2002, Theorem 8.64], for a nonnegative measurable function f on G, we have

$$\int_{g\in G} f(g)\,dv_G=\sum_{i=1}^{l_0}\frac{1}{|W(H_i,G)|}\int_{\gamma\in H_i}\bigg(\int_{g\in H_i\backslash G} f(g^{-1}\gamma g)\,dv_{H_i\backslash G}\bigg)\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|\,dv_{H_i}. \tag{3-33}$$

3D. *Regular elements.* For $0\leqslant j\leqslant m+n-\mathrm{rk}_{\mathbb{C}}(G)$, let D_j be the analytic function on G such that, for $t\in\mathbb{R}$ and $\gamma\in G$, we have

$$\det(t+1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}}=t^{\mathrm{rk}_{\mathbb{C}}(G)}\bigg(\sum_{j=0}^{m+n-\mathrm{rk}_{\mathbb{C}}(G)}D_j(\gamma)t^j\bigg). \tag{3-34}$$

If $\gamma\in H_i$, then

$$D_0(\gamma)=\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}. \tag{3-35}$$

We call $\gamma\in G$ regular if $D_0(\gamma)\neq 0$. Let $G'\subset G$ be the subset of regular elements of G. Then G' is open in G such that $G-G'$ has zero measure with respect to the Riemannian volume dv_G on G induced by $-B(\cdot,\theta\cdot)$. For $1\leqslant i\leqslant l_0$, set

$$H_i'=H_i\cap G',\quad G_i'=\bigcup_{g\in G}g^{-1}H_i'g. \tag{3-36}$$

By [Knapp 1986, Theorem 5.22(d)], G_i' is open, and we have the disjoint union

$$G'=\coprod_{1\leqslant i\leqslant l_0}G_i'. \tag{3-37}$$

4. Orbital integrals and Selberg trace formula

The purpose of this section is to recall the semisimple orbital integral formula of [Bismut 2011, Theorem 6.1.1] and the Selberg trace formula.

This section is organized as follows. In Section 4A, we introduce the Riemannian symmetric space $X=G/K$, and we give a formula for its Euler form.

In Section 4B, we recall the definition of semisimple orbital integrals.

In Section 4C, we recall Bismut's explicit formula for the semisimple orbital integrals associated to the heat operator of the Casimir element.

In Section 4D, we introduce a discrete torsion-free cocompact subgroup Γ of G. We state the Selberg trace formula.

Finally, in Section 4E, we recall Bismut's proof of a vanishing result on the analytic torsion in the case $\delta(G)\neq 1$, which is originally due to Moscovici and Stanton [1991].

4A. *The symmetric space.* We use the notation of Section 3. Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, and let $\omega^{\mathfrak{p}},\omega^{\mathfrak{k}}$ be its components in $\mathfrak{p}$, $\mathfrak{k}$, so that

$$\omega^{\mathfrak{g}}=\omega^{\mathfrak{p}}+\omega^{\mathfrak{k}}. \tag{4-1}$$

Let $X = G/K$ be the associated symmetric space. Then

$$p : G \to X = G/K \tag{4-2}$$

is a K-principle bundle, equipped with the connection form $\omega^{\mathfrak{k}}$. By (3-2) and (4-1), the curvature of $\omega^{\mathfrak{k}}$ is given by

$$\Omega^{\mathfrak{k}} = -\tfrac{1}{2}[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}]. \tag{4-3}$$

Let τ be a finite-dimensional orthogonal representation of K on the real Euclidean space E_τ. Then $\mathcal{E}_\tau = G \times_K E_\tau$ is a real Euclidean vector bundle on X, which is naturally equipped with a Euclidean connection $\nabla^{\mathcal{E}_\tau}$. The space of smooth sections $C^\infty(X, \mathcal{E}_\tau)$ on X can be identified with the K-invariant subspace $C^\infty(G, E_\tau)^K$ of smooth E_τ-valued functions on G. Let $C^{\mathfrak{g},X,\tau}$ be the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$. Then $C^{\mathfrak{g},X,\tau}$ is a formally self-adjoint second-order elliptic differential operator which is bounded from below.

Observe that K acts isometrically on $\mathfrak{p}$. Using the above construction, the tangent bundle $TX = G \times_K \mathfrak{p}$ is equipped with a Euclidean metric g^{TX} and a Euclidean connection ∇^{TX}. Also, ∇^{TX} is the Levi-Civita connection on (TX, g^{TX}) with curvature R^{TX}. Classically, (X, g^{TX}) is a Riemannian manifold of nonpositive sectional curvature. For $x, y \in X$, we denote by $d_X(x, y)$ the Riemannian distance on X.

If $E_\tau = \Lambda^\bullet(\mathfrak{p}^*)$, then $C^\infty(X, \mathcal{E}_\tau) = \Omega^\bullet(X)$. In this case, we write $C^{\mathfrak{g},X} = C^{\mathfrak{g},X,\tau}$. By [Bismut 2011, Proposition 7.8.1], $C^{\mathfrak{g},X}$ coincides with the Hodge Laplacian acting on $\Omega^\bullet(X)$.

Let us state a formula for $e(TX, \nabla^{TX})$. Let $o(TX)$ be the orientation line of TX. Let dv_X be the G-invariant Riemannian volume form on X. If $\alpha \in \Omega^\bullet(X, o(TX))$ is of maximal degree and G-invariant, set $[\alpha]^{\max} \in \mathbb{R}$ such that

$$\alpha = [\alpha]^{\max} \, dv_X. \tag{4-4}$$

Recall that if G has compact center, then U is the compact form of G. If $\delta(G) = 0$, by (3-25), G has compact center. In this case, T is a maximal torus of both U and K. Let $W(T, U)$, $W(T, K)$ be the respective the Weyl groups. Let $\mathrm{vol}(U/K)$ be the volume of U/K with respect to the volume form induced by $-B$.

Proposition 4.1. *If* $\delta(G) \neq 0$, *then* $[e(TX, \nabla^{TX})]^{\max} = 0$. *If* $\delta(G) = 0$, *then*

$$[e(TX, \nabla^{TX})]^{\max} = (-1)^{\frac{m}{2}} \frac{|W(T, U)|/|W(T, K)|}{\mathrm{vol}(U/K)}. \tag{4-5}$$

Proof. If G has noncompact center (thus $\delta(G) \neq 0$), it is trivial that $[e(TX, \nabla^{TX})]^{\max} = 0$. Assume now, G has compact center. By Hirzebruch proportionality [1966] (see Theorem 22.3.1 of that paper for a proof for Hermitian symmetric spaces; the proof for general case is identical), we have

$$[e(TX, \nabla^{TX})]^{\max} = (-1)^{\frac{m}{2}} \frac{\chi(U/K)}{\mathrm{vol}(U/K)}. \tag{4-6}$$

Proposition 4.1 is a consequence of (4-6), Bott's formula [1965, p. 175], Theorem II of the same paper and of the fact that $\delta(G) = \mathrm{rk}_{\mathbb{C}}(U) - \mathrm{rk}_{\mathbb{C}}(K)$. □

Let $\gamma \in G$ be a semisimple element as in (3-9). Let

$$X(\gamma) = Z(\gamma)/K(\gamma) \tag{4-7}$$

be the associated symmetric space. Clearly,

$$X(\gamma) = Z^0(\gamma)/K^0(\gamma). \tag{4-8}$$

Suppose that γ is nonelliptic. Set

$$X^{a,\perp}(\gamma) = Z^{a,\perp,0}(\gamma)/K^0(\gamma). \tag{4-9}$$

By (3-14), (4-8) and (4-9), we have

$$X(\gamma) = \mathbb{R} \times X^{a,\perp}(\gamma), \tag{4-10}$$

so that the action e^{ta} on $X(\gamma)$ is just the translation by $t|a|$ on $\mathbb{R}$.

4B. *The semisimple orbital integrals.* Recall that τ is a finite-dimensional orthogonal representation of K on the real Euclidean space E_τ, and that $C^{\mathfrak{g},X,\tau}$ is the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$.

Let $p_t^{X,\tau}(x,x')$ be the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to the Riemannian volume dv_X on X. Classically, for $t > 0$, there exist $c > 0$ and $C > 0$ such that, for $x, x' \in X$,

$$|p_t^{X,\tau}(x,x')| \leqslant C \exp(-c\, d_X^2(x,x')). \tag{4-11}$$

Set

$$p_t^{X,\tau}(g) = p_t^{X,\tau}(p1, pg). \tag{4-12}$$

For $g \in G$ and $k, k' \in K$, we have

$$p_t^{X,\tau}(kgk') = \tau(k) p_t^{X,\tau}(g) \tau(k'). \tag{4-13}$$

Also, we can recover $p_t^{X,\tau}(x,x')$ by

$$p_t^{X,\tau}(x,x') = p_t^{X,\tau}(g^{-1}g'), \tag{4-14}$$

where $g, g' \in G$ are such that $pg = x,\ pg' = x'$.

In the sequel, we do not distinguish $p_t^{X,\tau}(x,x')$ and $p_t^{X,\tau}(g)$. We refer to both of them as being the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$.

Let $dv_{K^0(\gamma)\backslash K}$ and $dv_{Z^0(\gamma)\backslash G}$ be the Riemannian volumes on $K^0(\gamma)\backslash K$ and $Z^0(\gamma)\backslash G$ induced by $-B(\cdot, \theta\,\cdot)$. Let $\mathrm{vol}(K^0(\gamma)\backslash K)$ be the volume of $K^0(\gamma)\backslash K$ with respect to $dv_{K^0(\gamma)\backslash K}$.

Definition 4.2. Let $\gamma \in G$ be semisimple. The orbital integral of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ is defined by

$$\mathrm{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)] = \frac{1}{\mathrm{vol}(K^0(\gamma)\backslash K)} \int_{g \in Z^0(\gamma)\backslash G} \mathrm{Tr}^{E_\tau}[p_t^{X,\tau}(g^{-1}\gamma g)]\, dv_{Z^0(\gamma)\backslash G}. \tag{4-15}$$

Remark 4.3. Definition 4.2 is equivalent to [Bismut 2011, Definition 4.2.2], where the volume forms are normalized such that $\mathrm{vol}(K^0(\gamma)\backslash K) = 1$.

Remark 4.4. As the notation $\operatorname{Tr}^{[\gamma]}$ indicates, the orbital integral only depends on the conjugacy class of γ in G. However, the notation $[\gamma]$ (see Section 4D) will be used later for the conjugacy class in the discrete group Γ.

Remark 4.5. We will also consider the case where E_τ is a $\mathbb{Z}_2$-graded or virtual representation of K. We will use the notation $\operatorname{Tr_s}^{[\gamma]}[q]$ when the trace on the right-hand side of (4-15) is replaced by the supertrace on E_τ.

4C. *Bismut's formula for semisimple orbital integrals.* Let us first recall the explicit formula for $\operatorname{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]$ for any semisimple element $\gamma\in G$, obtained by Bismut [2011, Theorem 6.1.1].

Let $\gamma = e^a k^{-1}\in G$ be semisimple as in (3-9). Set

$$\mathfrak{z}_0=\mathfrak{z}(a),\quad \mathfrak{p}_0=\mathfrak{z}(a)\cap\mathfrak{p},\quad \mathfrak{k}_0=\mathfrak{z}(a)\cap\mathfrak{k}. \tag{4-16}$$

Then

$$\mathfrak{z}_0=\mathfrak{p}_0\oplus\mathfrak{k}_0. \tag{4-17}$$

By (3-10), (3-11) and (4-16), we have $\mathfrak{p}(\gamma)\subset\mathfrak{p}_0$ and $\mathfrak{k}(\gamma)\subset\mathfrak{k}_0$. Let $\mathfrak{p}_0^\perp(\gamma)$, $\mathfrak{k}_0^\perp(\gamma)$, $\mathfrak{z}_0^\perp(\gamma)$ be the orthogonal spaces of $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$, $\mathfrak{z}(\gamma)$ in $\mathfrak{p}_0$, $\mathfrak{k}_0$, $\mathfrak{z}_0$. Let $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$, $\mathfrak{z}_0^\perp$ be the orthogonal spaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$, $\mathfrak{z}_0$ in $\mathfrak{p}$, $\mathfrak{k}$, $\mathfrak{z}$. Then we have

$$\mathfrak{p}=\mathfrak{p}(\gamma)\oplus\mathfrak{p}_0^\perp(\gamma)\oplus\mathfrak{p}_0^\perp,\quad \mathfrak{k}=\mathfrak{k}(\gamma)\oplus\mathfrak{k}_0^\perp(\gamma)\oplus\mathfrak{k}_0^\perp. \tag{4-18}$$

Recall that $\widehat{A}$ is the function defined in (2-2).

Definition 4.6. For $Y\in\mathfrak{k}(\gamma)$, put

$$J_\gamma(Y)=\frac{1}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac12}}\frac{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{k}(\gamma)})}\times\left[\frac{1}{\det(1-\operatorname{Ad}(k^{-1}))|_{\mathfrak{z}_0^\perp(\gamma)}}\frac{\det(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1}))|_{\mathfrak{k}_0^\perp(\gamma)}}{\det(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1}))|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{\frac12}. \tag{4-19}$$

As explained in [Bismut 2011, Section 5.5], there is a natural choice for the square root in (4-19). Moreover, J_γ is an $\operatorname{Ad}(K^0(\gamma))$-invariant analytic function on $\mathfrak{k}(\gamma)$, and there exist $c_\gamma>0$, $C_\gamma>0$, such that, for $Y\in\mathfrak{k}(\gamma)$,

$$|J_\gamma(Y)|\leqslant C_\gamma\exp(c_\gamma|Y|). \tag{4-20}$$

By (4-19), we have

$$J_1(Y)=\frac{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{p}})}{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{k}})}. \tag{4-21}$$

For $Y\in\mathfrak{k}(\gamma)$, let dY be the Lebesgue measure on $\mathfrak{k}(\gamma)$ induced by $-B$. Recall that $C^{\mathfrak{k},\mathfrak{p}}$ and $C^{\mathfrak{k},\mathfrak{k}}$ are defined in (3-8). The main result of [Bismut 2011, Theorem 6.1.1] is the following.

Theorem 4.7. *For $t > 0$, we have*

$$\operatorname{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)] = \frac{1}{(2\pi t)^{\dim \mathfrak{z}(\gamma)/2}} \exp\Big(-\frac{|a|^2}{2t} + \frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\Big)$$
$$\times \int_{Y\in\mathfrak{k}(\gamma)} J_\gamma(Y)\operatorname{Tr}^{E_\tau}[\tau(k^{-1})\exp(-i\tau(Y))]\exp(-|Y|^2/(2t))\,dY. \tag{4-22}$$

4D. ***A discrete subgroup of G.*** Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G. By [Selberg 1960, Lemma 1], Γ contains the identity element and nonelliptic semisimple elements. Also, Γ acts isometrically on the left on X. This action lifts to all the homogeneous Euclidean vector bundles $\mathcal{E}_\tau$ constructed in Section 4A, and preserves the corresponding connections.

Take $Z = \Gamma\backslash X = \Gamma\backslash G/K$. Then Z is a connected closed orientable Riemannian locally symmetric manifold with nonpositive sectional curvature. Since X is contractible, $\pi_1(Z) = \Gamma$ and X is the universal cover of Z. We denote by $\hat{p} : \Gamma\backslash G \to Z$ and $\hat{\pi} : X \to Z$ the natural projections, so that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "p"] & \Gamma\backslash G \arrow[d, "\hat{p}"] \\ X \arrow[r, "\hat{\pi}"] & Z \end{tikzcd} \tag{4-23}$$

commutes.

The Euclidean vector bundle $\mathcal{E}_\tau$ descends to a Euclidean vector bundle $\mathcal{F}_\tau = \Gamma\backslash\mathcal{E}_\tau$ on Z. Take $r \in \mathbb{N}^*$. Let $\rho : \Gamma \to \mathrm{U}(r)$ be a unitary representation of Γ. Let (F, ∇^F, g^F) be the unitarily flat vector bundle on Z associated to ρ. Let $C^{\mathfrak{g},Z,\tau,\rho}$ be the Casimir element of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbb{R}} F)$. As in Section 4A, when $E_\tau = \Lambda^\bullet(\mathfrak{p}^*)$, we write $C^{\mathfrak{g},Z,\rho} = C^{\mathfrak{g},Z,\tau,\rho}$. Then,

$$\square^Z = C^{\mathfrak{g},Z,\rho}. \tag{4-24}$$

Recall that $p_t^{X,\tau}(x,x')$ is the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to dv_X.

Proposition 4.8. *There exist $c > 0$, $C > 0$ such that, for $t > 0$ and $x \in X$, we have*

$$\sum_{\gamma\in\Gamma-\{1\}} |p_t^{X,\tau}(x,\gamma x)| \leqslant C\exp\Big(-\frac{c}{t} + Ct\Big). \tag{4-25}$$

Proof. By [Milnor 1968a, Remark p. 1, Lemma 2] or [Ma and Marinescu 2015, Equation (3.19)], there is $C > 0$ such that, for all $r \geqslant 0$, $x \in X$, we have

$$\big|\{\gamma \in \Gamma : d_X(x,\gamma x) \leqslant r\}\big| \leqslant Ce^{Cr}. \tag{4-26}$$

We claim that there exist $c > 0$, $C > 0$ and $N \in \mathbb{N}$ such that, for $t > 0$ and $x, x' \in X$, we have

$$|p_t^{X,\tau}(x,x')| \leqslant \frac{C}{t^N}\exp\Big(-c\frac{d_X^2(x,x')}{t} + Ct\Big). \tag{4-27}$$

Indeed, if $\tau = \mathbf{1}$, then $p_t^{X,\mathbf{1}}(x,x')$ is the heat kernel for the Laplace–Beltrami operator. In this case, (4-27) is a consequence of the Li–Yau estimate [1986, Corollary 3.1] and of the fact that X is a symmetric space.

For general τ, using the Itô formula as in [Bismut and Zhang 1992, Equation (12.30)], we can show that there is $C > 0$ such that

$$|p_t^{X,\tau}(x,x')| \leqslant Ce^{Ct}p_t^{X,1}(x,x'), \tag{4-28}$$

from which we get (4-27).[2]

Note that there exists $c_0 > 0$ such that, for all $\gamma \in \Gamma - \{1\}$ and $x \in X$,

$$d_X(x,\gamma x) \geqslant c_0. \tag{4-29}$$

By (4-27) and (4-29), there exist $c_1 > 0$, $c_2 > 0$ and $C > 0$ such that, for $t > 0$, $x \in X$ and $\gamma \in \Gamma - \{1\}$, we have

$$|p_t^{X,\tau}(x,\gamma x)| \leqslant C\exp\Big(-\frac{c_1}{t} - c_2\frac{d_X^2(x,\gamma x)}{t} + Ct\Big). \tag{4-30}$$

By (4-26) and (4-30), for $t > 0$ and $x \in X$, we have

$$\begin{aligned}\sum_{\gamma\in\Gamma-\{1\}} |p_t^{X,\tau}(x,\gamma x)| &\leqslant C\sum_{\gamma\in\Gamma}\exp\Big(-\frac{c_1}{t} - c_2\frac{d_X^2(x,\gamma x)}{t} + Ct\Big)\\ &= c_2C\exp\Big(-\frac{c_1}{t} + Ct\Big)\sum_{\gamma\in\Gamma}\int_{d_X^2(x,\gamma x)/t}^{\infty}\exp(-c_2r)\,dr\\ &= c_2C\exp\Big(-\frac{c_1}{t} + Ct\Big)\int_0^{\infty}\big|\{\gamma\in\Gamma : d_X(x,\gamma x)\leqslant\sqrt{rt}\}\big|\exp(-c_2r)\,dr\\ &\leqslant C'\exp\Big(-\frac{c_1}{t} + Ct\Big)\int_0^{\infty}\exp(-c_2r + C\sqrt{rt})\,dr. \end{aligned}\tag{4-31}$$

From (4-31), we get (4-25). □

For $\gamma \in \Gamma$, set

$$\Gamma(\gamma) = Z(\gamma)\cap\Gamma. \tag{4-32}$$

Let $[\gamma]$ be the conjugacy class of γ in Γ. Let $[\Gamma]$ be the set of all the conjugacy classes of Γ.

The following proposition is [Selberg 1960, Lemma 2]. We include a proof for the sake of completeness.

Proposition 4.9. *If $\gamma \in \Gamma$, then $\Gamma(\gamma)$ is cocompact in $Z(\gamma)$.*

Proof. Since Γ is discrete, $[\gamma]$ is closed in G. The inverse image of $[\gamma]$ by the continuous map $g \in G \to g\gamma g^{-1} \in G$ is $\Gamma\cdot Z(\gamma)$. Then $\Gamma\cdot Z(\gamma)$ is closed in G. Since $\Gamma\backslash G$ is compact, the closed subset $\Gamma\backslash\Gamma\cdot Z(\gamma) \subset \Gamma\backslash G$ is then compact.

The group $Z(\gamma)$ acts transitively on the right on $\Gamma\backslash\Gamma\cdot Z(\gamma)$. The stabilizer at $[1] \in \Gamma\backslash\Gamma\cdot Z(\gamma)$ is $\Gamma(\gamma)$. Hence $\Gamma(\gamma)\backslash Z(\gamma) \simeq \Gamma\backslash\Gamma\cdot Z(\gamma)$ is compact. □

Let $\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))$ be the volume of $\Gamma(\gamma)\backslash X(\gamma)$ with respect to the volume form induced by $dv_{X(\gamma)}$. Clearly, $\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))$ depends only on the conjugacy class $[\gamma] \in [\Gamma]$.

[2] See [Ma and Marinescu 2015, Theorem 4] for another proof of (4-27) using finite propagation speed of solutions of hyperbolic equations.

By the property of heat kernels on compact manifolds, the operator $\exp(-tC^{\mathfrak{g},Z,\tau,\rho}/2)$ is trace class. Its trace is given by the Selberg trace formula:

Theorem 4.10. *There exist $c>0$, $C>0$ such that, for $t>0$, we have*

$$\sum_{[\gamma]\in[\Gamma]-\{1\}} \operatorname{vol}(\Gamma(\gamma)\backslash X(\gamma))\left|\operatorname{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]\right| \leqslant C\exp\Big(-\frac{c}{t}+Ct\Big). \tag{4-33}$$

For $t>0$, the following identity holds:

$$\operatorname{Tr}[\exp(-tC^{\mathfrak{g},Z,\tau,\rho}/2)] = \sum_{[\gamma]\in[\Gamma]} \operatorname{vol}(\Gamma(\gamma)\backslash X(\gamma))\operatorname{Tr}[\rho(\gamma)]\operatorname{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]. \tag{4-34}$$

Proof. Let $F\subset X$ be a fundamental domain of Z in X. By [Bismut 2011, Equations (4.8.11), (4.8.15)], we have

$$\sum_{\gamma'\in[\gamma]}\int_{x\in F}\operatorname{Tr}^{E_\tau}[p_t^{X,\tau}(x,\gamma'x)]\,dx = \operatorname{vol}(\Gamma(\gamma)\backslash X(\gamma))\operatorname{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]. \tag{4-35}$$

By (4-25) and (4-35), we get (4-33). The proof of (4-34) is well known; see, for example, [Bismut 2011, Section 4.8]. □

4E. *A formula for* $\operatorname{Tr}_s^{[\gamma]}[N^{\Lambda^\bullet(T^*X)}\exp(-tC^{\mathfrak{g},X}/2)]$. Let $\gamma=e^ak^{-1}\in G$ be semisimple such that (3-9) holds. Let $\mathfrak{t}(\gamma)\subset\mathfrak{k}(\gamma)$ be a Cartan subalgebra of $\mathfrak{k}(\gamma)$. Set

$$\mathfrak{b}(\gamma)=\{Y\in\mathfrak{p}:\operatorname{Ad}(k)Y=Y,\ [Y,\mathfrak{t}(\gamma)]=0\}. \tag{4-36}$$

Then,

$$a\in\mathfrak{b}(\gamma). \tag{4-37}$$

By definition, $\dim\mathfrak{p}-\dim\mathfrak{b}(\gamma)$ is even.

Since k centralizes $\mathfrak{t}(\gamma)$, by [Knapp 1986, Theorem 4.21], there is $k'\in K$ such that

$$k'\mathfrak{t}(\gamma)k'^{-1}\subset\mathfrak{t},\quad k'kk'^{-1}\in T. \tag{4-38}$$

Up to a conjugation on γ, we can assume directly that $\gamma=e^ak^{-1}$ with

$$\mathfrak{t}(\gamma)\subset\mathfrak{t},\quad k\in T. \tag{4-39}$$

By (3-18), (4-36), and (4-39), we have

$$\mathfrak{b}\subset\mathfrak{b}(\gamma). \tag{4-40}$$

Proposition 4.11. *A semisimple element $\gamma\in G$ can be conjugated into H if and only if*

$$\dim\mathfrak{b}=\dim\mathfrak{b}(\gamma). \tag{4-41}$$

Proof. If $\gamma\in H$, then $\mathfrak{t}(\gamma)=\mathfrak{t}$. By (4-36), we get $\mathfrak{b}=\mathfrak{b}(\gamma)$, which implies (4-41).

Recall that $\mathfrak{h}(\gamma) \subset \mathfrak{z}(\gamma)$ is defined as in (3-20), when G is replaced by $Z^0(\gamma)$ and $\mathfrak{t}$ is replaced by $\mathfrak{t}(\gamma)$. It is a θ-invariant Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{z}(\gamma)$. Let $\mathfrak{h}(\gamma) = \mathfrak{h}(\gamma)_{\mathfrak{p}} \oplus \mathfrak{h}(\gamma)_{\mathfrak{k}}$ be the Cartan decomposition. Then,

$$\mathfrak{h}(\gamma)_{\mathfrak{p}} = \{Y \in \mathfrak{p}(\gamma) : [Y, \mathfrak{t}(\gamma)] = 0\} = \mathfrak{b}(\gamma) \cap \mathfrak{p}(\gamma), \quad \mathfrak{h}(\gamma)_{\mathfrak{k}} = \mathfrak{t}(\gamma). \tag{4-42}$$

From (3-26) and (4-42), we get

$$\dim \mathfrak{b} \leqslant \dim \mathfrak{h}(\gamma)_{\mathfrak{p}} \leqslant \dim \mathfrak{b}(\gamma). \tag{4-43}$$

By (4-43), if $\dim \mathfrak{b} = \dim \mathfrak{b}(\gamma)$, then $\dim \mathfrak{b} = \dim \mathfrak{h}(\gamma)_{\mathfrak{p}}$. By Proposition 3.3, γ can be conjugated into H. $\square$

The following theorem extends [Bismut 2011, Theorem 7.9.1].

Theorem 4.12. *Let $\gamma \in G$ be semisimple such that $\dim \mathfrak{b}(\gamma) \geqslant 2$. For $Y \in \mathfrak{k}(\gamma)$, we have*

$$\operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}[N^{\Lambda^\bullet(\mathfrak{p}^*)} \operatorname{Ad}(k^{-1}) \exp(-i \operatorname{ad}(Y))] = 0. \tag{4-44}$$

In particular, for $t > 0$, we have

$$\operatorname{Tr_s}^{[\gamma]}[N^{\Lambda^\bullet(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)] = 0. \tag{4-45}$$

Proof. Since the left-hand side of (4-44) is $\operatorname{Ad}(K^0(\gamma))$-invariant, it is enough to show (4-44) for $Y \in \mathfrak{t}(\gamma)$. If $Y \in \mathfrak{t}(\gamma)$, by [Bismut 2011, Equation (7.9.1)], we have

$$\operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}^*)}[N^{\Lambda^\bullet(\mathfrak{p}^*)} \operatorname{Ad}(k^{-1}) \exp(-i \operatorname{ad}(Y))] = \frac{\partial}{\partial b}\bigg|_{b=0} \det\big(1 - e^b \operatorname{Ad}(k) \exp(i \operatorname{ad}(Y))\big)\big|_{\mathfrak{p}}. \tag{4-46}$$

Since $\dim \mathfrak{b}(\gamma) \geqslant 2$, by (4-46), we get (4-44) for $Y \in \mathfrak{t}(\gamma)$.

By (4-22) and (4-44), we get (4-45). $\square$

In this way, [Bismut 2011, Theorem 7.9.3] recovered [Moscovici and Stanton 1991, Corollary 2.2].

Corollary 4.13. *Let F be a unitarily flat vector bundle on Z. Assume that $\dim Z$ is odd and $\delta(G) \neq 1$. Then for any $t > 0$, we have*

$$\operatorname{Tr_s}[N^{\Lambda^\bullet(T^*Z)} \exp(-t\square^Z/2)] = 0. \tag{4-47}$$

In particular,

$$T(F) = 1. \tag{4-48}$$

Proof. Since $\dim Z$ is odd, $\delta(G)$ is odd. Since $\delta(G) \neq 1$, we have $\delta(G) \geqslant 3$. By (4-40), $\dim \mathfrak{b}(\gamma) \geqslant \delta(G) \geqslant 3$, so (4-47) is a consequence of (4-24), (4-34) and (4-45). $\square$

Suppose $\delta(G) = 1$. Up to sign, we fix an element $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. As in Section 3B, set

$$M = Z^{a_1,\perp,0}(e^{a_1}), \quad K_M = K^0(e^{a_1}), \tag{4-49}$$

and

$$\mathfrak{m} = \mathfrak{z}^{a_1,\perp}(e^{a_1}), \quad \mathfrak{p}_{\mathfrak{m}} = \mathfrak{p}^{a_1,\perp}(e^{a_1}), \quad \mathfrak{k}_{\mathfrak{m}} = \mathfrak{k}(e^{a_1}). \tag{4-50}$$

As in Section 3B, M is a connected reductive group with Lie algebra $\mathfrak{m}$, with maximal compact subgroup K_M, and with Cartan decomposition $\mathfrak{m} = \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{k}_\mathfrak{m}$. Let

$$X_M = M/K_M \tag{4-51}$$

be the corresponding symmetric space. By definition, $T \subset M$ is a compact Cartan subgroup. Therefore $\delta(M) = 0$, and $\dim \mathfrak{p}_\mathfrak{m}$ is even.

Assume that $\delta(G) = 1$ and that G has noncompact center, so that $\dim \mathfrak{z}_\mathfrak{p} \geqslant 1$. By (3-19), we find that $a_1 \in \mathfrak{z}_\mathfrak{p}$, so that $Z^0(a_1) = G$. By (3-14) and (4-10), we have

$$G = \mathbb{R} \times M, \quad K = K_M, \quad X = \mathbb{R} \times X_M. \tag{4-52}$$

Let $\gamma \in G$ be a semisimple element such that $\dim \mathfrak{b}(\gamma) = 1$. By Proposition 4.11, we may assume that $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$ and $k \in T$.

Proposition 4.14. *We have*

$$\mathrm{Tr_s}^{[1]}[N^{\Lambda^\bullet(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)] = -\frac{1}{\sqrt{2\pi t}}[e(TX_M, \nabla^{TX_M})]^{\max}. \tag{4-53}$$

If $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$, $a \neq 0$, and $k \in T$, then

$$\mathrm{Tr}^{[\gamma]}[N^{\Lambda^\bullet(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)] = -\frac{1}{\sqrt{2\pi t}} e^{-\frac{|a|^2}{2t}} [e(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)})]^{\max}. \tag{4-54}$$

Proof. By (4-52), for $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$ and $k \in T$, we have

$$\mathrm{Tr}_s^{[\gamma]}[N^{\Lambda^\bullet(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)] = -\mathrm{Tr}^{[e^a]}[\exp(t\Delta^{\mathbb{R}}/2)]\, \mathrm{Tr}_s^{[k^{-1}]}[\exp(-tC^{\mathfrak{m},X_M}/2)], \tag{4-55}$$

where $\Delta^{\mathbb{R}}$ is the Laplace–Beltrami operator acting on $C^\infty(\mathbb{R})$.

Clearly,

$$\mathrm{Tr}^{[e^a]}[\exp(t\Delta^{\mathbb{R}}/2)] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|a|^2}{2t}}. \tag{4-56}$$

By [Bismut 2011, Theorem 7.8.13], we have

$$\mathrm{Tr}_s^{[1]}[\exp(-tC^{\mathfrak{m},X_M}/2)] = [e(TX_M, \nabla^{TX_M})]^{\max} \tag{4-57}$$

and

$$\mathrm{Tr}_s^{[k^{-1}]}[\exp(-tC^{\mathfrak{m},X_M}/2)] = [e(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)})]^{\max}. \tag{4-58}$$

By (4-55)–(4-58), we get (4-53) and (4-54). □

5. The solution to Fried conjecture

We use the notation in Sections 3 and 4. Also, we assume that $\dim \mathfrak{p}$ is odd. The purpose of this section is to introduce the Ruelle dynamical zeta function on Z and to state our main result, which contains the solution of the Fried conjecture in the case of locally symmetric spaces.

This section is organized as follows. In Section 5A, we describe the closed geodesics on Z.

In Section 5B, we define the dynamical zeta function and state Theorem 5.5, which is the main result of the article.

Finally, in Section 5C, we establish Theorem 5.5 when G has noncompact center and $\delta(G) = 1$.

5A. ***The space of closed geodesics.*** By [Duistermaat et al. 1979, Proposition 5.15], the set of nontrivial closed geodesics on Z consists of a disjoint union of smooth connected closed submanifolds

$$\coprod_{[\gamma]\in[\Gamma]-[1]} B_{[\gamma]}. \tag{5-1}$$

Moreover, $B_{[\gamma]}$ is diffeomorphic to $\Gamma(\gamma)\backslash X(\gamma)$. All the elements of $B_{[\gamma]}$ have the same length $|a|>0$ if γ can be conjugated to $e^a k^{-1}$ as in (3-9). Also, the geodesic flow induces a canonical locally free action of $\mathbb{S}^1$ on $B_{[\gamma]}$, so that $\mathbb{S}^1\backslash B_{[\gamma]}$ is a closed orbifold. The $\mathbb{S}^1$-action is not necessarily effective. Let

$$m_{[\gamma]} = \left|\ker(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]}))\right| \in \mathbb{N}^* \tag{5-2}$$

be the generic multiplicity.

Following [Satake 1957], if S is a closed Riemannian orbifold with Levi-Civita connection ∇^{TS}, then $e(TS,\nabla^{TS}) \in \Omega^{\dim S}(S, o(TS))$ is still well defined, and the Euler characteristic $\chi_{\mathrm{orb}}(S)\in\mathbb{Q}$ is given by

$$\chi_{\mathrm{orb}}(S) = \int_S e(TS,\nabla^{TS}). \tag{5-3}$$

Proposition 5.1. *For $\gamma\in\Gamma-\{1\}$, the following identity holds*:

$$\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} = \frac{\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))}{|a|}\left[e(TX^{a,\perp}(\gamma),\nabla^{TX^{a,\perp}(\gamma)})\right]^{\max}. \tag{5-4}$$

Proof. Take $\gamma\in\Gamma-\{1\}$. We can assume that $\gamma = e^a k^{-1}$ as in (3-9) with $a\neq 0$. By (3-10) and (4-32), for $t\in\mathbb{R}$, we know e^{ta} commutes with elements of $\Gamma(\gamma)$. Thus, e^{ta} acts on the left on $\Gamma(\gamma)\backslash X(\gamma)$. Since $e^a = \gamma k$, $\gamma\in\Gamma(\gamma)$, $k\in K(\gamma)$ and k commutes with elements of $Z(\gamma)$, we see that e^a acts as identity on $\Gamma(\gamma)\backslash X(\gamma)$. This induces an $\mathbb{R}/\mathbb{Z}\simeq\mathbb{S}^1$ action on $\Gamma(\gamma)\backslash X(\gamma)$ which coincides with the $\mathbb{S}^1$-action on $B_{[\gamma]}$. Therefore,

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = \mathrm{vol}(\mathbb{S}^1\backslash B_{[\gamma]})\left[e(TX^{a,\perp}(\gamma),\nabla^{TX^{a,\perp}(\gamma)})\right]^{\max} \tag{5-5}$$

and

$$\frac{\mathrm{vol}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} = \frac{\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))}{|a|}. \tag{5-6}$$

By (5-5) and (5-6), we get (5-4). □

Corollary 5.2. *Let $\gamma\in\Gamma-\{1\}$. If* $\dim\mathfrak{b}(\gamma)\geqslant 2$, *then*

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = 0. \tag{5-7}$$

Proof. By Propositions 4.1 and 5.1, it is enough to show that

$$\delta(Z^{a,\perp,0}(\gamma)) \geqslant 1. \tag{5-8}$$

By (3-14) and (3-26), we have

$$\delta(Z^{a,\perp,0}(\gamma)) = \delta(Z^0(\gamma)) - 1 \geqslant \delta(G) - 1. \tag{5-9}$$

Recall $\dim \mathfrak{p}$ is odd, therefore $\delta(G)$ is odd. If $\delta(G) \geqslant 3$, by (5-9), we get (5-8). If $\delta(G) = 1$, then $\dim \mathfrak{b}(\gamma) \geqslant 2 > \delta(G)$. By Propositions 3.3 and 4.11, the inequality in (5-9) is strict, which implies (5-8). □

Remark 5.3. By Theorem 4.12 and Corollary 5.2, we know both $\mathrm{Tr_s}^{[\gamma]}[N^{\Lambda^\bullet(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)]$ and $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})$ vanish when $\dim \mathfrak{b}(\gamma) \geqslant 2$.

5B. *Statement of the main result.* Recall that $\rho : \Gamma \to \mathrm{U}(r)$ is a unitary representation of Γ and that (F, ∇^F, g^F) is the unitarily flat vector bundle on Z associated with ρ.

Definition 5.4. The Ruelle dynamical zeta function $R_\rho(\sigma)$ is said to be well defined, if the following properties hold:

(1) For $\sigma \in \mathbb{C}$, $\mathrm{Re}(\sigma) \gg 1$, the sum

$$\Xi_\rho(\sigma) = \sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} \mathrm{Tr}[\rho(\gamma)] e^{-\sigma|a|} \tag{5-10}$$

converges to a holomorphic function.

(2) The function $R_\rho(\sigma) = \exp(\Xi_\rho(\sigma))$ has a meromorphic extension to $\sigma \in \mathbb{C}$.

If $\delta(G) \neq 1$, by Corollary 5.2,

$$R_\rho(\sigma) \equiv 1. \tag{5-11}$$

The main result of this article is the solution of the Fried conjecture. We restate Theorem 1.1 as follows.

Theorem 5.5. *The dynamical zeta function $R_\rho(\sigma)$ is well defined. There exist explicit constants $C_\rho \in \mathbb{R}^*$ and $r_\rho \in \mathbb{Z}$, see (7-75), such that, when $\sigma \to 0$ we have*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{5-12}$$

If $H^\bullet(Z, F) = 0$, then

$$C_\rho = 1, \quad r_\rho = 0, \tag{5-13}$$

so that

$$R_\rho(0) = T(F)^2. \tag{5-14}$$

Proof. When $\delta(G) \neq 1$, Theorem 5.5 is a consequence of (4-48) and (5-11). When $\delta(G) = 1$ and when G has noncompact center, we will show Theorem 5.5 in Section 5C. When $\delta(G) = 1$ and when G has compact center, we will show that $R_\rho(\sigma)$ is well defined such that (5-12) holds in Section 7, and we will show (5-13) in Section 8. □

5C. *Proof of Theorem 5.5 when G has noncompact center and $\delta(G)=1$.* We assume that $\delta(G)=1$ and that G has noncompact center. Let us show the following refined version of Theorem 5.5.

Theorem 5.6. *There is $\sigma_0>0$ such that*

$$\sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{|\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})|}{m_{[\gamma]}} e^{-\sigma_0|a|}<\infty. \tag{5-15}$$

The dynamical zeta function $R_\rho(\sigma)$ extends meromorphically to $\sigma\in\mathbb{C}$ such that

$$R_\rho(\sigma)=\exp\bigl(r\operatorname{vol}(Z)[e(TX_M,\nabla^{TX_M})]^{\max}\sigma\bigr)T(\sigma^2). \tag{5-16}$$

If $\chi'(Z,F)=0$, then $R_\rho(\sigma)$ is holomorphic at $\sigma=0$ and

$$R_\rho(0)=T(F)^2. \tag{5-17}$$

Proof. Following (2-5), for $(s,\sigma)\in\mathbb{C}\times\mathbb{R}$ such that $\operatorname{Re}(s)>m/2$ and $\sigma>0$, put

$$\begin{aligned}\theta_\rho(s,\sigma)&=-\operatorname{Tr}[N^{\Lambda^\bullet(T^*Z)}(C^{\mathfrak{g},Z,\rho}+\sigma)^{-s}]\\&=-\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr_s}\bigl[N^{\Lambda^\bullet(T^*Z)}\exp(-t(C^{\mathfrak{g},Z,\rho}+\sigma))\bigr]t^{s-1}\,dt.\end{aligned} \tag{5-18}$$

Let us show that there is $\sigma_0>0$ such that (5-15) holds true and that for $\sigma>\sigma_0$, we have

$$\Xi_\rho(\sigma)=\frac{\partial}{\partial s}\theta_\rho(0,\sigma^2)+r\operatorname{vol}(Z)[e(TX_M,\nabla^{TX_M})]^{\max}\sigma. \tag{5-19}$$

By (4-53), for $(s,\sigma)\in\mathbb{C}\times\mathbb{R}$ such that $\operatorname{Re}(s)>\frac12$ and $\sigma>0$, the function

$$\theta_{\rho,1}(s,\sigma)=-\frac{r\operatorname{vol}(Z)}{\Gamma(s)}\int_0^\infty \operatorname{Tr_s}^{[1]}\bigl[N^{\Lambda^\bullet(T^*X)}\exp(-t(C^{\mathfrak{g},X}+\sigma))\bigr]t^{s-1}\,dt \tag{5-20}$$

is well defined so that

$$\theta_{\rho,1}(s,\sigma)=\frac{r\operatorname{vol}(Z)}{2\sqrt{\pi}}[e(TX_M,\nabla^{TX_M})]^{\max}\frac{\Gamma\bigl(s-\frac12\bigr)}{\Gamma(s)}\sigma^{\frac12-s}. \tag{5-21}$$

Therefore, for $\sigma>0$ fixed, the function $s\to\theta_{\rho,1}(s,\sigma)$ has a meromorphic extension to $s\in\mathbb{C}$ which is holomorphic at $s=0$ so that

$$\frac{\partial}{\partial s}\theta_{\rho,1}(0,\sigma)=-r\operatorname{vol}(Z)[e(TX_M,\nabla^{TX_M})]^{\max}\sigma^{\frac12}. \tag{5-22}$$

For $(s,\sigma)\in\mathbb{C}\times\mathbb{R}$ such that $\operatorname{Re}(s)>m/2$ and $\sigma>0$, set

$$\theta_{\rho,2}(s,\sigma)=\theta_\rho(s,\sigma)-\theta_{\rho,1}(s,\sigma). \tag{5-23}$$

By (4-45), (4-54), (5-4), and (5-7), for $[\gamma]\in[\Gamma]-\{1\}$, we have

$$\operatorname{vol}(\Gamma(\gamma)\backslash X(\gamma))\operatorname{Tr_s}^{[\gamma]}[N^{\Lambda^\bullet(T^*X)}\exp(-tC^{\mathfrak{g},X})]=-\frac{1}{2\sqrt{\pi t}}\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}|a|\exp\Bigl(-\frac{|a|^2}{4t}\Bigr). \tag{5-24}$$

By (4-33) and (5-24), there exist $C_1 > 0$, $C_2 > 0$, and $C_3 > 0$ such that, for $t > 0$, we have

$$\sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{|\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})|}{m_{[\gamma]}} |a| \exp\Big(-\frac{|a|^2}{4t}\Big) \leqslant C_1 \exp\Big(-\frac{C_2}{t} + C_3 t\Big). \tag{5-25}$$

Take $\sigma_0 = \sqrt{2C_3}$. Since ρ is unitary, by (4-34), (5-18), (5-23), and (5-25), for $(s,\sigma) \in \mathbb{C}\times\mathbb{R}$ such that $\mathrm{Re}(s) > m/2$ and $\sigma \geqslant \sigma_0$, we have

$$\theta_{\rho,2}(s,\sigma^2) = \frac{1}{2\sqrt{\pi}\Gamma(s)} \int_0^\infty \sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} |a| \exp\Big(-\frac{|a|^2}{4t} - \sigma^2 t\Big) t^{s-\frac{3}{2}}\, dt. \tag{5-26}$$

Moreover, for $\sigma \geqslant \sigma_0$ fixed, the function $s \to \theta_{\rho,2}(s,\sigma^2)$ extends holomorphically to $\mathbb{C}$, so that

$$\frac{\partial}{\partial s}\theta_{\rho,2}(0,\sigma^2) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} |a| \exp\Big(-\frac{|a|^2}{4t} - \sigma^2 t\Big) \frac{dt}{t^{\frac{3}{2}}}. \tag{5-27}$$

Using the formula[3] that for $B_1 > 0$, $B_2 \geqslant 0$,

$$\int_0^\infty \exp\Big(-\frac{B_1}{t} - B_2 t\Big) \frac{dt}{t^{\frac{3}{2}}} = \sqrt{\frac{\pi}{B_1}} \exp(-2\sqrt{B_1 B_2}), \tag{5-28}$$

by (5-25), (5-27), and by Fubini's theorem, we get (5-15). Also, for $\sigma \geqslant \sigma_0$, we have

$$\frac{\partial}{\partial s}\theta_{\rho,2}(0,\sigma^2) = \Xi_\rho(\sigma). \tag{5-29}$$

By (5-22), (5-23), and (5-29), we get (5-19). By taking the exponentials, we get (5-16) for $\sigma \geqslant \sigma_0$. Since the right-hand side of (5-16) is meromorphic on $\sigma \in \mathbb{C}$, we know R_ρ has a meromorphic extension to $\mathbb{C}$. By (2-15) and (5-16), we get (5-17). □

6. Reductive groups G with compact center and $\delta(G) = 1$

In this section, we assume that $\delta(G) = 1$ and that G has compact center. The purpose of this section is to introduce some geometric objects associated with G. Their properties are proved by algebraic arguments based on the classification of real simple Lie algebras $\mathfrak{g}$ with $\delta(\mathfrak{g}) = 1$. The results of this section will be used in Section 7, in order to evaluate certain orbital integrals.

This section is organized as follows. In Section 6A, we introduce a splitting $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$, associated with the action of $\mathfrak{b}$ on $\mathfrak{g}$.

In Section 6B, we construct a natural compact Hermitian symmetric space $Y_{\mathfrak{b}}$, which will be used in the calculation of orbital integrals in Section 7A.

In Section 6C, we state one key result, which says that the action of K_M on $\mathfrak{n}$ lifts to K. The purpose of the following subsections is to prove this result.

[3] We give a proof of (5-28) when $B_1 = B_2 = 1$. Indeed, we have $\int_0^\infty \exp(-\frac{1}{t}-t)\frac{dt}{t^{3/2}} = \frac{1}{2}\int_0^\infty \exp(-\frac{1}{t}-t)\big(\frac{1}{t^{3/2}}+\frac{1}{t^{1/2}}\big)\,dt$. Using the change of variables $u = t^{1/2} - t^{-1/2}$, we get (5-28).

In Section 6D, we state a classification result of real simple Lie algebras $\mathfrak{g}$ with $\delta(\mathfrak{g}) = 1$, which asserts that they just contain $\mathfrak{sl}_3(\mathbb{R})$ and $\mathfrak{so}(p,q)$ with $pq > 1$ odd. This result has already been used by Moscovici and Stanton [1991].

In Sections 6E and 6F, we study the Lie groups $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SO}^0(p,q)$ with $pq > 1$ odd, and the structure of the associated Lie groups M, K_M.

In Section 6G, we study the connected component G_* of the identity of the isometry group of $X = G/K$. We show that G_* has a factor $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SO}^0(p,q)$ with $pq > 1$ odd.

Finally, in Sections 6H–6L, we show several unproven results stated in Sections 6A–6C. Most of the results are shown case by case for the groups $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SO}^0(p,q)$ with $pq > 1$ odd. We prove the corresponding results for general G using a natural morphism $i_G : G \to G_*$.

6A. ***A splitting of*** $\mathfrak{g}$**.** We use the notation in (4-49)–(4-51). Let $Z(\mathfrak{b}) \subset G$ be the stabilizer of $\mathfrak{b}$ in G, and let $\mathfrak{z}(\mathfrak{b}) \subset \mathfrak{g}$ be its Lie algebra.

We define $\mathfrak{p}(\mathfrak{b})$, $\mathfrak{k}(\mathfrak{b})$, $\mathfrak{p}^\perp(\mathfrak{b})$, $\mathfrak{k}^\perp(\mathfrak{b})$, $\mathfrak{z}^\perp(\mathfrak{b})$ in an obvious way as in Section 3B. By (4-50), we have

$$\mathfrak{p}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m}, \quad \mathfrak{k}(\mathfrak{b}) = \mathfrak{k}_\mathfrak{m}. \tag{6-1}$$

Also,

$$\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{p}^\perp(\mathfrak{b}), \quad \mathfrak{k} = \mathfrak{k}_\mathfrak{m} \oplus \mathfrak{k}^\perp(\mathfrak{b}). \tag{6-2}$$

Let $Z^0(\mathfrak{b})$ be the connected component of the identity in $Z(\mathfrak{b})$. By (3-14), we have

$$Z^0(\mathfrak{b}) = \mathbb{R} \times M. \tag{6-3}$$

The group K_M acts trivially on $\mathfrak{b}$. It also acts on $\mathfrak{p}_\mathfrak{m}$, $\mathfrak{p}^\perp(\mathfrak{b})$, $\mathfrak{k}_\mathfrak{m}$ and $\mathfrak{k}^\perp(\mathfrak{b})$, and preserves the splittings (6-2).

Recall that we have fixed $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. The choice of a_1 fixes an orientation of $\mathfrak{b}$. Let $\mathfrak{n} \subset \mathfrak{z}^\perp(\mathfrak{b})$ be the direct sum of the eigenspaces of $\mathrm{ad}(a_1)$ with the positive eigenvalues. Set $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Then $\bar{\mathfrak{n}}$ is the direct sum of the eigenspaces with negative eigenvalues, and

$$\mathfrak{z}^\perp(\mathfrak{b}) = \mathfrak{n} \oplus \bar{\mathfrak{n}}. \tag{6-4}$$

Clearly, $Z^0(\mathfrak{b})$ acts on $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ by adjoint action. Since K_M is fixed by θ, we have isomorphisms of representations of K_M

$$X \in \mathfrak{n} \to X - \theta X \in \mathfrak{p}^\perp(\mathfrak{b}), \quad X \in \mathfrak{n} \to X + \theta X \in \mathfrak{k}^\perp(\mathfrak{b}). \tag{6-5}$$

In the sequel, if $f \in \mathfrak{n}$, we define $\bar{f} = \theta f \in \bar{\mathfrak{n}}$.

By (6-2) and (6-5), we have $\dim \mathfrak{n} = \dim \mathfrak{p} - \dim \mathfrak{p}_\mathfrak{m} - 1$. Since $\dim \mathfrak{p}$ is odd and since $\dim \mathfrak{p}_\mathfrak{m}$ is even, $\dim \mathfrak{n}$ is even. Set

$$l = \tfrac{1}{2} \dim \mathfrak{n}. \tag{6-6}$$

Note that since G has compact center, we have $\mathfrak{b} \not\subset \mathfrak{z}_\mathfrak{g}$. Therefore, $\mathfrak{z}^\perp(\mathfrak{b}) \neq 0$ and $l > 0$.

Remark 6.1. Let $\mathfrak{q} \subset \mathfrak{g}$ be the direct sum of the eigenspaces of $\mathrm{ad}(a_1)$ with nonnegative eigenvalues. Then $\mathfrak{q}$ is a proper parabolic subalgebra of $\mathfrak{g}$, with Langlands decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{b} \oplus \mathfrak{n}$ [Knapp 2002, Section VII.7]. Let $Q \subset G$ be the corresponding parabolic subgroup of G, and let $Q = M_Q A_Q N_Q$ be the corresponding Langlands decomposition. Then M is the connected component of the identity in M_Q, and $\mathfrak{b}, \mathfrak{n}$ are the Lie algebras of A_Q and N_Q.

Proposition 6.2. *Any element of $\mathfrak{b}$ acts on $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ as a scalar; i.e., there exists $\alpha \in \mathfrak{b}^*$ such that, for $a \in \mathfrak{b}$, $f \in \mathfrak{n}$, we have*

$$[a, f] = \langle \alpha, a \rangle f, \quad [a, \bar{f}] = -\langle \alpha, a \rangle \bar{f}. \tag{6-7}$$

Proof. The proof of Proposition 6.2, based on the classification theory of real simple Lie algebras, will be given in Section 6H. □

Let $a_0 \in \mathfrak{b}$ be such that

$$\langle \alpha, a_0 \rangle = 1. \tag{6-8}$$

Proposition 6.3. *We have*

$$[\mathfrak{n}, \bar{\mathfrak{n}}] \subset \mathfrak{z}(\mathfrak{b}), \quad [\mathfrak{n}, \mathfrak{n}] = [\bar{\mathfrak{n}}, \bar{\mathfrak{n}}] = 0. \tag{6-9}$$

Also,

$$B|_{\mathfrak{n}\times\mathfrak{n}=0}, \quad B|_{\bar{\mathfrak{n}}\times\bar{\mathfrak{n}}} = 0. \tag{6-10}$$

Proof. By (6-7), $a \in \mathfrak{b}$ acts on $[\mathfrak{n}, \bar{\mathfrak{n}}]$, $[\mathfrak{n}, \mathfrak{n}]$, and $[\bar{\mathfrak{n}}, \bar{\mathfrak{n}}]$ by multiplication by 0, $2\langle \alpha, a \rangle$, and $-2\langle \alpha, a \rangle$. Equation (6-9) follows.

If $f_1, f_2 \in \mathfrak{n}$, by (6-7) and (6-8), we have

$$B(f_1, f_2) = B([a_0, f_1], f_2) = -B(f_1, [a_0, f_2]) = -B(f_1, f_2). \tag{6-11}$$

From (6-11), we get the first equation of (6-10). We obtain the second equation of (6-10) by the same argument. □

Remark 6.4. Clearly, we have

$$[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}(\mathfrak{b})] \subset \mathfrak{z}(\mathfrak{b}). \tag{6-12}$$

Since $\mathfrak{z}(\mathfrak{b})$ preserves B and since $\mathfrak{z}^{\perp}(\mathfrak{b})$ is the orthogonal space to $\mathfrak{z}(\mathfrak{b})$ in $\mathfrak{g}$ with respect to B, we have

$$[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}^{\perp}(\mathfrak{b})] \subset \mathfrak{z}^{\perp}(\mathfrak{b}). \tag{6-13}$$

By (6-4) and (6-9), we get

$$[\mathfrak{z}^{\perp}(\mathfrak{b}), \mathfrak{z}^{\perp}(\mathfrak{b})] \subset \mathfrak{z}(\mathfrak{b}). \tag{6-14}$$

We note the similarity between (3-2) and (6-12)–(6-14). In the sequel, We call such a pair $(\mathfrak{z}, \mathfrak{z}(\mathfrak{b}))$ a symmetric pair.

For $k \in K_M$, let $M(k) \subset M$ be the centralizer of k in M, and let $\mathfrak{m}(k)$ be its Lie algebra. Let $M^0(k)$ be the connected component of the identity in $M(k)$. Let $\mathfrak{p}_{\mathfrak{m}}(k)$ and $\mathfrak{k}_{\mathfrak{m}}(k)$ be the analogues of $\mathfrak{p}(\gamma)$ and $\mathfrak{k}(\gamma)$ in (3-11), so that

$$\mathfrak{m}(k) = \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k). \tag{6-15}$$

Since k is elliptic in M, we know $M^0(k)$ is reductive with maximal compact subgroup $K^0_M(k) = M^0(k)\cap K$ and with Cartan decomposition (6-15). Let

$$X_M(k) = M^0(k)/K^0_M(k) \tag{6-16}$$

be the corresponding symmetric space. Note that $\delta(M^0(k)) = 0$ and $\dim X_M(k)$ is even.

Clearly, if $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b},\ a \neq 0,\ k \in T$, then

$$\mathfrak{p}(\gamma) = \mathfrak{p}_{\mathfrak{m}}(k), \quad \mathfrak{k}(\gamma) = \mathfrak{k}_{\mathfrak{m}}(k), \quad Z^{a,\perp,0}(\gamma) = M^0(k), \quad K^0(\gamma) = K^0_M(k). \tag{6-17}$$

Proposition 6.5. *For $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b},\ a \neq 0,\ k \in T$, we have*

$$\begin{aligned}\left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac12} &= \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}[\mathrm{Ad}(k^{-1})] e^{(l-j)\langle\alpha,a\rangle} \\ &= \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}[\mathrm{Ad}(k^{-1})] e^{(l-j)|\alpha||a|}.\end{aligned} \tag{6-18}$$

Proof. We claim that

$$\left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac12} = e^{l\langle\alpha,a\rangle} \det(1-\mathrm{Ad}(\gamma))|_{\bar{\mathfrak{n}}}. \tag{6-19}$$

Indeed, since $\dim\bar{\mathfrak{n}}$ is even, the right-hand side of (6-19) is positive. By (6-4), we have

$$\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp} = \det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{n}} \det(1-\mathrm{Ad}(\gamma))|_{\bar{\mathfrak{n}}}. \tag{6-20}$$

Since $\bar{\mathfrak{n}} = \theta\mathfrak{n}$, we have

$$\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{n}} = \det(1-\mathrm{Ad}(\theta\gamma))|_{\bar{\mathfrak{n}}} = \det(\mathrm{Ad}(\theta\gamma))|_{\bar{\mathfrak{n}}} \det(\mathrm{Ad}(\theta\gamma)^{-1}-1)|_{\bar{\mathfrak{n}}}. \tag{6-21}$$

Since $\dim\bar{\mathfrak{n}} = 2l$ is even, and since $(\theta\gamma)^{-1} = e^a k$ and k acts unitarily on $\mathfrak{n}$, by (6-7) and (6-21), we have

$$\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{n}} = e^{2l\langle\alpha,a\rangle} \det(1-\mathrm{Ad}(e^a k))|_{\bar{\mathfrak{n}}} = e^{2l\langle\alpha,a\rangle} \det(1-\mathrm{Ad}(\gamma))|_{\bar{\mathfrak{n}}}. \tag{6-22}$$

By (6-20) and (6-22), we get (6-19).

Classically,

$$\det(1-\mathrm{Ad}(\gamma))|_{\bar{\mathfrak{n}}} = \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\bar{\mathfrak{n}})}[\mathrm{Ad}(k^{-1})] e^{-j\langle\alpha,a\rangle}. \tag{6-23}$$

Using the isomorphism of K_M-representations $\mathfrak{n}^* \simeq \bar{\mathfrak{n}}$, by (6-19), (6-23), we get the first equation of (6-18) and the second equation of (6-18) if a is positive in $\mathfrak{b}$. For the case a is negative in $\mathfrak{b}$, it is enough to remark that replacing γ by $\theta\gamma$ does not change the left-hand side of (6-18). □

6B. ***A compact Hermitian symmetric space $Y_{\mathfrak{b}}$.*** Let $\mathfrak{u}(\mathfrak{b}) \subset \mathfrak{u}$ and $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{u}$ be the compact forms of $\mathfrak{z}(\mathfrak{b})$ and $\mathfrak{m}$. Then,

$$\mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}}, \quad \mathfrak{u}_{\mathfrak{m}} = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{6-24}$$

Since $\delta(M) = 0$, we know M has compact center. By [Knapp 1986, Proposition 5.3], let U_M be the compact form of M.

Let $U(\mathfrak{b}) \subset U$, $A_0 \subset U$ be the connected subgroups of U associated with Lie algebras $\mathfrak{u}(\mathfrak{b})$, $\sqrt{-1}\mathfrak{b}$. By (6-24), A_0 is in the center of $U(\mathfrak{b})$, and

$$U(\mathfrak{b}) = A_0 U_M. \tag{6-25}$$

By [Knapp 2002, Corollary 4.51], the stabilizer of $\mathfrak{b}$ in U is a closed connected subgroup of U, and so it coincides with $U(\mathfrak{b})$.

Proposition 6.6. *The group A_0 is closed in U, and is diffeomorphic to a circle $\mathbb{S}^1$.*

Proof. The proof of Proposition 6.6, based on the classification theory of real simple Lie algebras, will be given in Section 6H. ☐

Set

$$Y_\mathfrak{b} = U/U(\mathfrak{b}). \tag{6-26}$$

We will see that $Y_\mathfrak{b}$ is a compact Hermitian symmetric space.

Recall that the bilinear form $-B$ induces an $\mathrm{Ad}(U)$-invariant metric on $\mathfrak{u}$. Let $\mathfrak{u}^\perp(\mathfrak{b})$ be the orthogonal space to $\mathfrak{u}(\mathfrak{b})$ in $\mathfrak{u}$ such that

$$\mathfrak{u} = \mathfrak{u}(\mathfrak{b}) \oplus \mathfrak{u}^\perp(\mathfrak{b}). \tag{6-27}$$

Also, we have

$$\mathfrak{u}^\perp(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}^\perp(\mathfrak{b}) \oplus \mathfrak{k}^\perp(\mathfrak{b}). \tag{6-28}$$

By (6-12)–(6-14), we have

$$[\mathfrak{u}(\mathfrak{b}), \mathfrak{u}(\mathfrak{b})] \subset \mathfrak{u}(\mathfrak{b}), \quad [\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})] \subset \mathfrak{u}^\perp(\mathfrak{b}), \quad [\mathfrak{u}^\perp(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})] \subset \mathfrak{u}(\mathfrak{b}). \tag{6-29}$$

Thus, $(\mathfrak{u}, \mathfrak{u}(\mathfrak{b}))$ is a symmetric pair.

Set

$$J = \sqrt{-1}\,\mathrm{ad}(a_0)|_{\mathfrak{u}^\perp(\mathfrak{b})} \in \mathrm{End}(\mathfrak{u}^\perp(\mathfrak{b})). \tag{6-30}$$

By (6-7)–(6-10), J is a $U(\mathfrak{b})$-invariant complex structure on $\mathfrak{u}^\perp(\mathfrak{b})$ which preserves the restriction $B|_{\mathfrak{u}^\perp(\mathfrak{b})}$. Moreover, $\mathfrak{n}_\mathbb{C} = \mathfrak{n} \otimes_\mathbb{R} \mathbb{C}$ and $\bar{\mathfrak{n}}_\mathbb{C} = \bar{\mathfrak{n}} \otimes_\mathbb{R} \mathbb{C}$ are the eigenspaces of J associated with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ such that

$$\mathfrak{u}^\perp(\mathfrak{b}) \otimes_\mathbb{R} \mathbb{C} = \mathfrak{n}_\mathbb{C} \oplus \bar{\mathfrak{n}}_\mathbb{C}. \tag{6-31}$$

The bilinear form $-B$ induces a Hermitian metric on $\mathfrak{n}_\mathbb{C}$ such that, for $f_1, f_2 \in \mathfrak{n}_\mathbb{C}$,

$$\langle f_1, f_2 \rangle_{\mathfrak{n}_\mathbb{C}} = -B(f_1, \bar{f_2}). \tag{6-32}$$

Since J commutes with the action of $U(\mathfrak{b})$, we know $U(\mathfrak{b})$ preserves the splitting (6-31). Therefore, $U(\mathfrak{b})$ acts on $\mathfrak{n}_\mathbb{C}$ and $\bar{\mathfrak{n}}_\mathbb{C}$. In particular, $U(\mathfrak{b})$ acts on $\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*)$. If $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ is the spinor of $(\mathfrak{u}^\perp(\mathfrak{b}), -B)$, by [Hitchin 1974], we have the isomorphism of representations of $U(\mathfrak{b})$,

$$\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*) \simeq S^{\mathfrak{u}^\perp(\mathfrak{b})} \otimes \det(\mathfrak{n}_\mathbb{C})^{\frac{1}{2}}. \tag{6-33}$$

Note that M has compact center Z_M. By [Knapp 1986, Proposition 5.5], M is a product of a connected semisimple Lie group and the connected component of the identity in Z_M. Since both of these two groups act trivially on $\det(\mathfrak{n})$, the same is true for M. Since the action of U_M on $\mathfrak{n}_{\mathbb{C}}$ can be obtained by the restriction of the induced action of $M_{\mathbb{C}}$ on $\mathfrak{n}_{\mathbb{C}}$, we know U_M acts trivially on $\det(\mathfrak{n}_{\mathbb{C}})$. By (6-33), we have the isomorphism of representations of U_M,

$$\Lambda^\bullet(\bar{\mathfrak{n}}_{\mathbb{C}}^*) \simeq S^{\mathfrak{u}^\perp(\mathfrak{b})}. \tag{6-34}$$

As in Section 4A, let $\omega^{\mathfrak{u}}$ be the canonical left invariant 1-form on U with values in $\mathfrak{u}$, and let $\omega^{\mathfrak{u}(\mathfrak{b})}$ and $\omega^{\mathfrak{u}^\perp(\mathfrak{b})}$ be the $\mathfrak{u}(\mathfrak{b})$ and $\mathfrak{u}^\perp(\mathfrak{b})$ components of $\omega^{\mathfrak{u}}$, so that

$$\omega^{\mathfrak{u}} = \omega^{\mathfrak{u}(\mathfrak{b})} + \omega^{\mathfrak{u}^\perp(\mathfrak{b})}. \tag{6-35}$$

Then, $U \to Y_{\mathfrak{b}}$ is a $U(\mathfrak{b})$-principle bundle, equipped with a connection form $\omega^{\mathfrak{u}(\mathfrak{b})}$. Let $\Omega^{\mathfrak{u}(\mathfrak{b})}$ be the curvature form. As in (4-3), we have

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = -\tfrac{1}{2}[\omega^{\mathfrak{u}^\perp(\mathfrak{b})}, \omega^{\mathfrak{u}^\perp(\mathfrak{b})}]. \tag{6-36}$$

The real tangent bundle

$$TY_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{u}^\perp(\mathfrak{b}) \tag{6-37}$$

is equipped with a Euclidean metric and a Euclidean connection $\nabla^{TY_{\mathfrak{b}}}$, which coincides with the Levi-Civita connection. By (6-30), J induces an almost complex structure on $TY_{\mathfrak{b}}$. Let $T^{(1,0)}Y_{\mathfrak{b}}$ and $T^{(0,1)}Y_{\mathfrak{b}}$ be the holomorphic and antiholomorphic tangent bundles. Then

$$T^{(1,0)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{n}_{\mathbb{C}}, \quad T^{(0,1)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \bar{\mathfrak{n}}_{\mathbb{C}}. \tag{6-38}$$

By (6-9) and (6-38), J is integrable.

The form $-B(\cdot, J\cdot)$ induces a Kähler form $\omega^{Y_{\mathfrak{b}}} \in \Omega^2(Y_{\mathfrak{b}})$ on $Y_{\mathfrak{b}}$. Clearly, $\omega^{Y_{\mathfrak{b}}}$ is closed, and therefore $(Y_{\mathfrak{b}}, \omega^{Y_{\mathfrak{b}}})$ is a Kähler manifold. Let $f_1, \dots, f_{2l} \in \mathfrak{n}$ be such that

$$-B(f_i, \bar{f}_j) = \delta_{ij}. \tag{6-39}$$

Then $f_1, \dots, f_{2l}$ is an orthogonal basis of $\mathfrak{n}_{\mathbb{C}}$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{n}_{\mathbb{C}}}$. Let $f^1, \dots, f^{2l}$ be the dual base of $\mathfrak{n}_{\mathbb{C}}^*$. The Kähler form $\omega^{Y_{\mathfrak{b}}}$ on $Y_{\mathfrak{b}}$ is given by

$$\omega^{Y_{\mathfrak{b}}} = -\sum_{1 \leqslant i,j \leqslant 2l} B(f_i, J\bar{f}_j) f^i \bar{f}^j = -\sqrt{-1} \sum_{1 \leqslant i \leqslant 2l} f^i \bar{f}^i. \tag{6-40}$$

Let us give a more explicit description of $Y_{\mathfrak{b}}$, although this description will not be needed in the following sections.

Proposition 6.7. *The homogenous space $Y_{\mathfrak{b}}$ is an irreducible compact Hermitian symmetric space of type AIII or BDI.*

Proof. The proof of Proposition 6.7, based on the classification theory of real simple Lie algebras, will be given in Section 6J. □

Since $U_{\mathfrak{m}}$ acts on $\mathfrak{u}_{\mathfrak{m}}$ and A_0 acts trivially on $\mathfrak{u}_{\mathfrak{m}}$, by (6-25), we have $U(\mathfrak{b})$ acts on $\mathfrak{u}_{\mathfrak{m}}$. Put

$$N_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{u}_{\mathfrak{m}}. \tag{6-41}$$

Then, $N_{\mathfrak{b}}$ is a Euclidean vector bundle on $Y_{\mathfrak{b}}$ equipped with a metric connection $\nabla^{N_{\mathfrak{b}}}$. We equip the trivial connection $\nabla^{\sqrt{-1}\mathfrak{b}}$ on the trivial line bundle $\sqrt{-1}\mathfrak{b}$ on $Y_{\mathfrak{b}}$. Since $U(\mathfrak{b})$ preserves the first splitting in (6-24), we have

$$\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}} = U \times_{U_{\mathfrak{b}}} U(\mathfrak{b}). \tag{6-42}$$

Moreover, the induced connection is given by

$$\nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}} = \nabla^{\sqrt{-1}\mathfrak{b}} \oplus \nabla^{N_{\mathfrak{b}}}. \tag{6-43}$$

By (6-27), (6-37), and (6-42), we have

$$TY_{\mathfrak{b}} \oplus \sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}} = \mathfrak{u}, \tag{6-44}$$

where $\mathfrak{u}$ stands for the corresponding trivial bundle on $Y_{\mathfrak{b}}$.

Proposition 6.8. *The following identity of closed forms holds on* $Y_{\mathfrak{b}}$:

$$\widehat{A}(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}})\widehat{A}(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}}) = 1. \tag{6-45}$$

Proof. Proceeding as in [Bismut 2011, Proposition 7.1.1], by (6-27), (6-37), and (6-42), we have

$$\widehat{A}(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}})\widehat{A}\big(\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}}, \nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}}\big) = 1. \tag{6-46}$$

By (6-43), we have

$$\widehat{A}\big(\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}}, \nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}}\big) = \widehat{A}(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}}). \tag{6-47}$$

By (6-46) and (6-47), we get (6-45). $\square$

Recall that the curvature form $\Omega^{\mathfrak{u}(\mathfrak{b})}$ is a 2-form on $Y_{\mathfrak{b}}$ with values in $U \times_{U(\mathfrak{b})} \mathfrak{u}(\mathfrak{b})$. Recall that $a_0 \in \mathfrak{b}$ is defined in (6-8). Let $\Omega^{\mathfrak{u}_{\mathfrak{m}}}$ be the $\mathfrak{u}_{\mathfrak{m}}$-component of $\Omega^{\mathfrak{u}(\mathfrak{b})}$. By (6-8), (6-36) and (6-40), we have

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = \sqrt{-1}\frac{a_0}{|a_0|^2} \otimes \omega^{Y_{\mathfrak{b}}} + \Omega^{\mathfrak{u}_{\mathfrak{m}}}. \tag{6-48}$$

By (6-48), the curvature of $(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}})$ is given by

$$R^{N_{\mathfrak{b}}} = \mathrm{ad}(\Omega^{\mathfrak{u}(\mathfrak{b})})|_{\mathfrak{u}_{\mathfrak{m}}} = \mathrm{ad}(\Omega^{\mathfrak{u}_{\mathfrak{m}}})|_{\mathfrak{u}_{\mathfrak{m}}}. \tag{6-49}$$

Also, $B(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})})$ and $B(\Omega^{\mathfrak{u}_{\mathfrak{m}}}, \Omega^{\mathfrak{u}_{\mathfrak{m}}})$ are well defined 4-forms on $Y_{\mathfrak{b}}$. We have an analogue of [Bismut 2011, Equation (7.5.19)].

Proposition 6.9. *The following identities hold*:

$$B(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}) = 0, \quad B(\Omega^{\mathfrak{u}_{\mathfrak{m}}}, \Omega^{\mathfrak{u}_{\mathfrak{m}}}) = \frac{\omega^{Y_{\mathfrak{b}},2}}{|a_0|^2}. \tag{6-50}$$

Proof. If $e_1, \dots, e_{4l}$ is an orthogonal basis of $\mathfrak{u}^{\perp}(\mathfrak{b})$, by (6-36), we have

$$B(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}) = \frac{1}{4} \sum_{1 \leqslant i,j,i',j' \leqslant 4l} B\big([e_i, e_j], [e_{i'}, e_{j'}]\big) e^i e^j e^{i'} e^{j'}$$
$$= \frac{1}{4} \sum_{1 \leqslant i,j,i',j' \leqslant 4l} B\big([[e_i, e_j], e_{i'}], e_{j'}\big) e^i e^j e^{i'} e^{j'}. \tag{6-51}$$

Using the Jacobi identity and (6-51), we get the first equation of (6-50).

The second equation of (6-50) is a consequence of (6-48) and the first equation of (6-50). ☐

6C. *Auxiliary virtual representations of K.* Let $\mathrm{RO}(K_M)$ and $\mathrm{RO}(K)$ be the real representation rings of K_M and K. Let $\iota : K_M \to K$ be the injection. We denote by

$$\iota^* : \mathrm{RO}(K) \to \mathrm{RO}(K_M) \tag{6-52}$$

the induced morphism of rings. Since K_M and K have the same maximal torus T, we know ι^* is injective.

Proposition 6.10. *The following identity in* $\mathrm{RO}(K_M)$ *holds*:

$$\iota^* \Bigg(\sum_{i=1}^{m} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) \Bigg) = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} \sum_{j=0}^{2l} (-1)^{i+j} \Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \otimes \Lambda^j(\mathfrak{n}^*). \tag{6-53}$$

Proof. For a representation V of K_M, we use the multiplication notation introduced by Hirzebruch. Put

$$\Lambda_y(V) = \sum_i y^i \Lambda^i(V), \tag{6-54}$$

a polynomial of y with coefficients in $\mathrm{RO}(K_M)$. In particular,

$$\Lambda_{-1}(V) = \sum_i (-1)^i \Lambda^i(V), \quad \Lambda'_{-1}(V) = \sum_i (-1)^{i-1} i \Lambda^i(V). \tag{6-55}$$

Denote by $\mathbf{1}$ the trivial representation. Since $\Lambda_1(\mathbf{1}) = 0$ and $\Lambda'_{-1}(\mathbf{1}) = \mathbf{1}$, we get

$$\Lambda'_{-1}(V \oplus \mathbf{1}) = \Lambda_{-1}(V). \tag{6-56}$$

By (6-2), (6-5), and the fact that K_M acts trivially on $\mathfrak{b}$, we have the isomorphism of K_M-representations

$$\mathfrak{p} \simeq \mathbf{1} \oplus \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{n}. \tag{6-57}$$

Taking $V = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{n}$, by (6-56) and (6-57), we get (6-53). ☐

The following theorem is crucial.

Theorem 6.11. *The adjoint representation of K_M on $\mathfrak{n}$ has a unique lift in* $\mathrm{RO}(K)$.

Proof. The injectivity of ι^* implies the uniqueness. The proof of the existence of the lifting of $\mathfrak{n}$, based on the classification theorem of real simple Lie algebras, will be given in Section 6I. ☐

Corollary 6.12. *For $i, j \in \mathbb{N}$, the adjoint representations of K_M on $\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*)$ and $\Lambda^j(\mathfrak{n}^*)$ have unique lifts in* $\mathrm{RO}(K)$.

Proof. As before, it is enough to show the existence of lifts. Since the representation of K_M on $\mathfrak{n}$ lifts to K, the same is true for the $\Lambda^j(\mathfrak{n}^*)$. By (6-57), this extends to the $\Lambda^i(\mathfrak{p}_\mathfrak{m}^*)$. □

Denote by η_j the adjoint representation of M on $\Lambda^j(\mathfrak{n}^*)$. Recall that by (6-31), $U(\mathfrak{b})$ acts on $\mathfrak{n}_\mathbb{C}$. Recall also that $C^{\mathfrak{u}_\mathfrak{m},\eta_j} \in \mathrm{End}(\Lambda^j(\mathfrak{n}_\mathbb{C}^*))$, $C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})} \in \mathrm{End}(\mathfrak{u}^\perp(\mathfrak{b}))$ are defined in (3-8).

Proposition 6.13. *For $0 \leqslant j \leqslant 2l$, the operator $C^{\mathfrak{u}_\mathfrak{m},\eta_j}$ is a scalar such that*

$$C^{\mathfrak{u}_\mathfrak{m},\eta_j} = \frac{1}{8}\,\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}] + (j-l)^2|\alpha|^2. \tag{6-58}$$

Proof. Equation (6-58) was proved in [Moscovici and Stanton 1991, Lemma 2.5]. We give here a more conceptual proof.

Recall that $(\mathfrak{u}, \mathfrak{u}(\mathfrak{b}))$ is a compact symmetric pair. Let $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ be the $\mathfrak{u}^\perp(\mathfrak{b})$-spinors [Bismut 2011, Section 7.2]. Let $C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}}$ be the Casimir element of $\mathfrak{u}(\mathfrak{b})$ acting on $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ defined as in (3-8). By (7.8.6) of the same paper, $C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}}$ is a scalar such that

$$C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}} = \frac{1}{8}\,\mathrm{Tr}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}]. \tag{6-59}$$

Let $C^{\mathfrak{u}_\mathfrak{m},\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*)}$ be the Casimir element of $\mathfrak{u}_\mathfrak{m}$ acting on $\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*)$. By (3-7), (6-33) and (6-34), we have

$$C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}} = C^{\mathfrak{u}_\mathfrak{m},\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*)} - (\mathrm{Ad}(a_1)|_{\Lambda^\bullet(\bar{\mathfrak{n}}_\mathbb{C}^*)\otimes\det^{-1/2}(\mathfrak{n}_\mathbb{C})})^2. \tag{6-60}$$

By (6-7), we have

$$\mathrm{Ad}(a_1)|_{\Lambda^j(\bar{\mathfrak{n}}_\mathbb{C}^*)\otimes\det^{-1/2}(\mathfrak{n}_\mathbb{C})} = (j-l)|\alpha|. \tag{6-61}$$

By (6-59)–(6-61), we get (6-58). □

Let $\gamma = e^a k^{-1} \in G$ be such that (3-9) holds. Since $\Lambda^\bullet(\mathfrak{p}_\mathfrak{m}^*) \in RO(K)$, for $Y \in \mathfrak{k}(\gamma)$, we know $\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}_\mathfrak{m}^*)}[k^{-1}\exp(-iY)]$ is well defined. We have an analogue of (4-44).

Proposition 6.14. *If $\dim \mathfrak{b}(\gamma) \geqslant 2$, then for $Y \in \mathfrak{k}(\gamma)$, we have*

$$\mathrm{Tr_s}^{\Lambda^\bullet(\mathfrak{p}_\mathfrak{m}^*)}[k^{-1}\exp(-iY)] = 0. \tag{6-62}$$

Proof. The proof of Proposition 6.14, based on the classification theory of real simple Lie algebras, will be given in Section 6L. □

6D. ***A classification of real reductive Lie algebra $\mathfrak{g}$ with $\delta(\mathfrak{g}) = 1$.*** Recall that G is a real reductive group with compact center such that $\delta(G) = 1$.

Theorem 6.15. *We have a decomposition of Lie algebras*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{6-63}$$

where

$$\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbb{R}) \quad \text{or} \quad \mathfrak{so}(p,q), \tag{6-64}$$

with $pq > 1$ odd, and $\mathfrak{g}_2$ is real reductive with $\delta(\mathfrak{g}_2) = 0$.

Proof. Since G has compact center, by (3-6), $\mathfrak{z}_{\mathfrak{p}} = 0$. By (3-25), we have

$$\delta([\mathfrak{g}, \mathfrak{g}]) = 1. \tag{6-65}$$

As in [Bismut 2011, Remark 7.9.2], by the classification theory of real simple Lie algebras, we have

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 \oplus \mathfrak{g}_2', \tag{6-66}$$

where

$$\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbb{R}) \quad \text{or} \quad \mathfrak{so}(p, q), \tag{6-67}$$

with $pq > 1$ odd, and where $\mathfrak{g}_2'$ is semisimple with $\delta(\mathfrak{g}_2') = 0$. Take

$$\mathfrak{g}_2 = \mathfrak{z}_{\mathfrak{k}} \oplus \mathfrak{g}_2'. \tag{6-68}$$

By (3-24), (6-66)–(6-68), we get (6-63). □

6E. *The group* $\mathrm{SL}_3(\mathbb{R})$. In this subsection, we assume that $G = \mathrm{SL}_3(\mathbb{R})$, so that $K = \mathrm{SO}(3)$. We have

$$\mathfrak{p} = \left\{ \begin{pmatrix} x & a_1 & a_2 \\ a_1 & y & a_3 \\ a_2 & a_3 & -x-y \end{pmatrix} : x, y, a_1, a_2, a_3 \in \mathbb{R} \right\}, \quad \mathfrak{k} = \left\{ \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}. \tag{6-69}$$

Let

$$T = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(2) \right\} \subset K \tag{6-70}$$

be a maximal torus of K.

By (3-18), (6-69) and (6-70), we have

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -2x \end{pmatrix} : x \in \mathbb{R} \right\} \subset \mathfrak{p}. \tag{6-71}$$

By (6-71), we get

$$\mathfrak{p}_{\mathfrak{m}} = \left\{ \begin{pmatrix} x & a_1 & 0 \\ a_1 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, a_1 \in \mathbb{R} \right\}, \quad \mathfrak{p}^{\perp}(\mathfrak{b}) = \left\{ \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ a_2 & a_3 & 0 \end{pmatrix} : a_2, a_3 \in \mathbb{R} \right\}. \tag{6-72}$$

Also,

$$\mathfrak{k}_{\mathfrak{m}} = \mathfrak{t}, \quad K_M = T, \quad M = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SL}_2(\mathbb{R}) \right\}. \tag{6-73}$$

By (6-71), we can orient $\mathfrak{b}$ by $x > 0$. Thus,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} : a_2, a_3 \in \mathbb{R} \right\}. \tag{6-74}$$

By (6-71) and (6-74), since for $x \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $a_3 \in \mathbb{R}$,

$$\left[\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -2x \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}\right] = 3x \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6-75}$$

we find that $\mathfrak{b}$ acts on $\mathfrak{n}$ as a scalar.

Denote by $\mathrm{Isom}^0(\mathrm{G/K})$ the connected component of the identity of the isometric group of $X = G/K$. Since G acts isometrically on G/K, we have the morphism of groups

$$i_G : G \to \mathrm{Isom}^0(\mathrm{G/K}). \tag{6-76}$$

Proposition 6.16. *The morphism i_G is an isomorphism; i.e.,*

$$\mathrm{SL}_3(\mathbb{R}) \simeq \mathrm{Isom}^0(\mathrm{SL}_3(\mathbb{R})/\,\mathrm{SO}(3)). \tag{6-77}$$

Proof. By [Helgason 1978, Theorem V.4.1], it is enough to show that K acts on $\mathfrak{p}$ effectively. Assume that $k \in K$ acts on $\mathfrak{p}$ as the identity. Thus, k fixes the elements of $\mathfrak{b}$. As in (6-73), there is $A \in \mathrm{GL}_2(\mathbb{R})$ such that

$$k = \begin{pmatrix} A & 0 \\ 0 & \det^{-1}(A) \end{pmatrix}. \tag{6-78}$$

Since k fixes also the elements of $\mathfrak{p}^{\perp}(\mathfrak{b})$, by (6-72) and (6-78), we get $A = 1$. Therefore, $k = 1$. □

6F. ***The group $G = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd.*** In this subsection, we assume that $G = \mathrm{SO}^0(p,q)$, so that $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$, with $pq > 1$ odd.

In the sequel, if $l, l' \in \mathbb{N}^*$, let $\mathrm{Mat}_{l,l'}(\mathbb{R})$ be the space of real matrices of l rows and l' columns. If $L \subset \mathrm{Mat}_{l,l}(\mathbb{R})$ is a matrix group, we denote by σ_l the standard representation of L on $\mathbb{R}^l$. We have

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} : B \in \mathrm{Mat}_{p,q}(\mathbb{R})\right\}, \quad \mathfrak{k} = \left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{so}(p),\ D \in \mathfrak{so}(q)\right\}. \tag{6-79}$$

Let

$$T_{p-1} = \left\{\begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{(p-1)/2} \end{pmatrix} : A_1, \dots, A_{(p-1)/2} \in \mathrm{SO}(2)\right\} \subset \mathrm{SO}(p-1) \tag{6-80}$$

be a maximal torus of $\mathrm{SO}(p-1)$. Then,

$$T = \left\{\begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & B \end{pmatrix} \in K : A \in T_{p-1},\ B \in T_{q-1}\right\} \subset K \tag{6-81}$$

is a maximal torus of K.

By (3-18) and (6-81), we have

$$\begin{aligned}
\mathfrak{b} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p} : x \in \mathbb{R} \right\}, \\
\mathfrak{p}_{\mathfrak{m}} &= \left\{ \begin{pmatrix} 0 & 0 & B \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ B^t & 0 & 0 \end{pmatrix} \in \mathfrak{p} : B \in \mathrm{Mat}_{p-1,q-1}(\mathbb{R}) \right\}, \\
\mathfrak{p}^{\perp}(\mathfrak{b}) &= \left\{ \begin{pmatrix} 0 & \begin{matrix} 0 & v_1 \end{matrix} & 0 \\ \begin{matrix} 0 \\ v_1^t \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} v_2^t \\ 0 \end{matrix} \\ 0 & \begin{matrix} v_2 & 0 \end{matrix} & 0 \end{pmatrix} \in \mathfrak{p} : v_1 \in \mathbb{R}^{p-1},\ v_2 \in \mathbb{R}^{q-1} \right\},
\end{aligned} \tag{6-82}$$

where v_1, v_2 are considered as column vectors. Also,

$$\mathfrak{k}_{\mathfrak{m}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in \mathfrak{k} : A \in \mathfrak{so}(p-1),\ D \in \mathfrak{so}(q-1) \right\}. \tag{6-83}$$

By (6-82) and (6-83), we get

$$\begin{aligned}
M &= \left\{ \begin{pmatrix} A & 0 & B \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ C & 0 & D \end{pmatrix} \in G : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}^0(p-1, q-1) \right\}, \\
K_M &= \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in K : A \in \mathrm{SO}(p-1), D \in \mathrm{SO}(q-1) \right\}.
\end{aligned} \tag{6-84}$$

By (6-82), we can orient $\mathfrak{b}$ by $x > 0$. Then,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & \begin{matrix} -v_1 & v_1 \end{matrix} & 0 \\ \begin{matrix} v_1^t \\ v_1^t \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} v_2^t \\ v_2^t \end{matrix} \\ 0 & \begin{matrix} v_2 & -v_2 \end{matrix} & 0 \end{pmatrix} \in \mathfrak{g} : v_1 \in \mathbb{R}^{p-1},\ v_2 \in \mathbb{R}^{q-1} \right\}. \tag{6-85}$$

By (6-82) and (6-85), since for $x \in \mathbb{R},\ v_1 \in \mathbb{R}^{p-1},\ v_2 \in \mathbb{R}^{q-1}$,

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \begin{matrix} -v_1 & v_1 \end{matrix} & 0 \\ \begin{matrix} v_1^t \\ v_1^t \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} v_2^t \\ v_2^t \end{matrix} \\ 0 & \begin{matrix} v_2 & -v_2 \end{matrix} & 0 \end{pmatrix} \right] = x \begin{pmatrix} 0 & \begin{matrix} -v_1 & v_1 \end{matrix} & 0 \\ \begin{matrix} v_1^t \\ v_1^t \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} v_2^t \\ v_2^t \end{matrix} \\ 0 & \begin{matrix} v_2 & -v_2 \end{matrix} & 0 \end{pmatrix}, \tag{6-86}$$

we find that $\mathfrak{b}$ acts on $\mathfrak{n}$ as a scalar.

Proposition 6.17. *We have the isomorphism of Lie groups*

$$\mathrm{SO}^0(p,q) \simeq \mathrm{Isom}^0\big(\mathrm{SO}^0(p,q)/\,\mathrm{SO}(p)\times\mathrm{SO}(q)\big), \tag{6-87}$$

where $pq > 1$ is odd.

Proof. As in the proof of Proposition 6.16, it is enough to show that K acts effectively on $\mathfrak{p}$. The representation of $K \simeq \mathrm{SO}(p)\times\mathrm{SO}(q)$ on $\mathfrak{p}$ is equivalent to $\sigma_p \boxtimes \sigma_q$. Assume that $(k_1,k_2)\in\mathrm{SO}(p)\times\mathrm{SO}(q)$ acts on $\mathbb{R}^p\boxtimes\mathbb{R}^q$ as the identity. If λ is any eigenvalue of k_1 and if μ is any eigenvalue of k_2, then

$$\lambda\mu = 1. \tag{6-88}$$

By (6-88), both k_1 and k_2 are scalars. Using the fact that $\det(k_1) = \det(k_2) = 1$ and that p,q are odd, we deduce $k_1 = 1$ and $k_2 = 1$. □

6G. *The isometry group of X.* We return to the general case, where G is only assumed to have compact center and be such that $\delta(G) = 1$.

Proposition 6.18. *The symmetric space G/K is of noncompact type.*

Proof. Let Z_G^0 be the connected component of the identity in Z_G, and let $G_{ss} \subset G$ be the connected subgroup of G associated with the Lie algebra $[\mathfrak{g},\mathfrak{g}]$. By [Knapp 1986, Proposition 5.5], G_{ss} is closed in G such that

$$G = Z_G^0 G_{ss}. \tag{6-89}$$

Moreover, G_{ss} is semisimple with finite center, with maximal compact subgroup $K_{ss} = G_{ss}\cap K$. Also, the imbedding $G_{ss}\to G$ induces the diffeomorphism

$$G_{ss}/K_{ss} \simeq G/K. \tag{6-90}$$

Therefore, X is a symmetric space of noncompact type. □

Put

$$G_* = \mathrm{Isom}^0(X), \tag{6-91}$$

and let $K_* \subset G_*$ be the stabilizer of $p1 \in X$ fixed. Then G_* is a semisimple Lie group with trivial center, and with maximal compact subgroup K_*. We denote by $\mathfrak{g}_*$ and $\mathfrak{k}_*$ the Lie algebras of G_* and K_*. Let

$$\mathfrak{g}_* = \mathfrak{p}_* \oplus \mathfrak{k}_* \tag{6-92}$$

be the corresponding Cartan decomposition. Clearly,

$$G_*/K_* \simeq X. \tag{6-93}$$

The morphism $i_G : G \to G_*$ defined in (6-76) induces a morphism $i_\mathfrak{g} : \mathfrak{g}\to\mathfrak{g}_*$ of Lie algebras. By (3-4) and (6-93), $i_\mathfrak{g}$ induces an isomorphism of vector spaces

$$\mathfrak{p} \simeq \mathfrak{p}_*. \tag{6-94}$$

By the property of $\mathfrak{k}_*$ and by (6-94), we have

$$\mathfrak{k}_* = [\mathfrak{p}_*,\mathfrak{p}_*] = i_\mathfrak{g}[\mathfrak{p},\mathfrak{p}] \subset i_\mathfrak{g}\mathfrak{k}. \tag{6-95}$$

Thus $i_G, i_\mathfrak{g}$ are surjective.

Proposition 6.19. *We have*

$$G_* = G_1 \times G_2, \tag{6-96}$$

where $G_1 = \mathrm{SL}_3(\mathbb{R})$ or $G_1 = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd, and where G_2 is a semisimple Lie group with trivial center with $\delta(G_2) = 0$.

Proof. By [Kobayashi and Nomizu 1963, Theorem IV.6.2], let $X = \prod_{i=1}^{l_1} X_l$ be the de Rham decomposition of (X, g^{TX}). Then every X_i is an irreducible symmetric space of noncompact type. By Theorem VI.3.5 of the same paper, we have

$$G_* = \prod_{i=1}^{l_1} \mathrm{Isom}^0(X_i). \tag{6-97}$$

By Theorem 6.15, (6-77), (6-87) and (6-97), Proposition 6.19 follows. □

6H. *Proof of Proposition 6.2.* By (6-63) and by the definitions of $\mathfrak{b}$ and $\mathfrak{n}$, we have

$$\mathfrak{b}, \mathfrak{n} \subset \mathfrak{g}_1. \tag{6-98}$$

Proposition 6.2 follows from (6-75) and (6-86). □

6I. *Proof of Theorem 6.11.* *The case $G = \mathrm{SL}_3(\mathbb{R})$.* By (6-73) and (6-74), the representation of $K_M \simeq \mathrm{SO}(2)$ on $\mathfrak{n}$ is just σ_2. Note that $K = \mathrm{SO}(3)$. We have the identity in $\mathrm{RO}(K_M)$

$$\iota^*(\sigma_3 - \mathbf{1}) = \sigma_2, \tag{6-99}$$

which says $\mathfrak{n}$ lifts to K.

The case $G = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd. By (6-84) and (6-85), the representation of $K_M \simeq \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ on $\mathfrak{n}$ is just $\sigma_{p-1} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \sigma_{q-1}$. Note that $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$. We have the identity in $\mathrm{RO}(K_M)$

$$\iota^*\big((\sigma_p - \mathbf{1}) \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes (\sigma_q - \mathbf{1})\big) = \sigma_{p-1} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \sigma_{q-1}, \tag{6-100}$$

which says $\mathfrak{n}$ lifts to K.

The case for G_*. This is a consequence of Proposition 6.19 and (6-98)–(6-100).

The general case. Recall that $i_G : G \to G_*$ is a surjective morphism of Lie groups. Therefore, the restriction $i_K : K \to K_*$ of i_G to K is surjective. By (6-94), we have the identity in $\mathrm{RO}(K)$

$$\mathfrak{p} = i_K^*(\mathfrak{p}_*). \tag{6-101}$$

Set

$$\mathfrak{t}_* = i_{\mathfrak{g}}(\mathfrak{t}) \subset \mathfrak{k}_*. \tag{6-102}$$

Since i_K is surjective, by [Bröcker and tom Dieck 1985, Theorem IV.2.9], $\mathfrak{t}_*$ is a Cartan subalgebra of $\mathfrak{k}_*$.

Let $\mathfrak{b}_* \subset \mathfrak{p}_*$ be the analogue of $\mathfrak{b}$ defined by $\mathfrak{t}_*$. Thus,

$$\dim \mathfrak{b}_* = 1, \quad \mathfrak{b}_* = i_{\mathfrak{g}}(\mathfrak{b}). \tag{6-103}$$

We denote by $K_{*,M}$, $\mathfrak{p}_*^{\perp}(\mathfrak{b}_*)$, $\mathfrak{n}_*$ the analogues of K_M, $\mathfrak{p}^{\perp}(\mathfrak{b})$, $\mathfrak{n}$. By (6-94), $i_{\mathfrak{g}}$ induces an isomorphism of vector spaces

$$\mathfrak{p}^{\perp}(\mathfrak{b}) \simeq \mathfrak{p}_*^{\perp}(\mathfrak{b}_*). \tag{6-104}$$

Let $i_{K_M} : K_M \to K_{*,M}$ be the restriction of i_G to K_M. We have the identity in $\mathrm{RO}(K_M)$

$$\mathfrak{p}^{\perp}(\mathfrak{b}) = i_{K_M}^*(\mathfrak{p}_*^{\perp}(\mathfrak{b}_*)). \tag{6-105}$$

Let $\iota' : K_{*,M} \to K_*$ be the imbedding. Then the diagram

$$\begin{tikzcd} K_M \arrow[r, "\iota"] \arrow[d, "i_{K_M}"] & K \arrow[d, "i_K"] \\ K_{*,M} \arrow[r, "\iota'"] & K_* \end{tikzcd} \tag{6-106}$$

commutes. It was proved in the previous step that there is $E \in \mathrm{RO}(K_*)$ such that the following identity in $\mathrm{RO}(K_{*,M})$ holds:

$$\iota'^*(E) = \mathfrak{n}_*. \tag{6-107}$$

By (6-5) and (6-105)–(6-107), we have the identity in $\mathrm{RO}(K_M)$,

$$\mathfrak{n} = \mathfrak{p}^{\perp}(\mathfrak{b}) = i_{K_M}^*(\mathfrak{p}_*^{\perp}(\mathfrak{b}_*)) = i_{K_M}^*(\mathfrak{n}_*) = i_{K_M}^*\iota'^*(E) = \iota^* i_K^*(E), \tag{6-108}$$

which completes the proof of our theorem. □

6J. ***Proof of Proposition 6.7.*** If $n \in \mathbb{N}$, consider the following closed subgroups:

$$\begin{aligned} A \in \mathrm{U}(2) &\to \begin{pmatrix} A & 0 \\ 0 & \det^{-1}(A) \end{pmatrix} \in \mathrm{SU}(3), \\ (A, B) \in \mathrm{SO}(n) \times \mathrm{SO}(2) &\to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{SO}(n+2). \end{aligned} \tag{6-109}$$

We state Proposition 6.7 in a more exact way.

Proposition 6.20. *We have the isomorphism of symmetric spaces*

$$Y_{\mathfrak{b}} \simeq \mathrm{SU}(3)/\mathrm{U}(2) \quad \text{or} \quad \mathrm{SO}(p+q)/\,\mathrm{SO}(p+q-2) \times \mathrm{SO}(2), \tag{6-110}$$

with $pq > 1$ odd.

Proof. Let U_* and $U_*(\mathfrak{b}_*)$ be the analogues of U and $U(\mathfrak{b})$ when G and $\mathfrak{b}$ are replaced by G_* and $\mathfrak{b}_*$. It is enough to show that

$$Y_{\mathfrak{b}} \simeq U_*/U_*(\mathfrak{b}_*). \tag{6-111}$$

Indeed, by the explicit constructions given in Sections 6E and 6F, by Proposition 6.19, and by (6-109), (6-111), we get (6-110).

Let $Z_U \subset U$ be the center of U, and let Z_U^0 be the connected component of the identity in Z_U. Let $U_{ss} \subset U$ be the connected subgroup of U associated to the Lie algebra $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}$. By [Knapp 1986, Proposition 4.32], U_{ss} is compact, and $U = U_{ss} Z_U^0$.

Let $U_{ss}(\mathfrak{b})$ be the analogue of $U(\mathfrak{b})$ when U is replaced by U_{ss}. Then $U(\mathfrak{b}) = U_{ss}(\mathfrak{b}) Z_U^0$, and the imbedding $U_{ss} \to U$ induces an isomorphism of homogeneous spaces

$$U_{ss}/U_{ss}(\mathfrak{b}) \simeq U/U(\mathfrak{b}). \tag{6-112}$$

Let $\widetilde{U}_{ss}$ be the universal cover of U_{ss}. Since U_{ss} is semisimple, $\widetilde{U}_{ss}$ is compact. We define $\widetilde{U}_{ss}(\mathfrak{b})$ similarly. The canonical projection $\widetilde{U}_{ss} \to U_{ss}$ induces an isomorphism of homogeneous spaces

$$\widetilde{U}_{ss}/\widetilde{U}_{ss}(\mathfrak{b}) \simeq U_{ss}/U_{ss}(\mathfrak{b}). \tag{6-113}$$

Similarly, since U_* is semisimple, if $\widetilde{U}_*$ is a universal cover of U_*, and if we define $\widetilde{U}_*(\mathfrak{b})$ in the same way, we have

$$\widetilde{U}_*/\widetilde{U}_*(\mathfrak{b}) \simeq U_*/U_*(\mathfrak{b}). \tag{6-114}$$

The surjective morphism of Lie algebras $i_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}_*$ induces a surjective morphism of the compact forms $i_{\mathfrak{u}} : \mathfrak{u} \to \mathfrak{u}_*$. Since $\mathfrak{u}_*$ is semisimple, the restriction of $i_{\mathfrak{u}}$ to $[\mathfrak{u}, \mathfrak{u}]$ is still surjective. It lifts to a surjective morphism of simply connected Lie groups

$$\widetilde{U}_{ss} \to \widetilde{U}_*. \tag{6-115}$$

Since any connected, simply connected, semisimple compact Lie group can be written as a product of connected, simply connected, simple compact Lie groups, we can assume that there is a connected and simply connected semisimple compact Lie group U' such that $\widetilde{U}_{ss} = \widetilde{U}_* \times U'$, and that the morphism (6-115) is the canonical projection. Therefore,

$$\widetilde{U}_{ss}/\widetilde{U}_{ss}(\mathfrak{b}) \simeq \widetilde{U}_*/\widetilde{U}_*(\mathfrak{b}_*). \tag{6-116}$$

From (6-26), (6-112)–(6-114) and (6-116), we get (6-111). □

Remark 6.21. The Hermitian symmetric spaces on the right-hand side of (6-110) are irreducible and respectively of type AIII and type BDI in the classification of Cartan [Helgason 1978, p. 518, Table V].

6K. ***Proof of Proposition 6.6.*** We use the notation in Section 6J. By definition, $A_0 \subset U_{ss}$. Let $\widetilde{A}_0 \subset \widetilde{U}_{ss}$ and $A_{*0} \subset U_*$ be the analogues of A_0 when U is replaced by $\widetilde{U}_{ss}$ and U_*. As in the proof of Proposition 6.7, we can show that $\widetilde{A}_0$ is a finite cover of A_0 and A_{*0}.

On the other hand, by the explicit constructions given in Sections 6E, 6F, and by Proposition 6.19, A_{*0} is a circle $\mathbb{S}^1$. Therefore, both $\widetilde{A}_0, A_0$ are circles.

6L. ***Proof of Proposition 6.14.*** We use the notation in Section 4E. Let $\gamma \in G$ be such that $\dim \mathfrak{b}(\gamma) \geqslant 2$. As in (4-39), we assume that $\gamma = e^a k^{-1}$ is such that

$$\mathfrak{t}(\gamma) \subset \mathfrak{t}, \quad k \in T. \tag{6-117}$$

It is enough to show (6-62) for $Y \in \mathfrak{t}(\gamma)$.

For $Y \in \mathfrak{t}(\gamma)$, since $k^{-1}\exp(-iY) \in T$ and $T \subset K^M$, we have

$$\operatorname{Tr_s}^{\Lambda^\bullet(\mathfrak{p}_\mathfrak{m})}[k^{-1}\exp(-iY)] = \det\bigl(1 - \operatorname{Ad}(k)\exp(i\,\operatorname{ad}(Y))\bigr)\big|_{\mathfrak{p}_\mathfrak{m}}. \tag{6-118}$$

It is enough to show

$$\dim \mathfrak{b}(\gamma) \cap \mathfrak{p}_\mathfrak{m} \geqslant 1. \tag{6-119}$$

Note that $a \neq 0$, otherwise $\dim \mathfrak{b}(\gamma) = 1$. Let

$$a = a^1 + a^2 + a^3 \in \mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{p}^\perp(\mathfrak{b}). \tag{6-120}$$

Since the decomposition $\mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{p}^\perp(\mathfrak{b})$ is preserved by $\operatorname{ad}(\mathfrak{t})$ and $\operatorname{Ad}(T)$, it is also preserved by $\operatorname{ad}(\mathfrak{t}(\gamma))$ and $\operatorname{Ad}(k)$. Since $a \in \mathfrak{b}(\gamma)$, the a_i, $1 \leqslant i \leqslant 3$, all lie in $\mathfrak{b}(\gamma)$. If $a^2 \neq 0$, we get (6-119). If $a^2 = 0$ and $a^3 = 0$, we have $a \in \mathfrak{b}$. Since $a \neq 0$, we have $\mathfrak{b}(\gamma) = \mathfrak{b}$, which is impossible since $\dim b(\gamma) \geqslant 2$.

It remains to consider the case

$$a^2 = 0, \quad a^3 \neq 0. \tag{6-121}$$

We will follow the steps in the proof of Theorem 6.11.

The case $G = \mathrm{SL}_3(\mathbb{R})$. By (6-70) and (6-72), the representation of $T \simeq \mathrm{SO}(2)$ on $\mathfrak{p}^\perp(\mathfrak{b})$ is equivalent to σ_2. A nontrivial element of T never fixes a^3. Therefore,

$$k = 1. \tag{6-122}$$

Since $a \notin \mathfrak{b}$, we know a does not commute with all the elements of $\mathfrak{t}$. From (6-117), we get

$$\dim \mathfrak{t}(\gamma) < \dim \mathfrak{t} = 1. \tag{6-123}$$

Therefore,

$$\mathfrak{t}(\gamma) = 0. \tag{6-124}$$

By (4-36), (6-122) and (6-124), we see that $\mathfrak{b}(\gamma) = \mathfrak{p}$. Therefore,

$$\dim \mathfrak{b}(\gamma) \cap \mathfrak{p}_\mathfrak{m} = \dim \mathfrak{p}_\mathfrak{m}. \tag{6-125}$$

By (6-72) and (6-125), we get (6-119).

The case $G = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd. By (6-82) and (6-84), the representations of $K_M \simeq \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ on $\mathfrak{p}_\mathfrak{m}$ and $\mathfrak{p}^\perp(\mathfrak{b})$ are equivalent to $\sigma_{p-1} \boxtimes \sigma_{q-1}$ and $\sigma_{p-1} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \sigma_{q-1}$. We identify $a^3 \in \mathfrak{p}^\perp(\mathfrak{b})$ with

$$v^1 + v^2 \in \mathbb{R}^{p-1} \oplus \mathbb{R}^{q-1}. \tag{6-126}$$

Then v^1 and v^2 are fixed by $\operatorname{Ad}(k)$ and commute with $\mathfrak{t}(\gamma)$.

If $v^1 \neq 0$ and $v^2 \neq 0$, by (4-36), the nonzero element $v^1 \boxtimes v^2 \in \mathbb{R}^{p-1} \boxtimes \mathbb{R}^{q-1} \simeq \mathfrak{p}_\mathfrak{m}$ is in $\mathfrak{b}(\gamma)$. It implies (6-119).

If $v^2 = 0$, we will show that γ can be conjugated into H by an element of K, which implies $\dim \mathfrak{b}(\gamma) = 1$ and contradicts $\dim \mathfrak{b}(\gamma) \geqslant 2$. (The proof for the case $v^1 = 0$ is similar.) Without loss of generality, assume that there exist $s \in \mathbb{N}$ with $1 \leqslant s \leqslant (p-1)/2$ and nonzero complex numbers $\lambda_s, \dots, \lambda_{(p-1)/2} \in \mathbb{C}$ such that

$$v^1 = (0, \dots, 0, \lambda_s, \dots, \lambda_{(p-1)/2}) \in \mathbb{C}^{(p-1)/2} \simeq \mathbb{R}^{p-1}. \tag{6-127}$$

Then there exists $x \in \mathbb{R}$ such that

$$a = \begin{pmatrix} 0 & 0 \; v^1 & 0 \\ 0 & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & 0 \\ v^{1t} & & 0 \\ 0 & 0 \;\; 0 & 0 \end{pmatrix} \in \mathfrak{p}. \tag{6-128}$$

By (6-81) and (6-117), there exist $A \in T_{p-1}$ and $D \in T_{q-1}$ such that

$$k = \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in T. \tag{6-129}$$

If we identify $T_{p-1} \simeq U(1)^{(p-1)/2}$, there are $\theta_1, \dots, \theta_{(p-1)/2} \in \mathbb{R}$ such that

$$A = (e^{2\sqrt{-1}\pi\theta_1}, \dots, e^{2\sqrt{-1}\pi\theta_{(p-1)/2}}). \tag{6-130}$$

Since k fixes a, by (6-127)–(6-130), for $i = s, \dots, (p-1)/2$, we have

$$e^{2\sqrt{-1}\pi\theta_i} = 1. \tag{6-131}$$

If $W \in \mathfrak{so}(p-2s+2)$, set

$$l(W) = \begin{array}{c} \overbrace{}^{\text{p col.}} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \in \mathfrak{k}. \tag{6-132}$$

By (6-129)–(6-132), we have

$$kl(W) = l(W)k. \tag{6-133}$$

Put $w = (\lambda_s, \dots, \lambda_{(p-1)/2}, x) \in \mathbb{C}^{(p-2s+1)/2} \oplus \mathbb{R} \simeq \mathbb{R}^{p-2s+2}$. There exists $W \in \mathfrak{so}(p-2s+2)$ such that

$$\exp(W)w = (0, \dots, 0, |w|), \tag{6-134}$$

where $|w|$ is the Euclidean norm of w.

Put

$$k' = \exp(l(W)) \in K. \tag{6-135}$$

By (6-82), (6-133) and (6-134), we have

$$\mathrm{Ad}(k')a \in \mathfrak{b}, \quad k'kk'^{-1} = k. \tag{6-136}$$

Thus, γ is conjugated by k' into H.

The general case. By (6-63), $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbb{R})$ or $\mathfrak{g}_1 = \mathfrak{so}(p,q)$ with $pq > 1$ odd. By (6-98) and (6-121), we have $a \in \mathfrak{g}_1$. The arguments in (6-122)–(6-126) extend directly. We only need to take care of the case $\mathfrak{g}_1 = \mathfrak{so}(p,q)$ and $a^2 = 0$, $v^1 \neq 0$ and $v^2 = 0$. In this case, the arguments in (6-128)–(6-134) extend to the group of isometries G_*. In particular, there is $W_* \in \mathfrak{k}_*$ such that

$$\mathrm{Ad}(\exp(W_*))i_{\mathfrak{g}}(a) \in \mathfrak{b}_*, \quad \mathrm{Ad}(i_G(k))W_* = W_*. \tag{6-137}$$

By (6-94), $\ker(i_{\mathfrak{g}}) \subset \mathfrak{k}$. Let $\ker(i_{\mathfrak{g}})^{\perp}$ be the orthogonal space of $\ker(i_{\mathfrak{g}})$ in $\mathfrak{k}$. Then,

$$\mathfrak{k} = \ker(i_{\mathfrak{g}}) \oplus \ker(i_{\mathfrak{g}})^{\perp}, \quad \ker(i_{\mathfrak{g}})^{\perp} \simeq \mathfrak{k}_*. \tag{6-138}$$

Take $W = (0, W_*) \in \mathfrak{k}$. Put

$$k' = \exp(W) \in K. \tag{6-139}$$

By (6-94), (6-137) and (6-139), we get (6-136). Thus, γ is conjugate by k' into H. The proof of (6-62) is completed. □

7. Selberg and Ruelle zeta functions

In this section, we assume that $\delta(G) = 1$ and that G has compact center. The purpose of this section is to establish the first part of our main result, Theorem 5.5.

In Section 7A, we introduce a class of representations η of M such that $\eta|_{K_M}$ lifts as an element of $\mathrm{RO}(K)$. In particular, η_j is in this class. Take $\hat{\eta} = \Lambda^{\bullet}(\mathfrak{p}_{\mathfrak{m}}^*) \otimes \eta \in \mathrm{RO}(K)$. Using the explicit formulas for orbital integrals of Theorem 4.7, we give an explicit geometric formula for $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)]$, whose proof is given in Section 7B.

In Section 7C, we introduce a Selberg zeta function $Z_{\eta,\rho}$ associated with η and ρ. Using the result in Section 7A, we express $Z_{\eta,\rho}$ in terms of the regularized determinant of the resolvent of $C^{\mathfrak{g},Z,\hat{\eta},\rho}$, and we prove that $Z_{\eta,\rho}$ is meromorphic and satisfies a functional equation.

Finally, in Section 7D, we show that the dynamical zeta function $R_\rho(\sigma)$ is equal to an alternating product of $Z_{\eta_j,\rho}$, from which we deduce the first part of Theorem 5.5.

7A. *An explicit formula for* $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)]$.

We introduce a class of representations of M.

Assumption 7.1. *Let η be a real finite-dimensional representation of M such that*

(1) *the restriction $\eta|_{K_M}$ on K_M can be lifted into* $\mathrm{RO}(K)$;

(2) *the action of the Lie algebra $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ on $E_\eta \otimes_{\mathbb{R}} \mathbb{C}$, induced by complexification, can be lifted to an action of Lie group U_M;*

(3) *the Casimir element $C^{\mathfrak{u}_{\mathfrak{m}}}$ of $\mathfrak{u}_{\mathfrak{m}}$ acts on $E_\eta \otimes_{\mathbb{R}} \mathbb{C}$ as the scalar $C^{\mathfrak{u}_{\mathfrak{m}},\eta} \in \mathbb{R}$.*

By Corollary 6.12, let $\hat{\eta} = \hat{\eta}^+ - \hat{\eta}^- \in \mathrm{RO}(K)$ be the virtual real finite-dimensional representation of K on $E_{\hat{\eta}} = E_{\hat{\eta}}^+ - E_{\hat{\eta}}^-$ such that the following identity in $\mathrm{RO}(K_M)$ holds:

$$E_{\hat{\eta}}|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} (-1)^i \Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \otimes E_\eta|_{K_M}. \tag{7-1}$$

By Corollary 6.12 and by Proposition 6.13, η_j satisfies Assumption 7.1, so that the following identity in $\mathrm{RO}(K)$ holds

$$\sum_{i=1}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{2l} (-1)^j E_{\hat{\eta}_i}. \tag{7-2}$$

As in Section 4A, let $\mathcal{E}_{\hat{\eta}} = G \times_K E_{\hat{\eta}}$ be the induced virtual vector bundle on X. Let $C^{\mathfrak{g},X,\hat{\eta}}$ be the corresponding Casimir element of G acting on $C^\infty(X, \mathcal{E}_{\hat{\eta}})$. We will state an explicit formula for $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)]$.

By (6-25), the complex representation of U_M on $E_\eta \otimes_{\mathbb{R}} \mathbb{C}$ extends to a complex representation of $U(\mathfrak{b})$ such that A_0 acts trivially. Set

$$F_{\mathfrak{b},\eta} = U \times_{U(\mathfrak{b})} (E_\eta \otimes_{\mathbb{R}} \mathbb{C}). \tag{7-3}$$

Then $F_{\mathfrak{b},\eta}$ is a complex vector bundle on $Y_{\mathfrak{b}}$. It is equipped with a connection $\nabla^{F_{\mathfrak{b},\eta}}$, induced by $\omega^{\mathfrak{u}(\mathfrak{b})}$, with curvature $R^{F_{\mathfrak{b},\eta}}$.

Remark 7.2. When $\eta = \eta_j$, the above action of $U(\mathfrak{b})$ on $\Lambda^j(\mathfrak{n}_{\mathbb{C}}^*)$ is different from the adjoint action of $U(\mathfrak{b})$ on $\Lambda^j(\mathfrak{n}_{\mathbb{C}}^*)$ induced by (6-31).

Recall that T is the maximal torus of both K and U_M. Put

$$c_G = (-1)^{\frac{m-1}{2}} \frac{|W(T, U_M)|}{|W(T, K)|} \frac{\mathrm{vol}(K/K_M)}{\mathrm{vol}(U_M/K_M)}. \tag{7-4}$$

Recall that $X_M = M/K_M$. By Bott's formula [1965, p. 175],

$$\chi(K/K_M) = \frac{|W(T, K)|}{|W(T, K_M)|}, \tag{7-5}$$

and by (4-5) and (7-4), we have a more geometric expression

$$c_G = (-1)^l \frac{[e(TX_M, \nabla^{TX_M})]^{\max}}{[e(T(K/K_M), \nabla^{T(K/K_M)})]^{\max}}. \tag{7-6}$$

Note that $\dim \mathfrak{u}^\perp(\mathfrak{b}) = 2 \dim \mathfrak{n} = 4l$. If $\beta \in \Lambda^\bullet(\mathfrak{u}^{\perp,*}(\mathfrak{b}))$, let $[\beta]^{\max} \in \mathbb{R}$ be such that

$$\beta - [\beta]^{\max} \frac{\omega^{Y_{\mathfrak{b}},2l}}{(2l)!} \tag{7-7}$$

is of degree smaller than $4l$.

Theorem 7.3. *For $t > 0$, we have*

$$\begin{aligned} \mathrm{Tr_s}^{[1]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] = \frac{c_G}{\sqrt{2\pi t}} \exp\left(\frac{t}{16} \mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}},\eta}\right) \\ \times \left[\exp\left(-\frac{\omega^{Y_{\mathfrak{b}},2}}{8\pi^2 |a_0|^2 t}\right) \widehat{A}(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}}) \operatorname{ch}(F_{\mathfrak{b},\eta}, \nabla^{F_{\mathfrak{b},\eta}})\right]^{\max}. \end{aligned} \tag{7-8}$$

If $\gamma = e^a k^{-1} \in H$ *with* $a \in \mathfrak{b}$, $a \neq 0$, $k \in T$, *for* $t > 0$, *we have*

$$\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] = \frac{1}{\sqrt{2\pi t}}[e(TX_M(k), \nabla^{TX_M(k)})]^{\max} \times\exp\left(-\frac{|a|^2}{2t}+\frac{t}{16}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}]-\frac{t}{2}C^{\mathfrak{u}_\mathfrak{m},\eta}\right)\frac{\mathrm{Tr}^{E_\eta}[\eta(k^{-1})]}{\left|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac{1}{2}}}. \tag{7-9}$$

If $\dim \mathfrak{b}(\gamma) \geqslant 2$, *for* $t > 0$, *we have*

$$\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] = 0. \tag{7-10}$$

Proof. The proof of (7-8) and (7-9) will be given in Section 7B. Equation (7-10) is a consequence of (4-22), (6-62) and (7-1). □

7B. ***The proof of* (7-8) *and* (7-9).** Let us recall some facts about Lie algebras. Let $\Delta(\mathfrak{t},\mathfrak{k}) \subset \mathfrak{t}^*$ be the real root system [Bröcker and tom Dieck 1985, Definition V.1.3]. We fix a set of positive roots $\Delta^+(\mathfrak{t},\mathfrak{k}) \subset \Delta(\mathfrak{t},\mathfrak{k})$. Set

$$\rho^\mathfrak{k} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})}\alpha. \tag{7-11}$$

By Kostant's strange formula [1976] or [Bismut 2011, Proposition 7.5.1], we have

$$4\pi^2|\rho^\mathfrak{k}|^2 = -\tfrac{1}{24}\mathrm{Tr}^\mathfrak{k}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{7-12}$$

Let $\pi_\mathfrak{k} : \mathfrak{t} \to \mathbb{C}$ be the polynomial function such that, for $Y \in \mathfrak{t}$,

$$\pi_\mathfrak{k}(Y) = \prod_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})} 2i\pi\langle\alpha, Y\rangle. \tag{7-13}$$

Let $\sigma_\mathfrak{k} : \mathfrak{t} \to \mathbb{C}$ be the denominator in the Weyl character formula. For $Y \in \mathfrak{t}$, we have

$$\sigma_\mathfrak{k}(Y) = \prod_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})}(e^{i\pi\langle\alpha,Y\rangle} - e^{-i\pi\langle\alpha,Y\rangle}). \tag{7-14}$$

The Weyl group $W(T,K)$ acts isometrically on $\mathfrak{t}$. For $w \in W(T,K)$, set $\epsilon_w = \det(w)|_\mathfrak{t}$. The Weyl denominator formula asserts for $Y \in \mathfrak{t}$, we have

$$\sigma_\mathfrak{k}(Y) = \sum_{w\in W(T,K)}\epsilon_w\exp(2i\pi\langle\rho^\mathfrak{k}, wY\rangle). \tag{7-15}$$

Let $\widehat{K}$ be the set of equivalence classes of complex irreducible representations of K. There is a bijection between $\widehat{K}$ and the set of dominant and analytic integral elements in $\mathfrak{t}^*$ [Bröcker and tom Dieck 1985, Section VI, (1.7)]. If $\lambda \in \mathfrak{t}^*$ is dominant and analytic integral, the character χ_λ of the corresponding complex irreducible representation is given by the Weyl character formula: for $Y \in \mathfrak{t}$,

$$\sigma_\mathfrak{k}(Y)\chi_\lambda(\exp(Y)) = \sum_{w\in W(T,K)}\epsilon_w\exp(2i\pi\langle\rho^\mathfrak{k}+\lambda, wY\rangle). \tag{7-16}$$

Let us recall the Weyl integral formula for Lie algebras. Let $dv_{K/T}$ be the Riemannian volume on K/T induced by $-B$, and let dY be the Lebesgue measure on $\mathfrak{k}$ or $\mathfrak{t}$ induced by $-B$. By [Knapp 1986, Lemma 11.4], if $f \in C_c(\mathfrak{k})$, we have

$$\int_{Y\in\mathfrak{k}} f(Y)\,dY = \frac{1}{|W(T,K)|}\int_{Y\in\mathfrak{t}} |\pi_{\mathfrak{k}}(Y)|^2\left(\int_{k\in K/T} f(\mathrm{Ad}(k)Y)\,dv_{K/T}\right)dY. \tag{7-17}$$

Clearly, the formula (7-17) extends to $L^1(\mathfrak{k})$.

Proof of (7-8). By (3-3), (4-22) and (7-17), we have

$$\begin{aligned}\mathrm{Tr_s}^{[1]}&[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)]\\ &= \frac{1}{(2\pi t)^{(m+n)/2}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\right)\\ &\qquad\times\frac{\mathrm{vol}(K/T)}{|W(T,K)|}\int_{Y\in\mathfrak{t}}|\pi_{\mathfrak{k}}(Y)|^2 J_1(Y)\,\mathrm{Tr_s}^{E_{\hat{\eta}}}[\exp(-i\hat{\eta}(Y))]\exp(-|Y|^2/(2t))\,dY.\end{aligned} \tag{7-18}$$

As $\delta(M)=0$, we have $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}$. We will use (7-17) again to write the integral on the last line of (7-18) as an integral over $\mathfrak{u}_{\mathfrak{m}}$.

By (6-5), we have the isomorphism of representations of K_M,

$$\mathfrak{p}^{\perp}(\mathfrak{b}) \simeq \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{7-19}$$

By (4-21) and (7-19), for $Y\in\mathfrak{t}$, we have

$$J_1(Y) = \frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}_{\mathfrak{m}}})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}_{\mathfrak{m}}})}. \tag{7-20}$$

By (7-1), for $Y\in\mathfrak{t}$, we have

$$\mathrm{Tr_s}^{E_{\hat{\eta}}}[\exp(-i\hat{\eta}(Y))] = \det(1-\exp(i\,\mathrm{ad}(Y)))|_{\mathfrak{p}_{\mathfrak{m}}}\mathrm{Tr}^{E_{\eta}}[\exp(-i\eta(Y))]. \tag{7-21}$$

By (7-13), (7-20) and (7-21), for $Y\in\mathfrak{t}$, we have

$$\begin{aligned}\frac{|\pi_{\mathfrak{k}}(Y)|^2}{|\pi_{\mathfrak{u}_{\mathfrak{m}}}(Y)|^2}&J_1(Y)\,\mathrm{Tr_s}^{E_{\hat{\eta}}}[\exp(-i\hat{\eta}(Y))]\\ &= (-1)^{\frac{\dim\mathfrak{p}_{\mathfrak{m}}}{2}}\det(\mathrm{ad}(Y))|_{\mathfrak{k}^{\perp}(\mathfrak{b})}\widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}})\,\mathrm{Tr}^{E_{\eta}}[\exp(-i\eta(Y))].\end{aligned} \tag{7-22}$$

Using (6-5), for $Y\in\mathfrak{t}$, we have

$$\det(\mathrm{ad}(Y))|_{\mathfrak{k}^{\perp}(\mathfrak{b})} = \det(\mathrm{ad}(Y))|_{\mathfrak{n}_{\mathbb{C}}}. \tag{7-23}$$

By the second condition of Assumption 7.1 and by (7-23), the function on the right-hand side of (7-22) extends naturally to an $\mathrm{Ad}(U_M)$-invariant function defined on $\mathfrak{u}_{\mathfrak{m}}$. By (7-4), (7-17), (7-18), (7-22) and

(7-23), we have

$$\begin{aligned}
&\mathrm{Tr_s}^{[1]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \\
&\quad = \frac{(-1)^l c_G}{(2\pi t)^{(m+n)/2}} \exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\right) \\
&\qquad \times \int_{Y\in\mathfrak{u}_{\mathfrak{m}}} \det(\mathrm{ad}(Y))|_{\mathfrak{n}_{\mathbb{C}}} \widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}})\,\mathrm{Tr}^{E_\eta}[\exp(-i\eta(Y))]\exp(-|Y|^2/(2t))\,dY. \qquad (7\text{-}24)
\end{aligned}$$

It remains to evaluate the integral on the last line of (7-24). We use the method in [Bismut 2011, Section 7.5]. For $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$|Y|^2 = -B(Y,Y). \tag{7-25}$$

By (6-32), (6-36) and (6-48), for $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$B(Y,\Omega^{\mathfrak{u}_{\mathfrak{m}}}) = -\sum_{1\leqslant i,j\leqslant 2l} B(\mathrm{ad}(Y)f_i,\bar{f}_j)f^i\wedge\bar{f}^j = \sum_{1\leqslant i,j\leqslant 2l}\langle \mathrm{ad}(Y)f_i,f_j\rangle_{\mathfrak{n}_{\mathbb{C}}} f^i\wedge\bar{f}^j. \tag{7-26}$$

By (6-40), (7-7) and (7-26), for $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$\frac{\det(\mathrm{ad}(Y))|_{\mathfrak{n}_{\mathbb{C}}}}{(2\pi t)^{2l}} = (-1)^l\left[\exp\left(\frac{1}{t}B\left(Y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right)\right)\right]^{\max}. \tag{7-27}$$

As $\dim\mathfrak{u}_{\mathfrak{m}} = \dim\mathfrak{m} = m+n-2l-1$, from (7-24) and (7-27), we get

$$\begin{aligned}
&\mathrm{Tr_s}^{[1]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \\
&\quad = \frac{c_G}{\sqrt{2\pi t}} \exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\right) \\
&\qquad \times \exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_{\mathfrak{m}}}\right)\left\{\widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}})\,\mathrm{Tr}^{E_\eta}[\exp(-i\eta(Y))]\exp\left(\frac{1}{t}B\left(Y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right)\right)\right\}^{\max}\Bigg|_{Y=0}. \qquad (7\text{-}28)
\end{aligned}$$

Using

$$B\left(Y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right) + \frac{1}{2}B(Y,Y) = \frac{1}{2}B\left(Y+\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}, Y+\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right) - \frac{1}{2}B\left(\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi},\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right), \tag{7-29}$$

by (6-50) and (7-28), we have

$$\begin{aligned}
&\mathrm{Tr_s}^{[1]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \\
&\quad = \frac{c_G}{\sqrt{2\pi t}} \exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\right) \\
&\qquad \times\left\{\exp\left(-\frac{\omega^{Y_{\mathfrak{b}},2}}{8\pi^2|a_0|^2 t}\right)\exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_{\mathfrak{m}}}\right)(\widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}})\,\mathrm{Tr}^{E_\eta}[\exp(-i\eta(Y))])\right\}^{\max}\Bigg|_{Y=-\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}}. \qquad (7\text{-}30)
\end{aligned}$$

We claim that the $\mathrm{Ad}(U_M)$-invariant function

$$Y\in\mathfrak{u}_{\mathfrak{m}} \to \widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}})\mathrm{Tr}^{E_\eta}[\exp(-i\eta(Y))] \tag{7-31}$$

is an eigenfunction of $\Delta^{\mathfrak{u}_\mathfrak{m}}$ with eigenvalue

$$-C^{\mathfrak{u}_\mathfrak{m},\eta} - \tfrac{1}{24}\operatorname{Tr}^{\mathfrak{u}_\mathfrak{m}}[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}]. \tag{7-32}$$

Indeed, if f is an $\mathrm{Ad}(U_M)$-invariant function on $\mathfrak{u}_\mathfrak{m}$, when restricted to $\mathfrak{t}$, it is well known, for example [Bismut 2011, Equation (7.5.22)], that

$$\Delta^{\mathfrak{u}_\mathfrak{m}} f = \frac{1}{\pi_{\mathfrak{u}_\mathfrak{m}}}\Delta^{\mathfrak{t}}\pi_{\mathfrak{u}_\mathfrak{m}} f. \tag{7-33}$$

Therefore, it is enough to show that the function

$$Y \in \mathfrak{t} \to \pi_{\mathfrak{u}_\mathfrak{m}}(Y)\widehat{A}^{-1}(i\,\mathrm{ad}(Y))|_{\mathfrak{u}_\mathfrak{m}}\operatorname{Tr}^{E_\eta}[\exp(-i\eta(Y))] \tag{7-34}$$

is an eigenfunction of $\Delta^{\mathfrak{t}}$ with eigenvalue (7-32). For $Y \in \mathfrak{t}$, we have

$$\widehat{A}^{-1}(i\,\mathrm{ad}(Y))|_{\mathfrak{u}_\mathfrak{m}} = \frac{\sigma_{\mathfrak{u}_\mathfrak{m}}(iY)}{\pi_{\mathfrak{u}_\mathfrak{m}}(iY)}. \tag{7-35}$$

By (7-35), for $Y \in \mathfrak{t}$, we have

$$\pi_{\mathfrak{u}_\mathfrak{m}}(Y)\widehat{A}^{-1}(i\,\mathrm{ad}(Y))|_{\mathfrak{u}_\mathfrak{m}} = i^{|\Delta^+(\mathfrak{t},\mathfrak{u}_\mathfrak{m})|}\sigma_{\mathfrak{u}_\mathfrak{m}}(-iY). \tag{7-36}$$

If $E_\eta \otimes_\mathbb{R} \mathbb{C}$ is an irreducible representation of U_M with the highest weight $\lambda \in \mathfrak{t}^*$, by the Weyl character formula (7-16), we have

$$\sigma_{\mathfrak{u}_\mathfrak{m}}(-iY)\operatorname{Tr_s}^{E_\eta}[\exp(-i\eta(Y))] = \sum_{w\in W(T,U_M)} \epsilon_w \exp(2\pi\langle \rho^{\mathfrak{u}_\mathfrak{m}} + \lambda, wY\rangle). \tag{7-37}$$

By (7-36) and (7-37), the function (7-34) is an eigenfunction of $\Delta^{\mathfrak{t}}$ with eigenvalue

$$4\pi^2|\rho^{\mathfrak{u}_\mathfrak{m}} + \lambda|^2. \tag{7-38}$$

By Assumption 7.1, the Casimir of $\mathfrak{u}_\mathfrak{m}$ acts as the scalar $C^{\mathfrak{u}_\mathfrak{m},\eta}$. Therefore,

$$-C^{\mathfrak{u}_\mathfrak{m},\eta} = 4\pi^2(|\rho^{\mathfrak{u}_\mathfrak{m}} + \lambda|^2 - |\rho^{\mathfrak{u}_\mathfrak{m}}|^2). \tag{7-39}$$

By (7-12) and (7-39), the eigenvalue (7-38) is equal to (7-32). If $E_\eta \otimes_\mathbb{R} \mathbb{C}$ is not irreducible, it is enough to decompose $E_\eta \otimes_\mathbb{R} \mathbb{C}$ as a sum of irreducible representations of U_M.

Since the function (7-34) and its derivations of any order satisfy estimations similar to (4-20), by (6-49) and (7-30), we get

$$\begin{aligned}\operatorname{Tr_s}^{[1]}&[\exp(-tC^{\mathfrak{g},X,\hat\eta}/2)]\\ &= \frac{c_G}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] - \frac{t}{48}\operatorname{Tr}^{\mathfrak{u}_\mathfrak{m}}[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}] - \frac{t}{2}C^{\mathfrak{u}_\mathfrak{m},\eta}\right)\\ &\qquad\times\left\{\exp\left(-\frac{\omega^{Y_\mathfrak{b},2}}{8\pi^2|a_0|^2 t}\right)\widehat{A}^{-1}\left(\frac{R^{N_\mathfrak{b}}}{2i\pi}\right)\operatorname{Tr}^{E_\eta}\left[\exp\left(-\frac{R^{F_{\mathfrak{b},\eta}}}{2i\pi}\right)\right]\right\}^{\max}.\end{aligned} \tag{7-40}$$

Since $\widehat{A}$ is an even function, by (2-3), we have

$$\widehat{A}\left(\frac{R^{N_{\mathfrak{b}}}}{2i\pi}\right) = \widehat{A}(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}}). \tag{7-41}$$

We claim that

$$\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] - \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{u}_{\mathfrak{m}}}[C^{\mathfrak{u}_{\mathfrak{m}},\mathfrak{u}_{\mathfrak{m}}}] = \operatorname{Tr}^{\mathfrak{u}^{\perp}(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}]. \tag{7-42}$$

Indeed, by [Bismut 2011, Proposition 2.6.1], we have

$$\begin{aligned} \operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] &= \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{u}}[C^{\mathfrak{u},\mathfrak{u}}],\\ \operatorname{Tr}^{\mathfrak{u}^{\perp}(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}] + \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{u}(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})}] &= \tfrac{1}{3}\operatorname{Tr}^{\mathfrak{u}}[C^{\mathfrak{u},\mathfrak{u}}]. \end{aligned} \tag{7-43}$$

By (6-24), it is trivial that

$$\operatorname{Tr}^{\mathfrak{u}(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})}] = \operatorname{Tr}^{\mathfrak{u}_{\mathfrak{m}}}[C^{\mathfrak{u}_{\mathfrak{m}},\mathfrak{u}_{\mathfrak{m}}}]. \tag{7-44}$$

From (7-43) and (7-44), we get (7-42).

By (2-4), (6-45) and (7-40)–(7-42), we get (7-8). □

Let $U_M(k)$ be the centralizer of k in U_M, and let $\mathfrak{u}_{\mathfrak{m}}(k)$ be its Lie algebra. Then

$$\mathfrak{u}_{\mathfrak{m}}(k) = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k). \tag{7-45}$$

Let $U_M^0(k)$ be the connected component of the identity in $U_M(k)$. Clearly, $U_M^0(k)$ is the compact form of $M^0(k)$.

Proof of (7-9). Since $\gamma \in H$, we know $\mathfrak{t} \subset \mathfrak{k}(\gamma)$ is a Cartan subalgebra of $\mathfrak{k}(\gamma)$. By (4-22), (6-17) and (7-17),

$$\begin{aligned} &\operatorname{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)]\\ &= \frac{1}{(2\pi t)^{\dim \mathfrak{z}(\gamma)/2}} \exp\left(-\frac{|a|^2}{2t} + \frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\right)\\ &\quad\times \frac{\operatorname{vol}(K_M^0(k)/T)}{|W(T,K_M^0(k))|}\int_{Y\in\mathfrak{t}} |\pi_{\mathfrak{k}_{\mathfrak{m}}(k)}(Y)|^2 J_\gamma(Y)\operatorname{Tr_s}^{E_{\hat{\eta}}}[\hat{\eta}(k^{-1})\exp(-i\hat{\eta}(Y))]\exp(-|Y|^2/(2t))\,dY. \end{aligned} \tag{7-46}$$

Since $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}(k)$, as in the proof of (7-8), we will write the integral on the last line of (7-46) as an integral over $\mathfrak{u}_{\mathfrak{m}}(k)$.

As $k \in T$ and $T \subset K_M$, by (7-1), for $Y \in \mathfrak{t}$, we have

$$\operatorname{Tr_s}^{E_{\hat{\eta}}}[\hat{\eta}(k^{-1})\exp(-i\hat{\eta}(Y))] = \det\big(1-\operatorname{Ad}(k)\exp(i\operatorname{ad}(Y))\big)\big|_{\mathfrak{p}_{\mathfrak{m}}}\operatorname{Tr}^{E_\eta}[\eta(k^{-1})\exp(-i\eta(Y))]. \tag{7-47}$$

By (4-19), (7-13) and (7-47), for $Y \in \mathfrak{t}$, we have

$$\begin{aligned} &\frac{|\pi_{\mathfrak{k}_{\mathfrak{m}}(k)}(Y)|^2}{|\pi_{\mathfrak{u}_{\mathfrak{m}}(k)}(Y)|^2} J_\gamma(Y)\operatorname{Tr_s}^{E_{\hat{\eta}}}[\hat{\eta}(k^{-1})\exp(-i\hat{\eta}(Y))]\\ &\qquad= \frac{(-1)^{\frac{\dim \mathfrak{p}_{\mathfrak{m}}(k)}{2}}}{\big|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^{\perp}}\big|^{\frac{1}{2}}}\widehat{A}^{-1}(i\operatorname{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)})\left[\frac{\det\big(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\big)\big|_{\mathfrak{z}_0^{\perp}(\gamma)}}{\det(1-\operatorname{Ad}(k^{-1}))|_{\mathfrak{z}_0^{\perp}(\gamma)}}\right]^{\frac{1}{2}}\\ &\qquad\quad\times\operatorname{Tr}^{E_\eta}[\eta(k^{-1})\exp(-i\eta(Y))]. \end{aligned} \tag{7-48}$$

Let $\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)$ be the orthogonal space to $\mathfrak{u}_{\mathfrak{m}}(k)$ in $\mathfrak{u}_{\mathfrak{m}}$. Then

$$\mathfrak{u}_{\mathfrak{m}}^{\perp}(k) = \sqrt{-1}\mathfrak{p}_0^{\perp}(\gamma) \oplus \mathfrak{k}_0^{\perp}(\gamma). \tag{7-49}$$

By (7-49), for $Y \in \mathfrak{t}$, we have

$$\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)\big|_{\mathfrak{z}_0^{\perp}(\gamma)}}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{z}_0^{\perp}(\gamma)}} = \frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)\big|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}. \tag{7-50}$$

By Assumption 7.1 and (7-50), the right-hand side of (7-48) extends naturally to an $\mathrm{Ad}(U_M^0(k))$-invariant function defined on $\mathfrak{u}_{\mathfrak{m}}(k)$. By (4-5), (7-17), (7-46) and (7-48), we have

$$\begin{aligned}
&\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \\
&= \frac{1}{\sqrt{2\pi t}} \frac{[e(TX_M(k),\nabla^{TX_M(k)})]^{\max}}{\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^{\perp}}\big|^{\frac{1}{2}}} \exp\Big(-\frac{|a|^2}{2t}+\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]\Big) \exp\Big(\frac{t}{2}\Delta^{\mathfrak{u}_{\mathfrak{m}}(k)}\Big) \\
&\quad \Bigg\{\widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)})\Bigg[\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)\big|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}\Bigg]^{\frac{1}{2}} \mathrm{Tr}^{E_\eta}[\eta(k^{-1})\exp(-i\eta(Y))]\Bigg\}\Bigg|_{Y=0}.
\end{aligned} \tag{7-51}$$

As before, we claim that the function

$$\begin{aligned}
Y \in \mathfrak{u}_{\mathfrak{m}}(k) \to \widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)})&\Bigg[\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)\big|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}{\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}\Bigg]^{\frac{1}{2}} \\
&\times \mathrm{Tr}^{E_\eta}[\eta(k^{-1})\exp(-i\eta(Y))]
\end{aligned} \tag{7-52}$$

is an eigenfunction of $\Delta^{\mathfrak{u}_{\mathfrak{m}}(k)}$ with eigenvalue (7-32). Indeed, it is enough to remark that, as in (7-37), up to a sign, if $k=\exp(\theta_1)$ for some $\theta_1 \in \mathfrak{t}$, we have

$$\begin{aligned}
\pi_{\mathfrak{u}_{\mathfrak{m}(k)}}(Y)\widehat{A}^{-1}(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}(k)}})\big[\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)\big|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}\big]^{\frac{1}{2}}& \\
= \pm i^{|\Delta^+(\mathfrak{t},\mathfrak{u}_{\mathfrak{m}})|}\sigma_{\mathfrak{u}}(-iY-\theta_1).&
\end{aligned} \tag{7-53}$$

Also, if $E_\eta \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible representation of U_M with the highest weight $\lambda \in \mathfrak{t}^*$,

$$\sigma_{\mathfrak{u}_{\mathfrak{m}}}(-iY-\theta_1)\,\mathrm{Tr_s}^{E_\eta}[\eta(k^{-1})\exp(-i\eta(Y))] = \sum_{w\in W(T,U_M)} \epsilon_w \exp\big(2\pi\langle\rho_{\mathfrak{u}_{\mathfrak{m}}}+\lambda, w(Y-i\theta_1)\rangle\big). \tag{7-54}$$

Proceeding as in the proof of (7-8), we get (7-9). □

7C. *Selberg zeta functions.* Recall that $\rho:\Gamma\to \mathrm{U}(r)$ is a unitary representation of Γ and that (F,∇^F,g^F) is the unitarily flat vector bundle on Z associated with ρ.

Definition 7.4. For $\sigma \in \mathbb{C}$, we define a formal sum

$$\Xi_{\eta,\rho}(\sigma) = -\sum_{[\gamma]\in[\Gamma]-\{1\}} \operatorname{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} \frac{\operatorname{Tr}^{E_\eta}[\eta(k^{-1})]}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac{1}{2}}} e^{-\sigma|a|} \tag{7-55}$$

and a formal Selberg zeta function

$$Z_{\eta,\rho}(\sigma) = \exp(\Xi_{\eta,\rho}(\sigma)). \tag{7-56}$$

The formal Selberg zeta function is said to be well defined if the same conditions as in Definition 5.4 hold.

Remark 7.5. When $G = \mathrm{SO}^0(p,1)$ with $p \geqslant 3$ odd, up to a shift on σ, we know $Z_{\eta,\rho}$ coincides with Selberg zeta function in [Fried 1986, Section 3].

Recall that the Casimir operator $C^{\mathfrak{g},Z,\hat{\eta},\rho}$ acting on $C^\infty(Z, \mathcal{F}_{\hat{\eta}} \otimes_{\mathbb{C}} F)$ is a formally self-adjoint second-order elliptic operator, which is bounded from below. For $\lambda \in \mathbb{C}$, set

$$m_{\eta,\rho}(\lambda) = \dim_{\mathbb{C}} \ker(C^{\mathfrak{g},Z,\hat{\eta}^+,\rho} - \lambda) - \dim_{\mathbb{C}} \ker(C^{\mathfrak{g},Z,\hat{\eta}^-,\rho} - \lambda). \tag{7-57}$$

Write

$$r_{\eta,\rho} = m_{\eta,\rho}(0). \tag{7-58}$$

As in Section 2B, for $\sigma \in \mathbb{R}$ and $\sigma \gg 1$, set

$$\det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta},\rho} + \sigma) = \frac{\det(C^{\mathfrak{g},Z,\hat{\eta}^+,\rho} + \sigma)}{\det(C^{\mathfrak{g},Z,\hat{\eta}^-,\rho} + \sigma)}. \tag{7-59}$$

Then, $\det_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta},\rho} + \sigma)$ extends meromorphically to $\sigma \in \mathbb{C}$. Its zeros and poles belong to the set $\{-\lambda : \lambda \in \operatorname{Sp}(C^{\mathfrak{g},Z,\hat{\eta},\rho})\}$. If $\lambda \in \operatorname{Sp}(C^{\mathfrak{g},Z,\hat{\eta},\rho})$, the order of the zero at $\sigma = -\lambda$ is $m_{\eta,\rho}(\lambda)$.

Set

$$\sigma_\eta = \tfrac{1}{8} \operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}] - C^{\mathfrak{u}_{\mathfrak{m}},\eta}. \tag{7-60}$$

Set

$$P_\eta(\sigma) = c_G \sum_{j=0}^{l} (-1)^j \frac{\Gamma\left(-j-\frac{1}{2}\right)}{j!(4\pi)^{2j+\frac{1}{2}}|a_0|^{2j}} \left[\omega^{Y_{\mathfrak{b}},2j} \hat{A}(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}}) \operatorname{ch}(\mathcal{F}_{\mathfrak{b},\eta}, \nabla^{\mathcal{F}_{\mathfrak{b},\eta}})\right]^{\max} \sigma^{2j+1}. \tag{7-61}$$

Then $P_\eta(\sigma)$ is an odd polynomial function of σ. As the notation indicates, σ_η and $P_\eta(\sigma)$ do not depend on Γ or ρ.

Theorem 7.6. *There is $\sigma_0 > 0$ such that*

$$\sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{|\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})|}{m_{[\gamma]}} \frac{1}{\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\right|^{\frac{1}{2}}} e^{-\sigma_0|a|} < \infty. \tag{7-62}$$

The Selberg zeta function $Z_{\eta,\rho}(\sigma)$ has a meromorphic extension to $\sigma \in \mathbb{C}$ such that the following identity of meromorphic functions on $\mathbb{C}$ holds:

$$Z_{\eta,\rho}(\sigma) = \det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta},\rho} + \sigma_\eta + \sigma^2) \exp(r \operatorname{vol}(Z) P_\eta(\sigma)). \tag{7-63}$$

The zeros and poles of $Z_{\eta,\rho}(\sigma)$ *belong to the set* $\{\pm i\sqrt{\lambda+\sigma_\eta} : \lambda \in \mathrm{Sp}(C^{\mathfrak{g},Z,\hat{\eta},\rho})\}$. *If* $\lambda \in \mathrm{Sp}(C^{\mathfrak{g},Z,\hat{\eta},\rho})$ *and* $\lambda \neq -\sigma_\eta$, *the order of zero at* $\sigma = \pm i\sqrt{\lambda+\sigma_\eta}$ *is* $m_{\eta,\rho}(\lambda)$. *The order of zero at* $\sigma = 0$ *is* $2m_{\eta,\rho}(-\sigma_\eta)$. *Also*,

$$Z_{\eta,\rho}(\sigma) = Z_{\eta,\rho}(-\sigma)\exp\bigl(2r\,\mathrm{vol}(Z)P_\eta(\sigma)\bigr). \tag{7-64}$$

Proof. Proceeding as in the proof of Theorem 5.6, by Proposition 5.1, Corollary 5.2, and Theorem 7.3, we get the first two statements of our theorem. By (7-63), the zeros and poles of $Z_{\eta,\rho}(\sigma)$ coincide with that of $\det_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta},\rho} + \sigma_\eta + \sigma^2)$, from which we deduce the third statement of our theorem. Equation (7-64) is a consequence of (7-63) and of the fact that $P_\eta(\sigma)$ is an odd polynomial. ☐

7D. *The Ruelle dynamical zeta function.* We now consider the Ruelle dynamical zeta function $R_\rho(\sigma)$.

Theorem 7.7. *The dynamical zeta function* $R_\rho(\sigma)$ *is holomorphic for* $\mathrm{Re}(\sigma) \gg 1$ *and extends meromorphically to* $\sigma \in \mathbb{C}$ *such that*

$$R_\rho(\sigma) = \prod_{j=0}^{2l} Z_{\eta_j,\rho}(\sigma + (j-l)|\alpha|)^{(-1)^{j-1}}. \tag{7-65}$$

Proof. Clearly, there is $C > 0$ such that, for all $\gamma \in \Gamma$,

$$\bigl|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\bigr|^{\frac{1}{2}} \leqslant C\exp(C|a|). \tag{7-66}$$

By (7-62) and (7-66), for $\sigma \in \mathbb{C}$ and $\mathrm{Re}(\sigma) > \sigma_0 + C$, the sum in (5-10) converges absolutely to a holomorphic function. By (5-4), (5-7), (5-10), (6-18) and (7-55), for $\sigma \in \mathbb{C}$ and $\mathrm{Re}(\sigma) > \sigma_0 + C$, we have

$$\Xi_\rho(\sigma) = \sum_{j=0}^{2l}(-1)^{j-1}\,\Xi_{\eta_j,\rho}(\sigma + (j-l)|\alpha|). \tag{7-67}$$

By taking exponentials, we get (7-65) for $\mathrm{Re}(\sigma) > \sigma_0 + C$. Since the right-hand side of (7-65) is meromorphic, $R_\rho(\sigma)$ has a meromorphic extension to $\mathbb{C}$ such that (7-65) holds. ☐

Remark that for $0 \leqslant j \leqslant 2l$, we have the isomorphism of K_M-representations of $\eta_j \simeq \eta_{2l-j}$. By (7-1), we have the isomorphism of K-representations,

$$\hat{\eta}_j \simeq \hat{\eta}_{2l-j}. \tag{7-68}$$

Note that by (6-58) and (7-60), we have

$$\sigma_{\eta_j} = -(j-l)^2|\alpha|^2. \tag{7-69}$$

By (7-63), (7-68) and (7-69), we have

$$\begin{aligned}
Z_{\eta_j,\rho}&\bigl(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\bigr)Z_{\eta_{2l-j},\rho}\bigl(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\bigr)\\
&= Z_{\eta_j,\rho}\bigl(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\bigr)Z_{\eta_j,\rho}\bigl(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\bigr)\\
&= \det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta}_j,\rho}+\sigma^2)^2 = \det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta}_j,\rho}+\sigma^2)\det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta}_{2l-j},\rho}+\sigma^2).
\end{aligned} \tag{7-70}$$

Recall that $T(\sigma)$ is defined in (2-14).

Theorem 7.8. *The following identity of meromorphic functions on $\mathbb{C}$ holds*:

$$R_\rho(\sigma) = T(\sigma^2)\exp\big((-1)^{l-1} r \operatorname{vol}(Z) P_{\eta_l}(\sigma)\big) \times \prod_{j=0}^{l-1} \left(\frac{Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|) Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)}{Z_{\eta_j,\rho}\big(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big) Z_{\eta_{2l-j},\rho}\big(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big)}\right)^{(-1)^{j-1}}. \tag{7-71}$$

Proof. By (2-14), (4-24), and (7-59), we have the identity of meromorphic functions,

$$T(\sigma) = \prod_{j=0}^{2l} \det{}_{\mathrm{gr}}(C^{\mathfrak{g},Z,\hat{\eta}_j,\rho}+\sigma)^{(-1)^{j-1}}. \tag{7-72}$$

By (7-63), (7-70), and (7-72), we have

$$T(\sigma^2) = Z_{\eta_l,\rho}(\sigma)^{(-1)^{l-1}} \exp\big((-1)^l r \operatorname{vol}(Z) P_{\eta_l}(\sigma)\big) \times \prod_{j=0}^{l-1} \Big(Z_{\eta_j,\rho}\big(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big) Z_{\eta_{2l-j},\rho}\big(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big)\Big)^{(-1)^{j-1}}. \tag{7-73}$$

By (7-65) and (7-73), we get (7-71). ☐

For $0 \leqslant j \leqslant 2l$, as in (7-58), we write $r_j = r_{\eta_j,\rho}$. By (7-68) and (7-72), we have

$$\chi'(Z,F) = 2\sum_{j=0}^{l-1}(-1)^{j-1} r_j + (-1)^{l-1} r_l. \tag{7-74}$$

Set

$$C_\rho = \prod_{j=0}^{l-1}\big(-4(l-j)^2|\alpha|^2\big)^{(-1)^{j-1} r_j}, \qquad r_\rho = 2\sum_{j=0}^{l}(-1)^{j-1} r_j. \tag{7-75}$$

Proof of (5-12). By Proposition 6.13 and Theorem 7.6, for $0 \leqslant j \leqslant l-1$, the orders of the zeros at $\sigma = 0$ of the functions $Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|)$ and $Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)$ are equal to r_j. Therefore, for $0 \leqslant j \leqslant l-1$, there are $A_j \neq 0$, $B_j \neq 0$ such that, as $\sigma \to 0$,

$$\begin{aligned} Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|) &= A_j \sigma^{r_j} + \mathcal{O}(\sigma^{r_j+1}),\\ Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|) &= B_j \sigma^{r_j} + \mathcal{O}(\sigma^{r_j+1}), \end{aligned} \tag{7-76}$$

and

$$\begin{aligned} Z_{\eta_j,\rho}\big(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big) &= A_j\left(\frac{-\sigma^2}{2(l-j)|\alpha|}\right)^{r_j} + \mathcal{O}(\sigma^{2r_j+2}),\\ Z_{\eta_{2l-j},\rho}\big(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big) &= B_j\left(\frac{\sigma^2}{2(l-j)|\alpha|}\right)^{r_j} + \mathcal{O}(\sigma^{2r_j+2}). \end{aligned} \tag{7-77}$$

By (7-76) and (7-77), as $\sigma \to 0$,

$$\frac{Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|)Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)}{Z_{\eta_j,\rho}\big(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big)Z_{\eta_{2l-j},\rho}\big(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\big)} \to (-4(l-j)^2|\alpha|^2)^{r_j}\sigma^{-2r_j}+\mathcal{O}(\sigma^{-2r_j+1}). \tag{7-78}$$

By (7-61), (7-71), (7-74), (7-75) and (7-78), we get (5-12). □

Remark 7.9. When $G=\mathrm{SO}^0(p,1)$ with $p \geqslant 3$ odd, we recover [Fried 1986, Theorem 3].

Remark 7.10. If we scale the form B with the factor $a>0$, then $R_\rho(\sigma)$ is replaced by $R_\rho(\sqrt{a}\sigma)$. By (5-12), as $\sigma \to 0$,

$$R_\rho(\sqrt{a}\sigma)=a^{\frac{r_\rho}{2}}C_\rho T(F)^2\sigma^{r_\rho}+\mathcal{O}(\sigma^{r_\rho+1}). \tag{7-79}$$

On the other hand, C_ρ should become $a^{\sum_{j=0}^{l-1}(-1)^j r_j}C_\rho$, and $T(F)$ should scale by $a^{\chi'(Z,F)/2}$. This is only possible if

$$r_\rho = 2\sum_{j=0}^{l-1}(-1)^j r_j + 2\chi'(Z,F), \tag{7-80}$$

which is just (7-74).

8. A cohomological formula for r_j

The purpose of this section is to establish (5-13) when G has compact center and is such that $\delta(G)=1$. We rely on some deep results from the representation theory of reductive Lie groups.

This section is organized as follows. In Section 8A, we recall the constructions of the infinitesimal and global characters of Harish-Chandra modules. We also recall some properties of $(\mathfrak{g},K)$-cohomology and $\mathfrak{n}$-homology of Harish-Chandra modules.

In Section 8B, we give a formula relating r_j with an alternating sum of the dimensions of Lie algebra cohomologies of certain Harish-Chandra modules, and we establish (5-13).

8A. ***Some results from representation theory.*** In this section, we do not assume that $\delta(G)=1$. We use the notation in Section 3 and the convention of real root systems introduced in Section 7B.

8A1. *Infinitesimal characters.* Let $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be the center of the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. A morphism of algebras $\chi:\mathcal{Z}(\mathfrak{g}_\mathbb{C})\to\mathbb{C}$ will be called a character of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$.

Recall that $\mathfrak{h}_1,\dots,\mathfrak{h}_{l_0}$ form all the nonconjugated θ-stable Cartan subalgebras of $\mathfrak{g}$. Let $\mathfrak{h}_{i\mathbb{C}}=\mathfrak{h}_i\otimes_\mathbb{R}\mathbb{C}$ and $\mathfrak{h}_{i\mathbb{R}}=\sqrt{-1}\mathfrak{h}_{i\mathfrak{p}}\oplus\mathfrak{h}_{i\mathfrak{k}}$ be the complexification and real form of $\mathfrak{h}_i$. For $\alpha\in\mathfrak{h}_{i\mathbb{R}}^*$, we extend α to a $\mathbb{C}$-linear form on $\mathfrak{h}_{i\mathbb{C}}$ by $\mathbb{C}$-linearity. In this way, we identify $\mathfrak{h}_{i\mathbb{R}}^*$ to a subset of $\mathfrak{h}_{i\mathbb{C}}^*$.

For $1\leqslant i\leqslant l_0$, let $S(\mathfrak{h}_{i\mathbb{C}})$ be the symmetric algebra of $\mathfrak{h}_{i\mathbb{C}}$. The algebraic Weyl group $W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})$ acts isometrically on $\mathfrak{h}_{i\mathbb{R}}$. By $\mathbb{C}$-linearity, $W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})$ acts on $\mathfrak{h}_{i\mathbb{C}}$. Therefore, $W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})$ acts on $S(\mathfrak{h}_{i\mathbb{C}})$. Let $S(\mathfrak{h}_{i\mathbb{C}})^{W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})}\subset S(\mathfrak{h}_{i\mathbb{C}})$ be the $W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})$-invariant subalgebra of $S(\mathfrak{h}_{i\mathbb{C}})$. Let

$$\gamma_i:\mathcal{Z}(\mathfrak{g}_\mathbb{C})\simeq S(\mathfrak{h}_{i\mathbb{C}})^{W(\mathfrak{h}_{i\mathbb{R}},\mathfrak{u})} \tag{8-1}$$

be the Harish-Chandra isomorphism [Knapp 2002, Section V.5]. For $\Lambda \in \mathfrak{h}^*_{i\mathbb{C}}$, we can associate to it a character χ_Λ of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ as follows: for $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$,

$$\chi_\Lambda(z) = \langle \gamma_i(z), 2\sqrt{-1}\pi\Lambda \rangle. \tag{8-2}$$

By [Knapp 2002, Theorem 5.62], every character of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ is of the form χ_Λ for some $\Lambda \in \mathfrak{h}^*_{i\mathbb{C}}$. Also, Λ is uniquely determined up to an action of $W(\mathfrak{h}_{i\mathbb{R}}, \mathfrak{u})$. Such an element $\Lambda \in \mathfrak{h}^*_{i\mathbb{C}}$ is called the Harish-Chandra parameter of the character. In particular, $\chi_\Lambda = 0$ if and only if there is $w \in W(\mathfrak{h}_{i\mathbb{R}}, \mathfrak{u})$ such that

$$w\Lambda = \rho^{\mathfrak{u}}_i, \tag{8-3}$$

where $\rho^{\mathfrak{u}}_i$ is defined as in (7-11) with respect to $(\mathfrak{h}_{i\mathbb{R}}, \mathfrak{u})$.

Definition 8.1. A complex representation of $\mathfrak{g}_\mathbb{C}$ is said to have infinitesimal character χ if $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ acts as a scalar $\chi(z) \in \mathbb{C}$.

A complex representation of $\mathfrak{g}_\mathbb{C}$ is said to have generalized infinitesimal character χ if $z - \chi(z)$ acts nilpotently for all $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$, i.e., $(z - \chi(z))^i$ acts like 0 for $i \gg 1$.

If $\lambda \in \mathfrak{h}^*_{i\mathbb{R}}$ is algebraically integral and dominant, let V_λ be the complex finite-dimensional irreducible representation of the Lie algebra $\mathfrak{g}_\mathbb{C}$ with the highest weight λ. Then V_λ possesses an infinitesimal character with Harish-Chandra parameter $\lambda + \rho^{\mathfrak{u}}_i \in \mathfrak{h}^*_{i\mathbb{R}}$.

8A2. *Harish-Chandra* $(\mathfrak{g}_\mathbb{C}, K)$*-modules and admissible representations of* G. We follow [Hecht and Schmid 1983, pp. 54–55] and [Knapp 1986, p. 207].

Definition 8.2. We will say that a complex $U(\mathfrak{g}_\mathbb{C})$-module V, equipped with an action of K, is a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module, if the following conditions hold:

(1) The space V is finitely generated as a $U(\mathfrak{g}_\mathbb{C})$-module.

(2) Every $v \in V$ lies in a finite-dimensional, $\mathfrak{k}_\mathbb{C}$-invariant subspace.

(3) The actions of $\mathfrak{g}_\mathbb{C}$ and K are compatible.

(4) Each irreducible K-module occurs only finitely many times in V.

Let V be a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module. For a character χ of $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$, let $V_\chi \subset V$ be the largest submodule of V on which $z - \chi(z)$ acts nilpotently for all $z \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$. Then V_χ is a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-submodule of V with generalized infinitesimal character χ. By [Hecht and Schmid 1983, Equation (2.4)], we can decompose V as a finite sum of Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-submodules

$$V = \bigoplus_\chi V_\chi. \tag{8-4}$$

Any Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module V has a finite composition series in the following sense: there exist finitely many Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-submodules

$$V = V_{n_1} \supset V_{n_1 - 1} \supset \cdots \supset V_0 \supset V_{-1} = 0 \tag{8-5}$$

such that each quotient V_i/V_{i-1}, for $0 \leqslant i \leqslant n_1$, is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module. Moreover, the set of all irreducible quotients and their multiplicities are the same for all the composition series.

Definition 8.3. We say that a representation π of G on a Hilbert space is admissible if the following hold:

(1) When restricted to K, $\pi|_K$ is unitary.

(2) Each $\tau \in \widehat{K}$ occurs with only finite multiplicity in $\pi|_K$.

Let π be a finitely generated admissible representation of G on the Hilbert space V_π. If $\tau \in \widehat{K}$, let $V_\pi(\tau) \subset V_\pi$ be the τ-isotopic subspace of V_π. Then $V_\pi(\tau)$ is the image of the evaluation map

$$(f, v) \in \mathrm{Hom}_K(V_\tau, V_\pi) \otimes V_\tau \to f(v) \in V_\pi. \tag{8-6}$$

Let

$$V_{\pi,K} = \bigoplus_{\tau\in\widehat{K}} V_\pi(\tau) \subset V_\pi \tag{8-7}$$

be the algebraic sum of representations of K. By [Knapp 1986, Proposition 8.5], $V_{\pi,K}$ is a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module. It is explained in [Vogan 2008, Section 4] that, by results of Casselman, Harish-Chandra, Lepowsky and Wallach, any Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module V can be constructed in this way and the corresponding V_π is called a Hilbert globalization of V. Moreover, V is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module if and only if V_π is an irreducible admissible representation of G. In this case, V or V_π has an infinitesimal character.

We note that a Hilbert globalization of a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}}, K)$-module is not unique.

8A3. *Global characters.* We recall the definition of the space of rapidly decreasing functions $\mathcal{S}(G)$ on G [Wallach 1988, Section 7.1.2].

For $z \in U(\mathfrak{g})$, we denote by z_L and z_R respectively the corresponding left and right invariant differential operators on G. For $r \geqslant 0$, $z_1 \in U(\mathfrak{g})$, $z_2 \in U(\mathfrak{g})$ and $f \in C^\infty(G)$, put

$$\|f\|_{r,z_1,z_2} = \sup_{g\in G} e^{r d_X(p1,pg)} |z_{1L} z_{2R} f(g)|. \tag{8-8}$$

Let $\mathcal{S}(G)$ be the space of all $f \in C^\infty(G)$ such that, for all $r \geqslant 0$, $z_1 \in U(\mathfrak{g})$, $z_2 \in U(\mathfrak{g})$, we have $\|f\|_{r,z_1,z_2} < \infty$. We endow $\mathcal{S}(G)$ with the topology given by the above seminorms. By [Wallach 1988, Theorem 7.1.1], $\mathcal{S}(G)$ is a Fréchet space which contains $C_c^\infty(G)$ as a dense subspace.

Let π be a finitely generated admissible representation of G on the Hilbert space V_π. By [Wallach 1988, Lemma 2.A.2.2], there exists $C > 0$ such that, for $g \in G$, we have

$$\|\pi(g)\| \leqslant C e^{C d_X(p1,pg)}, \tag{8-9}$$

where $\|\cdot\|$ is the operator norm. By (8-9), if $f \in \mathcal{S}(G)$,

$$\pi(f) = \int_G f(g)\pi(g)\, dg \tag{8-10}$$

is a bounded operator on V_π. By [Wallach 1988, Lemma 8.1.1], $\pi(f)$ is trace class. The global character Θ^G_π of π is a continuous linear functional on $\mathcal{S}(G)$ such that, for $f \in \mathcal{S}(G)$,

$$\operatorname{Tr}[\pi(f)] = \langle \Theta^G_\pi, f\rangle. \tag{8-11}$$

If V is a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module, we can define the global character Θ^G_V of V by the global character of its Hilbert globalization. We note that the global character does not depend on the choice of Hilbert globalization [Hecht and Schmid 1983, p. 56].

By Harish-Chandra's regularity theorem [Knapp 1986, Theorems 10.25], there is an L^1_{loc} and $\mathrm{Ad}(G)$-invariant function $\Theta^G_\pi(g)$ on G, whose restriction to the regular set G' is analytic, such that, for $f \in C^\infty_c(G)$, we have

$$\langle \Theta^G_\pi, f\rangle = \int_{g\in G} \Theta^G_\pi(g) f(g)\, dv_G. \tag{8-12}$$

Proposition 8.4. *If $f \in \mathcal{S}(G)$, then $\Theta^G_\pi(g) f(g) \in L^1(G)$ such that*

$$\langle \Theta^G_\pi, f\rangle = \int_{g\in G} \Theta^G_\pi(g) f(g)\, dv_G. \tag{8-13}$$

Proof. It is enough to show that there exist $C > 0$ and a seminorm $\|\cdot\|$ on $\mathcal{S}(G)$ such that

$$\int_G |\Theta^G_\pi(g) f(g)|\, dg \leqslant C\|f\|. \tag{8-14}$$

Recall that H' is defined in (3-36). By (3-33), we need to show that there exist $C > 0$ and a seminorm $\|\cdot\|$ on $\mathcal{S}(G)$ such that, for $1 \leqslant i \leqslant l_0$, we have

$$\int_{\gamma\in H'_i} |\Theta^G_\pi(\gamma)| \Big(\int_{g\in H_i\backslash G} |f(g^{-1}\gamma g)|\, dv_{H_i\backslash G}\Big) \big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|\, dv_{H_i} \leqslant C\|f\|. \tag{8-15}$$

By [Knapp 1986, Theorem 10.35], there exist $C > 0$ and $r_0 > 0$ such that, for $\gamma = e^a k^{-1} \in H'_i$ with $a \in \mathfrak{h}_{i\mathfrak{p}}$, $k \in H_i \cap K$, we have

$$|\Theta^G_\pi(\gamma)| \big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|^{\frac12} \leqslant C e^{r_0|a|}. \tag{8-16}$$

We claim that there exist $r_1 > 0$ and $C > 0$, such that, for $\gamma \in H'_i$, we have

$$\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|^{\frac12} \int_{g\in H_i\backslash G} \exp\big(-r_1 d_X(p1, g^{-1}\gamma g\cdot p1)\big)\, dv_{H_i\backslash G} \leqslant C. \tag{8-17}$$

Indeed, let $\Xi(g)$ be the Harish-Chandra Ξ-function [Varadarajan 1977, Section II.8.5]. By Section II.12.2 and Corollary 5 of the same paper, there exist $r_2 > 0$ and $C > 0$, such that, for $\gamma \in H'_i$, we have

$$\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|^{\frac12} \int_{g\in H_i\backslash G} \Xi(g^{-1}\gamma g)\big(1 + d_X(p1, g^{-1}\gamma g\cdot p1)\big)^{-r_2}\, dv_{H_i\backslash G} \leqslant C. \tag{8-18}$$

By [Knapp 1986, Proposition 7.15(c)] and by (8-18), we get (8-17).

By [Bismut 2011, Equation (3.1.10)], for $g \in G$ and $\gamma = e^a k^{-1} \in H_i$, we have

$$d_X(p1, g^{-1}\gamma g\cdot p1) \geqslant |a|. \tag{8-19}$$

Take $r = 2r_0 + r_1$, $z_1 = z_2 = 1 \in U(\mathfrak{g})$. Since $f \in \mathcal{S}(G)$, by (8-8) and (8-19), for $\gamma = e^a k^{-1} \in H_i'$, we have

$$\begin{aligned} |f(g^{-1}\gamma g)| &\leqslant \|f\|_{r,z_1,z_2} \exp\bigl(-r d_X(p1, g^{-1}\gamma g \cdot p1)\bigr) \\ &\leqslant \|f\|_{r,z_1,z_2} \exp(-2r_0|a|) \exp\bigl(-r_1 d_X(p1, g^{-1}\gamma g \cdot p1)\bigr). \end{aligned} \tag{8-20}$$

By (8-16), (8-17) and (8-20), for $\gamma \in H_i'$, we have

$$|\Theta_\pi^G(\gamma)| \biggl(\int_{g \in H_i \backslash G} |f(g^{-1}\gamma g)|\, dv_{H_i\backslash G}\biggr) \bigl|\det(1 - \mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\bigr| \leqslant C \|f\|_{r,z_1,z_2} \exp(-r_0|a|). \tag{8-21}$$

By (8-21), we get (8-15). □

Let V be a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module, and let τ be a real finite-dimensional orthogonal representation of K on the real Euclidean space E_τ. Then the invariant subspace $(V \otimes_\mathbb{R} E_\tau)^K \subset V \otimes_\mathbb{R} E_\tau$ has a finite dimension. We will describe an integral formula for $\dim_\mathbb{C}(V \otimes_\mathbb{R} E_\tau)^K$, which extends [Barbasch and Moscovici 1983, Corollary 2.2].

Recall that $p_t^{X,\tau}(g)$ is the smooth integral kernel of $\exp(-tC^{X,\tau}/2)$. By the estimation on the heat kernel or by [Barbasch and Moscovici 1983, Proposition 2.4], $p_t^{X,\tau}(g) \in \mathcal{S}(G) \otimes \mathrm{End}(E_\tau)$. Recall that dv_G is the Riemannian volume on G induced by $-B(\cdot\,, \theta\,\cdot)$.

Proposition 8.5. *Let $f \in C^\infty(G, E_\tau)^K$. Assume that there exist $C > 0$ and $r > 0$ such that*

$$|f(g)| \leqslant C \exp(r d_X(p1, pg)). \tag{8-22}$$

The integral

$$\int_{g \in G} p_t^{X,\tau}(g) f(g)\, dv_G \in E_\tau \tag{8-23}$$

is well defined so that

$$\begin{gathered} \frac{\partial}{\partial t} \int_{g \in G} p_t^{X,\tau}(g) f(g)\, dv_G = -\frac{1}{2} \int_{g \in G} C^{\mathfrak{g}} p_t^{X,\tau}(g) f(g)\, dv_G, \\ \frac{1}{\mathrm{vol}(K)} \lim_{t \to 0} \int_{g \in G} p_t^{X,\tau}(g) f(g)\, dv_G = f(1). \end{gathered} \tag{8-24}$$

Proof. By (8-22), by the property of $\mathcal{S}(G)$ and by $\frac{\partial}{\partial t} p_t^{X,\tau}(g) = -\frac{1}{2} C^{\mathfrak{g}} p_t^{X,\tau}(g)$, the left-hand side of (8-23) and the right-hand side of the first equation of (8-24) are well defined so that the first equation of (8-24) holds true.

It remains to show the second equation of (8-24). Let $\phi_1 \in C_c^\infty(G)^K$ be such that $0 \leqslant \phi_1(g) \leqslant 1$ and

$$\phi_1(g) = \begin{cases} 1, & d_X(p1, pg) \leqslant 1, \\ 0, & d_X(p1, pg) \geqslant 2. \end{cases} \tag{8-25}$$

Set $\phi_2 = 1 - \phi_1$.

Since $\phi_1 f$ has compact support, it descends to an L^2-section on X with values in $G \times_K E_\tau$. We have

$$\frac{1}{\mathrm{vol}(K)} \lim_{t \to 0} \int_{g \in G} p_t^{X,\tau}(g) \phi_1(g) f(g)\, dv_G = f(1). \tag{8-26}$$

By (4-27), there exist $c>0$ and $C>0$ such that, for $g\in G$ with $d_X(p1,pg)\geqslant 1$ and for $t\in(0,1]$, we have

$$|p_t^{X,\tau}(g)|\leqslant C\exp\Big(-c\frac{d_X^2(p1,pg)}{t}\Big)\leqslant Ce^{-\frac{c}{2t}}\exp\Big(-c\frac{d_X^2(p1,pg)}{2t}\Big). \tag{8-27}$$

By (8-22) and (8-27), there exist $c>0$ and $C>0$ such that, for $t\in(0,1]$, we have

$$\int_{g\in G}\big|p_t^{X,\tau}(g)\phi_2(g)f(g)\,dv_G\big|\leqslant Ce^{-\frac{c}{2t}}. \tag{8-28}$$

By (8-26) and (8-28), we get the second equation of (8-24). □

Proposition 8.6. *Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbb{C}},K)$-module with generalized infinitesimal character χ. For $t>0$, we have*

$$\dim_{\mathbb{C}}(V\otimes_{\mathbb{R}}E_\tau)^K=\mathrm{vol}(K)^{-1}e^{\frac{t\chi(C^{\mathfrak{g}})}{2}}\int_{g\in G}\Theta_V^G(g)\,\mathrm{Tr}[p_t^{X,\tau}(g)]\,dv_G. \tag{8-29}$$

Proof. Let V_π be a Hilbert globalization of V. Then,

$$(V\otimes_{\mathbb{R}}E_\tau)^K=(V_\pi\otimes_{\mathbb{R}}E_\tau)^K. \tag{8-30}$$

As in (8-10), set

$$\pi(p_t^{X,\tau})=\frac{1}{\mathrm{vol}(K)}\int_{g\in G}\pi(g)\otimes_{\mathbb{R}}p_t^{X,\tau}(g)\,dv_G. \tag{8-31}$$

Then, $\pi(p_t^{X,\tau})$ is a bounded operator acting on $V_\pi\otimes_{\mathbb{R}}E_\tau$.

We follow [Barbasch and Moscovici 1983, pp. 160–161]. Let $(V_\pi\otimes_{\mathbb{R}}E_\tau)^{K,\perp}$ be the orthogonal space to $(V_\pi\otimes_{\mathbb{R}}E_\tau)^K$ in $V_\pi\otimes_{\mathbb{R}}E_\tau$ such that

$$V_\pi\otimes_{\mathbb{R}}E_\tau=(V_\pi\otimes_{\mathbb{R}}E_\tau)^K\oplus(V_\pi\otimes_{\mathbb{R}}E_\tau)^{K,\perp}. \tag{8-32}$$

Let $Q_{\pi,\tau}$ be the orthogonal projection from $V_\pi\otimes_{\mathbb{R}}E_\tau$ to $(V_\pi\otimes_{\mathbb{R}}E_\tau)^K$. Then,

$$Q_{\pi,\tau}=\frac{1}{\mathrm{vol}(K)}\int_{k\in K}\pi\otimes\tau(k)\,dv_K. \tag{8-33}$$

By (4-13), (8-31) and (8-33), we get

$$Q_{\pi,\tau}\pi(p_t^{X,\tau})Q_{\pi,\tau}=\pi(p_t^{X,\tau}). \tag{8-34}$$

In particular, $\pi(p_t^{X,\tau})$ is of finite rank.

Take $u\in(V_\pi\otimes_{\mathbb{R}}E_\tau)^K$ and $v\in V_\pi$. Define $\langle u,v\rangle\in E_\tau$ to be such that, for any $w\in E_\tau$,

$$\langle\langle u,v\rangle,w\rangle=\langle u,v\otimes_{\mathbb{R}}w\rangle. \tag{8-35}$$

By (8-9), the function $g\in G\to\langle\pi(g)\otimes_{\mathbb{R}}\mathrm{id}\cdot u,v\rangle\in E_\tau$ is of class $C^\infty(G,E_\tau)^K$ such that (8-22) holds.

By (8-31), we have

$$\langle \pi(p_t^{X,\tau})u, v\rangle = \frac{1}{\mathrm{vol}(K)} \int_{g\in G} p_t^{X,\tau}(g)\big\langle \pi(g)\otimes_{\mathbb{R}} \mathrm{id}\cdot u, v\big\rangle\, dv_G. \tag{8-36}$$

By Proposition 8.5 and (8-36), we have

$$\frac{\partial}{\partial t}\langle \pi(p_t^{X,\tau})u, v\rangle = -\frac{1}{2}\big\langle \pi(C^{\mathfrak{g}})\pi(p_t^{X,\tau})u, v\big\rangle, \quad \lim_{t\to 0}\langle \pi(p_t^{X,\tau})u, v\rangle = \langle u, v\rangle. \tag{8-37}$$

Since $C^{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$ and since $\pi(C^{\mathfrak{g}})$ preserves the splitting (8-32), by (8-34) and (8-37), under the splitting (8-32), we have

$$\pi(p_t^{X,\tau}) = \begin{pmatrix} e^{-t\pi(C^{\mathfrak{g}})/2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8-38}$$

Since V has a generalized infinitesimal character χ, by (8-38), we have

$$\mathrm{Tr}[\pi(p_t^{X,\tau})] = e^{-t\chi(C^{\mathfrak{g}})/2} \dim_{\mathbb{C}}(V_\pi \otimes_{\mathbb{R}} E_\tau)^K. \tag{8-39}$$

Let $(\xi_i)_{i=1}^{\infty}$ and $(\eta_j)_{j=1}^{\dim E_\tau}$ be orthogonal bases of V_π and E_τ. Then

$$\mathrm{Tr}[\pi(p_t^{X,\tau})] = \frac{1}{\mathrm{vol}(K)} \sum_{i=1}^{\infty} \sum_{j=1}^{\dim E_\tau} \int_{g\in G} \langle p_t^{X,\tau}(g)\eta_j, \eta_j\rangle \langle \pi(g)\xi_i, \xi_i\rangle\, dv_G \tag{8-40}$$

$$= \frac{1}{\mathrm{vol}(K)} \sum_{i=1}^{\infty} \int_{g\in G} \mathrm{Tr}[p_t^{X,\tau}(g)] \langle \pi(g)\xi_i, \xi_i\rangle\, dv_G. \tag{8-41}$$

Since $\mathrm{Tr}[p_t^{X,\tau}(g)] \in \mathcal{S}(G)$, by (8-13) and (8-40), we have

$$\mathrm{Tr}[\pi(p_t^{X,\tau})] = \frac{1}{\mathrm{vol}(K)} \int_{g\in G} \mathrm{Tr}[p_t^{X,\tau}(g)]\Theta_\pi^G(g)\, dv_G. \tag{8-42}$$

From (8-30), (8-39) and (8-42), we get (8-29). □

Proposition 8.7. *For $1 \leqslant i \leqslant l_0$, the function*

$$\gamma \in H_i' \to \mathrm{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]\Theta_\pi^G(\gamma)\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big| \tag{8-43}$$

is almost everywhere well defined and integrable on H_i' so that

$$\int_{g\in G} \mathrm{Tr}[p_t^{X,\tau}(g)]\Theta_\pi^G(g)\, dv_G$$
$$= \sum_{i=1}^{l_0} \frac{\mathrm{vol}(K\cap H_i\backslash K)}{|W(H_i,G)|} \int_{\gamma\in H_i'} \mathrm{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]\Theta_\pi^G(g)\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big|\, dv_{H_i}. \tag{8-44}$$

Proof. Since $\mathrm{Tr}[p_t^{X,\tau}(g)]\Theta_\pi^G(g) \in L^1(G)$, by (3-33) and by Fubini's theorem, the function

$$\gamma \in H_i \to \left(\int_{g\in H_i\backslash G} \mathrm{Tr}^{E_\tau}[p_t^{X,\tau}(g^{-1}\gamma g)]\, dv_{H_i\backslash G}\right)\Theta_\pi^G(\gamma)\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\big| \tag{8-45}$$

is almost everywhere well defined and integrable on H_i.

Take $\gamma \in H_i'$. Since H_i is abelian, we have

$$Z^0(\gamma) = H_i^0 \subset H_i \subset Z(\gamma). \tag{8-46}$$

We have a finite covering space $H_i^0\backslash G \to H_i\backslash G$. Note that

$$[H_i : H_i^0] = [K \cap H_i : K \cap H_i^0]. \tag{8-47}$$

By (4-15), (8-46) and (8-47), if $\gamma \in H_i'$, we have

$$\begin{aligned}\int_{H_i\backslash G} \mathrm{Tr}^{E_\tau}[p_t^{X,\tau}(g^{-1}\gamma g)]\,dv_{H_i\backslash G} &= \frac{\mathrm{vol}(K^0(\gamma)\backslash K)}{[H_i : H_i^0]}\,\mathrm{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)]\\ &= \mathrm{vol}(K \cap H_i\backslash K)\,\mathrm{Tr}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\tau}/2)].\end{aligned} \tag{8-48}$$

Since $H_i - H_i'$ has zero measure, and by (8-45) and (8-48), the function (8-43) defines an L^1-function on H_i'. By (3-33) and (8-48), we get (8-44). □

8A4. *The* $(\mathfrak{g}, K)$*-cohomology.* If V is a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module, let $H^\bullet(\mathfrak{g}, K; V)$ be the $(\mathfrak{g}, K)$-cohomology of V [Borel and Wallach 2000, Section I.1.2]. The following two theorems are the essential algebraic ingredients in our proof of (5-13).

Theorem 8.8. *Let V be a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module with generalized infinitesimal character χ. Let W be a finite-dimensional $\mathfrak{g}_\mathbb{C}$-module with infinitesimal character. Let χ^{W^*} be the infinitesimal character of W^*. If $\chi \neq \chi^{W^*}$, then*

$$H^\bullet(\mathfrak{g}, K; V \otimes W) = 0. \tag{8-49}$$

Proof. If χ is the infinitesimal character of V, then (8-49) is a consequence of [Borel and Wallach 2000, Theorem I.5.3(ii)].

In general, let

$$V = V_{n_1} \supset V_{n_1-1} \supset \cdots \supset V_0 \supset V_{-1} = 0 \tag{8-50}$$

be the composition series of V. Then for $0 \leqslant i \leqslant n_1$, we have V_i/V_{i-1} is an irreducible Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module with infinitesimal character χ. Therefore, for all $0 \leqslant i \leqslant n_1$, we have

$$H^\bullet(\mathfrak{g}, K; (V_i/V_{i-1}) \otimes W) = 0. \tag{8-51}$$

We will show by induction that, for all $0 \leqslant i \leqslant n_1$,

$$H^\bullet(\mathfrak{g}, K; V_i \otimes W) = 0. \tag{8-52}$$

By (8-51), equation (8-52) holds for $i = 0$. Assume that (8-52) holds for some i with $0 \leqslant i \leqslant n_1$. Using the short exact sequence of Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-modules

$$0 \to V_i \to V_{i+1} \to V_{i+1}/V_i \to 0, \tag{8-53}$$

we get the long exact sequence of cohomologies

$$\cdots \to H^j(\mathfrak{g}, K; V_i \otimes W) \to H^j(\mathfrak{g}, K; V_{i+1} \otimes W) \to H^j(\mathfrak{g}, K; (V_{i+1}/V_i) \otimes W) \to \cdots. \tag{8-54}$$

By (8-51), (8-54) and by the induction hypotheses, (8-52) holds for $i+1$, which completes the proof of (8-52). □

We denote by $\widehat{G}_u$ the unitary dual of G, that is, the set of equivalence classes of complex irreducible unitary representations π of G on Hilbert spaces V_π. If $(\pi, V_\pi) \in \widehat{G}_u$, by [Knapp 1986, Theorem 8.1], π is irreducible admissible. Let χ_π be the corresponding infinitesimal character.

Theorem 8.9. *If* $(\pi, V_\pi) \in \widehat{G}_u$, *then*

$$\chi_\pi \neq 0 \quad \Longleftrightarrow \quad H^\bullet(\mathfrak{g}, K; V_{\pi,K}) = 0. \tag{8-55}$$

Proof. The direction "$\Longrightarrow$" of (8-55) is (8-49). The direction "$\Longleftarrow$" of (8-55) is a consequence of [Vogan and Zuckerman 1984; Vogan 1984; Salamanca-Riba 1999]. Indeed, the irreducible unitary representations with nonvanishing $(\mathfrak{g}, K)$-cohomology are classified and constructed in [Vogan and Zuckerman 1984; Vogan 1984]. By [Salamanca-Riba 1999], the irreducible unitary representations with vanishing infinitesimal character are in the class specified by Vogan and Zuckerman, which implies that their $(\mathfrak{g}, K)$-cohomology are nonvanishing. □

Remark 8.10. The condition that π is unitary is crucial in (8-55). See [Wallach 1988, Section 9.8.3] for a counterexample.

8A5. *The Hecht–Schmid character formula.* Let us recall the main result of [Hecht and Schmid 1983]. Let $Q \subset G$ be a standard parabolic subgroup of G with Lie algebra $\mathfrak{q} \subset \mathfrak{g}$. Let

$$Q = M_Q A_Q N_Q, \quad \mathfrak{q} = \mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q} \oplus \mathfrak{n}_\mathfrak{q} \tag{8-56}$$

be the corresponding Langlands decompositions [Knapp 1986, Section V.5].

Put $\Delta^+(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q})$ to be the set of all linear forms $\alpha \in \mathfrak{a}_\mathfrak{q}^*$ such that there exists a nonzero element $Y \in \mathfrak{n}_\mathfrak{q}$ such that, for all $a \in \mathfrak{a}_\mathfrak{q}$,

$$\operatorname{ad}(a)Y = \langle \alpha, a\rangle Y. \tag{8-57}$$

Set

$$\mathfrak{a}_\mathfrak{q}^- = \{a \in \mathfrak{a}_\mathfrak{q} : \langle \alpha, a\rangle < 0 \text{ for all } \alpha \in \Delta^+(\mathfrak{a}, \mathfrak{n})\}. \tag{8-58}$$

Put $(M_Q A_Q)^-$ to be the interior in $M_Q A_Q$ of the set

$$\{g \in M_Q A_Q : \det(1 - \operatorname{Ad}(ge^a))|_{\mathfrak{n}_\mathfrak{q}} \geqslant 0 \text{ for all } a \in \mathfrak{a}_\mathfrak{q}^-\}. \tag{8-59}$$

If V is a Harish-Chandra $(\mathfrak{g}_\mathbb{C}, K)$-module, let $H_\bullet(\mathfrak{n}_\mathfrak{q}, V)$ be the $\mathfrak{n}_\mathfrak{q}$-homology of V. By [Hecht and Schmid 1983, Proposition 2.24], $H_\bullet(\mathfrak{n}_\mathfrak{q}, V)$ is a Harish-Chandra $(\mathfrak{m}_{\mathfrak{q}\mathbb{C}} \oplus \mathfrak{a}_{\mathfrak{q}\mathbb{C}}, K \cap M_Q)$-module. We denote by $\Theta^{M_Q A_Q}_{H_\bullet(\mathfrak{n}_\mathfrak{q}, V)}$ the corresponding global character. Also, $M_Q A_Q$ acts on $\mathfrak{n}_\mathfrak{q}$. We denote by $\Theta^{M_Q A_Q}_{\Lambda^\bullet(\mathfrak{n}_\mathfrak{q})}$ the character of $\Lambda^\bullet(\mathfrak{n}_\mathfrak{q})$. By [Hecht and Schmid 1983, Theorem 3.6], the following identity of analytic functions on $(M_Q A_Q)^- \cap G'$ holds:

$$\Theta^G_V|_{(M_Q A_Q)^- \cap G'} = \left.\frac{\sum_{i=0}^{\dim \mathfrak{n}_\mathfrak{q}} (-1)^i \,\Theta^{M_Q A_Q}_{H_i(\mathfrak{n}_\mathfrak{q}, V)}}{\sum_{i=0}^{\dim \mathfrak{n}_\mathfrak{q}} (-1)^i \,\Theta^{M_Q A_Q}_{\Lambda^i(\mathfrak{n}_\mathfrak{q})}}\right|_{(M_Q A_Q)^- \cap G'}. \tag{8-60}$$

Take a θ-stable Cartan subalgebra $\mathfrak{h}^{\mathfrak{m}_\mathfrak{q}}$ of $\mathfrak{m}_\mathfrak{q}$. Set $\mathfrak{h}_\mathfrak{q} = \mathfrak{h}^{\mathfrak{m}_\mathfrak{q}} \oplus \mathfrak{a}_\mathfrak{q}$. Then $\mathfrak{h}_\mathfrak{q}$ is a θ-stable Cartan subalgebra of both $\mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q}$ and $\mathfrak{g}$. Put $\mathfrak{u}_\mathfrak{q}$ to be the compact form of $\mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q}$. Then $\mathfrak{h}_{\mathfrak{q}\mathbb{R}}$, the real form of $\mathfrak{h}_\mathfrak{q}$, is a Cartan subalgebra of both $\mathfrak{u}_\mathfrak{q}$ and $\mathfrak{u}$. The real root system of $\Delta(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u}_\mathfrak{q})$ is a subset of $\Delta(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u})$ consisting of the elements whose restrictions to $\mathfrak{a}_\mathfrak{q}$ vanish. The set of positive real roots $\Delta^+(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u}) \subset \Delta(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u})$ determines a set of positive real roots $\Delta^+(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u}_\mathfrak{q}) \subset \Delta(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u}_\mathfrak{q})$. Let $\rho^{\mathfrak{u}}_\mathfrak{q}$ and $\rho^{\mathfrak{u}_\mathfrak{q}}_\mathfrak{q}$ be the corresponding half sums of positive real roots.

If V possesses an infinitesimal character with Harish-Chandra parameter $\Lambda \in \mathfrak{h}^*_{\mathfrak{q}\mathbb{C}}$, by [Hecht and Schmid 1983, Corollary 3.32], $H_\bullet(\mathfrak{n}_\mathfrak{q}, V)$ can be decomposed in the sense of (8-4), where the generalized infinitesimal characters are given by

$$\chi_{w\Lambda + \rho^{\mathfrak{u}}_\mathfrak{q} - \rho^{\mathfrak{u}_\mathfrak{q}}_\mathfrak{q}} \tag{8-61}$$

for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u})$.

Also, $H_\bullet(\mathfrak{n}, V)$ is a Harish-Chandra $(\mathfrak{m}_{\mathfrak{q}\mathbb{C}}, K \cap M_Q)$-module. For $\nu \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$, let $H_\bullet(\mathfrak{n}, V)_{[\nu]}$ be the largest submodule of $H_\bullet(\mathfrak{n}, V)$ on which $z - \langle 2\sqrt{-1}\pi\nu, z\rangle$ acts nilpotently for all $z \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}$. Then,

$$H_\bullet(\mathfrak{n}, V) = \bigoplus_\nu H_\bullet(\mathfrak{n}, V)_{[\nu]}, \tag{8-62}$$

where $\nu = (w\Lambda + \rho^{\mathfrak{u}}_\mathfrak{q} - \rho^{\mathfrak{u}_\mathfrak{q}}_\mathfrak{q})|_{\mathfrak{a}_{\mathfrak{q}\mathbb{C}}}$ for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u})$. Let $\Theta^{M_Q}_{H_\bullet(\mathfrak{n},V)}$ and $\Theta^{M_Q}_{H_\bullet(\mathfrak{n},V)_{[\nu]}}$ be the corresponding global characters. We have the identities of L^1_{loc}-functions: for $m \in M_Q$ and $a \in \mathfrak{a}_\mathfrak{q}$,

$$\Theta^{M_Q A_Q}_{H_\bullet(\mathfrak{n},V)}(me^a) = \sum_\nu e^{2\sqrt{-1}\pi\langle\nu,a\rangle}\Theta^{M_Q}_{H_\bullet(\mathfrak{n},V)_{[\nu]}}(m), \quad \Theta^{M_Q}_{H_\bullet(\mathfrak{n},V)}(m) = \sum_\nu \Theta^{M_Q}_{H_\bullet(\mathfrak{n},V)_{[\nu]}}(m), \tag{8-63}$$

where $\nu = (w\Lambda + \rho^{\mathfrak{u}}_\mathfrak{q} - \rho^{\mathfrak{u}_\mathfrak{q}}_\mathfrak{q})|_{\mathfrak{a}_{\mathfrak{q}\mathbb{C}}}$ for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbb{R}}, \mathfrak{u})$.

Suppose now G has compact center and is such that $\delta(G) = 1$. Use the notation in Section 6A. Take $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{b} \oplus \mathfrak{n}$, and let $Q = M_Q A_Q N_Q \subset G$ be the corresponding parabolic subgroup. Then M is the connected component of the identity in M_Q. Since $K \cap M_Q$ has a finite number of connected components, $H_\bullet(\mathfrak{n}, V)$ is still a Harish-Chandra $(\mathfrak{m}_\mathbb{C} \oplus \mathfrak{b}_\mathbb{C}, K_M)$-module. Also, it is a Harish-Chandra $(\mathfrak{m}_\mathbb{C}, K_M)$-module. Let $\Theta^{MA_Q}_{H_\bullet(\mathfrak{n},V)}$ and $\Theta^{M}_{H_\bullet(\mathfrak{n},V)}$ be the respective global characters.

Recall that $H = \exp(\mathfrak{b})T \subset MA_Q$ is the Cartan subgroup of MA_Q.

Proposition 8.11. *We have*

$$\bigcup_{g \in MA_Q} gH'g^{-1} \subset (M_Q A_Q)^- \cap G'. \tag{8-64}$$

Proof. Put $L' = \bigcup_{g\in MA_Q} gH'g^{-1} \subset MA_Q \cap G'$. Then L' is an open subset of MA_Q. It is enough to show that L' is a subset of (8-59).

By (6-19) and (6-22), for $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}$ and $k \in T$, we have $\det(1 - \mathrm{Ad}(\gamma))|_\mathfrak{n} \geqslant 0$. Therefore, L' is a subset of (8-59). ☐

8B. ***Formulas for $r_{\eta,\rho}$ and r_j.*** Recall that $\hat{p} : \Gamma\backslash G \to Z$ is the natural projection. The group G acts unitarily on the right on $L^2(\Gamma\backslash G, \hat{p}^*F)$. By [Gel'fand et al. 1969, p. 23, Theorem], we can decompose

$L^2(\Gamma\backslash G, \hat{p}^* F)$ into a direct sum of unitary representations of G,

$$L^2(\Gamma\backslash G, \hat{p}^* F) = \bigoplus_{\pi\in\widehat{G}_u}^{\mathrm{Hil}} n_\rho(\pi) V_\pi, \tag{8-65}$$

with $n_\rho(\pi) < \infty$.

Recall that τ is a real finite-dimensional orthogonal representation of K on the real Euclidean space E_τ, and that $C^{\mathfrak{g},Z,\tau,\rho}$ is the Casimir element of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbb{C}} F)$. By (8-65), we have

$$\ker C^{\mathfrak{g},Z,\tau,\rho} = \bigoplus_{\substack{\pi\in\widehat{G}_u \\ \chi_\pi(C^{\mathfrak{g}})=0}} n_\rho(\pi)(V_{\pi,K} \otimes_{\mathbb{R}} E_\tau)^K. \tag{8-66}$$

By the properties of elliptic operators, the sum on right-hand side of (8-66) is finite.

We will give two applications of (8-66). In our first application, we take $E_\tau = \Lambda^\bullet(\mathfrak{p}^*)$.

Proposition 8.12. *We have*

$$H^\bullet(Z, F) = \bigoplus_{\substack{\pi\in\widehat{G}_u \\ \chi_\pi=0}} n_\rho(\pi) H^\bullet(\mathfrak{g}, K; V_{\pi,K}). \tag{8-67}$$

If $H^\bullet(Z, F) = 0$, *then for any* $\pi \in \widehat{G}_u$ *such that* $\chi_\pi = 0$, *we have*

$$n_\rho(\pi) = 0. \tag{8-68}$$

Proof. By Hodge theory, and by (4-24) and (8-66), we have

$$H^\bullet(Z, F) = \bigoplus_{\substack{\pi\in\widehat{G}_u \\ \chi_\pi(C^{\mathfrak{g}})=0}} n_\rho(\pi)(V_{\pi,K} \otimes_{\mathbb{R}} \Lambda^\bullet(\mathfrak{p}^*))^K. \tag{8-69}$$

By Hodge theory for Lie algebras [Borel and Wallach 2000, Proposition II.3.1], if $\chi_\pi(C^{\mathfrak{g}}) = 0$, we have

$$(V_{\pi,K} \otimes_{\mathbb{R}} \Lambda^\bullet(\mathfrak{p}^*))^K = H^\bullet(\mathfrak{g}, K; V_{\pi,K}). \tag{8-70}$$

From (8-69) and (8-70), we get

$$H^\bullet(Z, F) = \bigoplus_{\substack{\pi\in\widehat{G}_u \\ \chi_\pi(C^{\mathfrak{g}})=0}} n_\rho(\pi) H^\bullet(\mathfrak{g}, K; V_{\pi,K}). \tag{8-71}$$

By (8-49) and (8-71), we get (8-67).

By Theorem 8.9, and by (8-67), we get (8-68). □

Remark 8.13. Equation (8-67) is [Borel and Wallach 2000, Proposition VII.3.2]. When ρ is a trivial representation, (8-71) is originally due to Matsushima [1967].

In the rest of this section, G is assumed to have compact center and satisfy $\delta(G) = 1$. Recall that η is a real finite-dimensional representation of M satisfying Assumption 7.1, and that $\hat{\eta}$ is defined in (7-1). In our second application of (8-66), we take $\tau = \hat{\eta}$.

Theorem 8.14. *If* $(\pi, V_\pi) \in \widehat{G}_u$, *then*

$$\dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^+)^K - \dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^-)^K = \frac{1}{\chi(K/K_M)} \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} \sum_{j=0}^{2l} (-1)^{i+j} \dim_{\mathbb{C}} H^i\big(\mathfrak{m}, K_M; H_j(\mathfrak{n}, V_{\pi,K}) \otimes_{\mathbb{R}} E_\eta\big). \tag{8-72}$$

Proof. Let $\Lambda(\pi) \in \mathfrak{h}_{\mathbb{C}}^*$ be the Harish-Chandra parameter of the infinitesimal character of π. By (8-29), for $t > 0$, we have

$$\dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^+)^K - \dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^-)^K = \mathrm{vol}(K)^{-1} e^{t\chi_\pi(C^{\mathfrak{g}})/2} \int_{g \in G} \Theta_\pi^G(g) \operatorname{Tr_s}[p_t^{X,\hat{\eta}}(g)]\, dv_G. \tag{8-73}$$

By (7-10), by Proposition 8.7 and by $H \cap K = T$, we have

$$\int_G \Theta_\pi^G(g) \operatorname{Tr_s}[p_t^{X,\hat{\eta}}(g)]\, dv_G = \frac{\mathrm{vol}(T\backslash K)}{|W(H,G)|} \int_{\gamma \in H'} \Theta_\pi^G(\gamma) \operatorname{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}}\big|\, dv_H. \tag{8-74}$$

Since $\gamma = e^a k^{-1} \in H'$ implies $T = K_M(k) = M^0(k)$, by (7-9), (8-73) and (8-74), we have

$$\begin{aligned}\dim_{\mathbb{C}}&(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^+)^K - \dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^-)^K \\ &= \frac{1}{|W(H,G)|\,\mathrm{vol}(T)} \frac{1}{\sqrt{2\pi t}} \exp\Big(\frac{t}{16} \operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}},\eta} + \frac{t}{2}\chi_\pi(C^{\mathfrak{g}})\Big) \\ &\quad \times \int_{\gamma = e^a k^{-1} \in H'} \Theta_\pi^G(\gamma) \exp(-|a|^2/(2t)) \operatorname{Tr}^{E_\eta}[\eta(k^{-1})] \frac{\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}}\big|}{\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\big|^{\frac{1}{2}}}\, dv_H. \end{aligned} \tag{8-75}$$

By (6-19), for $\gamma = e^a k^{-1} \in H'$, we have

$$\frac{\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}}\big|}{\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{n}} \big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\big|^{\frac{1}{2}}} = e^{-l\langle\alpha,a\rangle} \big|\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{m}/\mathfrak{t}}\big|. \tag{8-76}$$

By (8-60), (8-64), (8-75) and (8-76), we have

$$\begin{aligned}\dim_{\mathbb{C}}&(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^+)^K - \dim_{\mathbb{C}}(V_{\pi,K} \otimes_{\mathbb{R}} \hat{\eta}^-)^K \\ &= \frac{1}{|W(H,G)|\,\mathrm{vol}(T)} \frac{1}{\sqrt{2\pi t}} \exp\Big(\frac{t}{16} \operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}},\eta} + \frac{t}{2}\chi_\pi(C^{\mathfrak{g}})\Big) \\ &\quad \times \sum_{j=0}^{2l} (-1)^j \int_{\gamma = e^a k^{-1} \in H'} \Theta_{H_j(\mathfrak{n},V_{\pi,K})}^{MA_Q}(\gamma) \exp(-|a|^2/(2t) - l\langle\alpha,a\rangle) \\ &\qquad \operatorname{Tr}^{E_\eta}[\eta(k^{-1})] \big|\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{m}/\mathfrak{t}}\big|\, dv_H. \end{aligned} \tag{8-77}$$

By (8-16), there exist $C > 0$ and $c > 0$ such that, for $\gamma = e^a k^{-1} \in H'$, we have

$$|\Theta_{H_j(\mathfrak{n},V_{\pi,K})}^{MA_Q}(\gamma)| \big|\det(1-\mathrm{Ad}(k^{-1}))|_{\mathfrak{m}/\mathfrak{t}}\big|^{\frac{1}{2}} \leqslant Ce^{c|a|}. \tag{8-78}$$

By (8-63), (8-77), (8-78), and by letting $t\to 0$, we get

$$\begin{aligned}\dim_{\mathbb{C}}(V_{\pi,K}\otimes_{\mathbb{R}}\hat{\eta}^{+})^{K}-\dim_{\mathbb{C}}(V_{\pi,K}\otimes_{\mathbb{R}}\hat{\eta}^{-})^{K}\\ =\frac{1}{|W(H,G)|\operatorname{vol}(T)}\sum_{j=0}^{2l}(-1)^{j}\int_{\gamma\in T'}\Theta^{M}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\operatorname{Tr}^{E_\eta}[\eta(\gamma)]\big|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{m}/\mathfrak{t}}\big|\,dv_T,\end{aligned}\tag{8-79}$$

where T' is the set of the regular elements of M in T.

We claim that, for $0\leqslant j\leqslant 2l$, we have

$$\begin{aligned}\sum_{i=0}^{\dim\mathfrak{p}_{\mathfrak{m}}}(-1)^{i}\dim_{\mathbb{C}}\big(H_j(\mathfrak{n},V_{\pi,K})\otimes_{\mathbb{R}}\Lambda^{i}(\mathfrak{p}_{\mathfrak{m}}^{*})\otimes_{\mathbb{R}}E_\eta\big)^{K_M}\\ =\frac{1}{|W(T,M)|\operatorname{vol}(T)}\int_{\gamma\in T'}\Theta^{M}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\operatorname{Tr}^{E_\eta}[\eta(\gamma)]\big|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{m}/\mathfrak{t}}\big|\,dv_T.\end{aligned}\tag{8-80}$$

Indeed, consider $H_j(\mathfrak{n},V_{\pi,K})$ as a Harish-Chandra $(\mathfrak{m}_{\mathbb{C}},K_M)$-module. We can decompose $H_j(\mathfrak{n},V_{\pi,K})$ in the sense of (8-4), where the generalized infinitesimal characters are given by

$$\chi_{(w\Lambda(\pi)+\rho^{\mathfrak{u}}-\rho^{\mathfrak{u}(\mathfrak{b})})|_{\mathfrak{t}_{\mathbb{C}}}}\tag{8-81}$$

for some $w\in W(\mathfrak{h}_{\mathbb{R}},\mathfrak{u})$. Therefore, it is enough to show (8-80) when $H_j(\mathfrak{n},V_{\pi,K})$ is replaced by any Harish-Chandra $(\mathfrak{m}_{\mathbb{C}},K_M)$-module with generalized infinitesimal character $\chi_{(w\Lambda(\pi)+\rho^{\mathfrak{u}}-\rho^{\mathfrak{u}(\mathfrak{b})})|_{\mathfrak{t}_{\mathbb{C}}}}$. Let (π^M,V_{π^M}) be a Hilbert globalization of such a Harish-Chandra $(\mathfrak{m}_{\mathbb{C}},K_M)$-module. As before, let $C^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}$ be the Casimir element of M acting on $C^\infty(M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta)^{K_M}$, and let $p_t^{X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}(g)$ be the smooth integral kernel of the heat operator $\exp(-tC^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}/2)$. Remark that by [Bismut et al. 2017, Proposition 8.4], $C^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}-C^{\mathfrak{m},E_\eta}$ is the Hodge Laplacian on X_M acting on the differential forms with values in the homogeneous flat vector bundle $M\times_{K_M}E_\eta$. Proceeding as in [Bismut 2011, Theorem 7.8.2], if $\gamma\in M$ is semisimple and nonelliptic, we have

$$\operatorname{Tr}^{[\gamma]}\big[\exp\big(-t(C^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}-C^{\mathfrak{m},E_\eta})/2\big)\big]=0.\tag{8-82}$$

Also, if $\gamma=k^{-1}\in K_M$, then

$$\operatorname{Tr}^{[\gamma]}\big[\exp\big(-t(C^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}-C^{\mathfrak{m},E_\eta})/2\big)\big]=\operatorname{Tr}^{E_\eta}[\eta(k^{-1})]e(X_M(k),\nabla^{TX_M(k)}).\tag{8-83}$$

Using (8-82), proceeding as in (8-73) and (8-74), we have

$$\begin{aligned}&\sum_{i=0}^{\dim\mathfrak{p}_{\mathfrak{m}}}(-1)^{i}\dim_{\mathbb{C}}\big(V_{\pi^M}\otimes_{\mathbb{R}}\Lambda^{i}(\mathfrak{p}_{\mathfrak{m}}^{*})\otimes_{\mathbb{R}}E_\eta\big)^{K_M}\\ &=\operatorname{vol}(K_M)^{-1}\exp(t\chi_{\pi^M}(C^{\mathfrak{m}})/2)\int_{g\in M}\Theta^{M}_{\pi^M}(g)\operatorname{Tr_s}[p_t^{X_M,\Lambda^\bullet(\mathfrak{p}^*_{\mathfrak{m}})\otimes E_\eta}(g)]\,dv_M\\ &=\frac{\exp(t\chi_{\pi^M}(C^{\mathfrak{m}})/2)}{|W(T,M)|\operatorname{vol}(T)}\int_{\gamma\in T'}\Theta^{M}_{\pi^M}(\gamma)\operatorname{Tr_s}^{[\gamma]}\big[\exp(-tC^{\mathfrak{m},X_M,\Lambda^\bullet(\mathfrak{p}_{\mathfrak{m}})\otimes E_\eta}/2)\big]\big|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{m}/\mathfrak{t}}\big|\,dv_T.\end{aligned}\tag{8-84}$$

By (8-83), (8-84), proceeding as in (8-75), and letting $t\to 0$, we get the desired equality (8-80).

The Euler formula asserts

$$\sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} (-1)^i \dim_\mathbb{C}\big(H_j(\mathfrak{n}, V_{\pi,K}) \otimes_\mathbb{R} \Lambda^i(\mathfrak{p}_\mathfrak{m}^*) \otimes_\mathbb{R} E_\eta\big)^{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} (-1)^i \dim_\mathbb{C} H^i\big(\mathfrak{m}, K_M; H_j(\mathfrak{n}, V_{\pi,K}) \otimes_\mathbb{R} E_\eta\big). \tag{8-85}$$

By (3-17), we have

$$W(H,G) = W(T,K), \quad W(T,M) = W(T,K_M). \tag{8-86}$$

By (7-5), (8-79), (8-80) and (8-85)–(8-86), we get (8-72). □

Corollary 8.15. *The following identity holds*:

$$r_{\eta,\rho} = \frac{1}{\chi(K/K_M)} \sum_{\substack{\pi\in\widehat{G}_u\\ \chi_\pi(C^\mathfrak{g})=0}} n_\rho(\pi) \sum_{i=0}^{\dim \mathfrak{p}_\mathfrak{m}} \sum_{j=0}^{2l} (-1)^{i+j} \dim_\mathbb{C} H^i\big(\mathfrak{m}, K_M; H_j(\mathfrak{n}, V_{\pi,K}) \otimes_\mathbb{R} E_\eta\big). \tag{8-87}$$

Proof. This is a consequence of (7-58), (8-66) and (8-72). □

Remark 8.16. When $G = \mathrm{SO}^0(p,1)$ with $p \geqslant 3$ odd, the formula (8-87) is compatible with [Juhl 2001, Theorem 3.11].

We will apply (8-87) to η_j. The following proposition allows us to reduce the first sum in (8-87) to the one over $\pi \in \widehat{G}_u$ with $\chi_\pi = 0$.

Proposition 8.17. *Let* $(\pi, V_\pi) \in \widehat{G}_u$. *Assume* $\chi_\pi(C^\mathfrak{g}) = 0$ *and*

$$H^\bullet\big(\mathfrak{m}, K_M; H_\bullet(\mathfrak{n}, V_\pi) \otimes_\mathbb{R} \Lambda^j(\mathfrak{n}^*)\big) \neq 0. \tag{8-88}$$

Then the infinitesimal character χ_π *vanishes.*

Proof. Recall that $\Lambda(\pi) \in \mathfrak{h}_\mathbb{C}^*$ is a Harish-Chandra parameter of π. We need to show that there is $w \in W(\mathfrak{h}_\mathbb{R}, \mathfrak{u})$ such that

$$w\Lambda(\pi) = \rho^\mathfrak{u}. \tag{8-89}$$

Let B^* be the bilinear form on $\mathfrak{g}^*$ induced by B. It extends to $\mathfrak{g}_\mathbb{C}^*$ and $\mathfrak{u}^*$ in an obvious way. Since $\chi_\pi(C^{\mathfrak{g},\pi}) = 0$, we have

$$B^*(\Lambda(\pi), \Lambda(\pi)) = B^*(\rho^\mathfrak{u}, \rho^\mathfrak{u}). \tag{8-90}$$

We identify $\mathfrak{h}_\mathbb{R}^* = \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^*$. By definition,

$$\rho^\mathfrak{u} = \left(\frac{l\alpha}{2\sqrt{-1}\pi}, \rho^{\mathfrak{u}_\mathfrak{m}}\right) \in \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^* \quad \text{and} \quad \rho^{\mathfrak{u}(\mathfrak{b})} = (0, \rho^{\mathfrak{u}_\mathfrak{m}}) \in \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^*. \tag{8-91}$$

By (8-49), (8-81) and (8-88), there exist $w \in W(\mathfrak{h}_\mathbb{R}, \mathfrak{u})$, $w' \in W(\mathfrak{t}, \mathfrak{u}_\mathfrak{m}) \subset W(\mathfrak{h}_\mathbb{R}, \mathfrak{u})$ and the highest real weight $\mu_j \in \mathfrak{t}^*$ of an irreducible subrepresentation of $\mathfrak{m}_\mathbb{C}$ on $\Lambda^j(\mathfrak{n}_\mathbb{C}) \simeq \Lambda^j(\bar{\mathfrak{n}}_\mathbb{C}^*)$ such that

$$w\Lambda(\pi)|_{\mathfrak{t}_\mathbb{C}} = w'(\mu_j + \rho^{\mathfrak{u}_\mathfrak{m}}). \tag{8-92}$$

By (6-58), (8-90) and (8-92), there exists $w'' \in W(\mathfrak{h}_{\mathbb{R}}, \mathfrak{u})$ such that

$$w''\Lambda(\pi) = \left(\pm\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j + \rho^{\mathfrak{u}_{\mathfrak{m}}}\right) = \left(\pm\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) + \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8-93}$$

In particular, $w''\Lambda(\pi) \in \mathfrak{h}^*_{\mathbb{R}}$.

Clearly, $\big((j-l)\alpha/(2\sqrt{-1}\pi), \mu_j\big) \in \mathfrak{h}^*_{\mathbb{R}}$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}$ on $\Lambda^j(\bar{\mathfrak{n}}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} (\det(\mathfrak{n}_{\mathbb{C}}))^{-\frac{1}{2}}$. By (6-33), $\big((j-l)\alpha/(2\sqrt{-1}\pi), \mu_j\big) \in \mathfrak{h}^*_{\mathbb{R}}$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}$ on $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$. By [Borel and Wallach 2000, Lemma II.6.9], there exists $w_1 \in W(\mathfrak{h}_{\mathbb{R}}, \mathfrak{u})$ such that

$$\left(\frac{(j-l)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) = w_1\rho^{\mathfrak{u}} - \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8-94}$$

Similarly, $\big((l-j)\alpha/(2\sqrt{-1}\pi), \mu_j\big) \in \mathfrak{h}^*_{\mathbb{R}}$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}$ on both $\Lambda^{2l-j}(\bar{\mathfrak{n}}^*_{\mathbb{C}}) \otimes_{\mathbb{C}} (\det(\mathfrak{n}_{\mathbb{C}}))^{-\frac{1}{2}}$ and $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$. Therefore, there exists $w_2 \in W(\mathfrak{h}_{\mathbb{R}}, \mathfrak{u})$ such that

$$\left(\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) = w_2\rho^{\mathfrak{u}} - \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8-95}$$

By (8-93)–(8-95), we get (8-89). $\square$

Corollary 8.18. *For* $0 \leqslant j \leqslant 2l$, *we have*

$$r_j = \frac{1}{\chi(K/K_M)} \sum_{\substack{\pi \in \widehat{G}_u \\ \chi_\pi = 0}} n_\rho(\pi) \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} \sum_{k=0}^{2l} (-1)^{i+k} \dim_{\mathbb{C}} H^i\big(\mathfrak{m}, K_M; H_k(\mathfrak{n}, V_{\pi,K}) \otimes_{\mathbb{R}} \Lambda^j(\mathfrak{n}^*)\big). \tag{8-96}$$

If $H^\bullet(Z, F) = 0$, *then for all* $0 \leqslant j \leqslant 2l$,

$$r_j = 0. \tag{8-97}$$

Proof. This is a consequence of Proposition 8.12, Corollary 8.15 and Proposition 8.17. $\square$

Remark 8.19. By (7-75) and (8-97), we get (5-13) when G has compact center and $\delta(G) = 1$.

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Index

$|\cdot|$, 11
$\|\cdot\|_{r,z_1,z_2}$, 58
$\langle\cdot,\cdot\rangle_{\mathfrak{n}_{\mathbb{C}}}$, 31

A_0, 31
$[\alpha]^{\max}$, 16
α, 29
$\widehat{A}$, 9
$\widehat{A}(E,\nabla^E)$, 9
a_0, 29

$\mathfrak{b}(\gamma)$, 21
B, 5, 11
$B_{[\gamma]}$, 3, 24
$[\beta]^{\max}$, 46
$\square^Z$, 2, 10
$\mathfrak{b}$, 13
$\mathfrak{b}_*$, 40

$C^{\mathfrak{g},X,\tau}$, 16
$C^{\mathfrak{g},X}$, 16
$C^{\mathfrak{g}}$, 12
$C^{\mathfrak{k},V}$, $C^{\mathfrak{k},\tau}$, 12
$C^{\mathfrak{k}}$, 12
$C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}$, 35
$C^{\mathfrak{u}_{\mathfrak{m}},\eta_j}$, 35
C_ρ, 55
$\chi(Z)$, 10
$\chi(Z,F)$, $\chi'(Z,F)$, 10
χ_λ, 47
χ_Λ, 57
χ_{orb}, 3, 24
$\mathrm{ch}(E',\nabla^{E'})$, 9
c_G, 46

$\det_{\mathrm{gr}}$, 53
D^Z, 10

$\Delta(\mathfrak{t},\mathfrak{k})$, $\Delta^+(\mathfrak{t},\mathfrak{k})$, 47
$\delta(G)$, 5, 13
$\delta(\mathfrak{g})$, 14
$\det(P)$, 9
$\det(P+\sigma)$, 9
d^Z, 10
$d^{Z,*}$, 10
$d_X(x,y)$, 16
dv_X, 16
dv_{H_i}, $dv_{H_i\backslash G}$, 15
$dv_{K^0(\gamma)\backslash K}$, $dv_{Z^0(\gamma)\backslash G}$, 17
dv_G, 60

E_τ, 16
$E_{\hat{\eta}}$, $E^+_{\hat{\eta}}$, $E^-_{\hat{\eta}}$, 45
η, 45
η_j, 35
$\hat{\eta}$, $\hat{\eta}^+$, $\hat{\eta}^-$, 45
$\mathcal{E}_\tau$, 16
$\mathcal{E}_{\hat{\eta}}$, 46
$e(E,\nabla^E)$, 9

F, 2, 5, 10, 19
$F_{\mathfrak{b},\eta}$, 46
$\mathcal{F}_\tau$, 19

G, 5, 11
G', 15
G_*, 39
G_1, G_2, 40
$G_{\mathbb{C}}$, 11
G'_i, 15
$[\Gamma]$, 3, 20
$[\gamma]$, 3, 20
Γ, 3, 5, 10, 19
$\Gamma(\gamma)$, 20
$\mathfrak{g}$, 5, 11

$\mathfrak{g}_*$, 39
$\mathfrak{g}_1$, $\mathfrak{g}_2$, 35
$\mathfrak{g}_{\mathbb{C}}$, 11
g^F, 2, 10
g^{TX}, 16
g^{TZ}, 2, 10

$\mathfrak{h}_{i\mathfrak{p}}$, $\mathfrak{h}_{i\mathfrak{k}}$, 13
$H^\bullet(Z,F)$, 10
$H^\bullet(\mathfrak{g},K;V)$, 63
H'_i, 15
$H_\bullet(\mathfrak{n}_{\mathfrak{q}},V)$, 64
$\mathfrak{h}$, 13
$\mathfrak{h}(\gamma)$, 14
$\mathfrak{h}_i$, H_i, 13
$\mathfrak{h}_{i\mathbb{C}}$, $\mathfrak{h}_{i\mathbb{R}}$, 56

ι, 34
i_G, 37

J, 31
J_γ, 18

K, 5, 11
K_*, 39
K_M, 22
$\mathfrak{k}$, 5, 11
$\mathfrak{k}(\gamma)$, 12
$\mathfrak{k}_*$, 39
$\mathfrak{k}_0$, 18
$\mathfrak{k}_0^\perp$, 18
$\mathfrak{k}_0^\perp(\gamma)$, 18
$\mathfrak{k}_{\mathfrak{m}}(k)$, 29
$\mathfrak{k}_{\mathfrak{m}}$, 22

l, 28
l_0, 13

$l_{[\gamma]}$, 3

M, 22
$M(k), M^0(k)$, 29
$\mathfrak{m}$, 22
$\mathfrak{m}(k)$, 29
m, 10, 11
$m_P(\lambda)$, 9
$m_{[\gamma]}$, 3, 24
$m_{\eta,\rho}(\lambda)$, 53

$N^{\Lambda^\bullet(T^*Z)}$, 2
$N_{\mathfrak{b}}$, 33
$\mathfrak{n}, \bar{\mathfrak{n}}$, 28
∇^{TX}, 16
n, 11

$\omega^{\mathfrak{g}}, \omega^{\mathfrak{p}}, \omega^{\mathfrak{k}}$, 15
$\omega^{Y_{\mathfrak{b}}}$, 32
$\omega^{\mathfrak{u}}, \omega^{\mathfrak{u}(\mathfrak{b})}, \omega^{\mathfrak{u}^\perp(\mathfrak{b})}$, 32
$\Omega^{\mathfrak{k}}$, 16
$\Omega^{\mathfrak{u}(\mathfrak{b})}$, 32
$\Omega^{\mathfrak{u}_{\mathfrak{m}}}$, 33
$\Omega^\bullet(Z, F)$, 2, 10
$o(TX)$, 16

$P_\eta(\sigma)$, 53
$\hat{p}, \hat{\pi}$, 19
$\mathfrak{p}$, 11
$\mathfrak{p}(\gamma)$, 12
$\mathfrak{p}^{a,\perp}(\gamma)$, 12
$\mathfrak{p}_*$, 39
$\mathfrak{p}_0$, 18
$\mathfrak{p}_0^\perp$, 18
$\mathfrak{p}_0^\perp(\gamma)$, 18
$\mathfrak{p}_{\mathfrak{m}}(k)$, 29
$\mathfrak{p}_{\mathfrak{m}}$, 22
$\pi_{\mathfrak{k}}(Y)$, 47
$p_t^{X,\tau}(g)$, 17
$p_t^{X,\tau}(x, x')$, 17
P, 16

$\mathfrak{q}$, 28

$R^{F_{\mathfrak{b},\eta}}$, 46
$R^{N_{\mathfrak{b}}}$, 33
$R_\rho(\sigma)$, 3, 25
$\mathrm{RO}(K_M), \mathrm{RO}(K)$, 6, 34
ρ, 5, 10, 19
$\rho^{\mathfrak{k}}$, 47
$\mathrm{rk}_{\mathbb{C}}$, 13
r, 10, 19
r_ρ, 55
r_j, 55
$r_{\eta,\rho}$, 53

$S^{\mathfrak{u}^\perp(\mathfrak{b})}$, 31
$\mathcal{S}(G)$, 58
σ_0, 26, 53
σ_η, 53
$\sigma_{\mathfrak{k}}(Y)$, 47

T, 13
$T(F)$, 2, 10
$T(\sigma)$, 7, 10
Θ_π^G, 59
$\mathrm{Tr}^{[\gamma]}[\cdot]$, 17
$\mathrm{Tr_s}^{[\gamma]}[\cdot]$, 18
$\mathfrak{t}(\gamma)$, 21
$\mathfrak{t}_*$, 40
τ, 12, 16
θ, 5, 11
$\theta(s)$, 2
$\theta_P(s)$, 9

U, 11
$U(\mathfrak{b}), U_M$, 31
$U(\mathfrak{g})$, 12
$U(\mathfrak{g}_{\mathbb{C}})$, 56
$U_M(k), U_M^0(k)$, 51
$\mathfrak{u}$, 11
$\mathfrak{u}(\mathfrak{b}), \mathfrak{u}(\mathfrak{m})$, 30
$\mathfrak{u}^\perp(\mathfrak{b})$, 31
$\mathfrak{u}_{\mathfrak{m}}(k)$, 51

$\mathrm{vol}(\cdot)$, 16, 17

$W(H_i, G)$, 13
$W(T, K)$, 16

X, 5, 16
$X(\gamma)$, 17
$X^{a,\perp}(\gamma)$, 17
$X_M(k)$, 30
X_M, 23
$\Xi(g)$, 59
Ξ_ρ, 25

$Y_{\mathfrak{b}}$, 31

Z, 2, 10, 19
$Z(\gamma)$, 12
$Z(a)$, 12
$Z^0(\gamma)$, 12
$Z^0(\mathfrak{b})$, 28
Z_G, 11
$\mathcal{Z}(\mathfrak{g})$, 12
$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, 56
$\mathfrak{z}(\gamma)$, 12
$\mathfrak{z}(a)$, 12
$\mathfrak{z}^{a,\perp}(\gamma)$, 12
$\mathfrak{z}_0$, 18
$\mathfrak{z}_0^\perp$, 18
$\mathfrak{z}_0^\perp(\gamma)$, 18
$\mathfrak{z}_{\mathfrak{g}}$, 11
$\mathfrak{z}_{\mathfrak{p}}, \mathfrak{z}_{\mathfrak{k}}$, 11

References

[Barbasch and Moscovici 1983] D. Barbasch and H. Moscovici, "L^2-index and the Selberg trace formula", *J. Funct. Anal.* **53**:2 (1983), 151–201. MR Zbl

[Berline et al. 2004] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer, 2004. MR Zbl

[Bismut 2005] J.-M. Bismut, "The hypoelliptic Laplacian on the cotangent bundle", *J. Amer. Math. Soc.* **18**:2 (2005), 379–476. MR Zbl

[Bismut 2011] J.-M. Bismut, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies **177**, Princeton University Press, 2011. MR Zbl

[Bismut and Goette 2004] J.-M. Bismut and S. Goette, "Equivariant de Rham torsions", *Ann. of Math.* (2) **159**:1 (2004), 53–216. MR Zbl

[Bismut and Zhang 1992] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205**, Société Mathématique de France, Paris, 1992. MR Zbl

[Bismut et al. 2017] J.-M. Bismut, X. Ma, and W. Zhang, "Asymptotic torsion and Toeplitz operators", *J. Inst. Math. Jussieu* **16**:2 (2017), 223–349. MR Zbl

[Borel and Wallach 2000] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd ed., Mathematical Surveys and Monographs **67**, Amer. Math. Society, Providence, RI, 2000. MR Zbl

[Bott 1965] R. Bott, "The index theorem for homogeneous differential operators", pp. 167–186 in *Differential and combinatorial topology: a symposium in honor of Marston Morse*, edited by S. S. Cairns, Princeton University Press, 1965. MR Zbl

[Bröcker and tom Dieck 1985] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics **98**, Springer, 1985. MR Zbl

[Cheeger 1979] J. Cheeger, "Analytic torsion and the heat equation", *Ann. of Math.* (2) **109**:2 (1979), 259–322. MR Zbl

[Duistermaat et al. 1979] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, "Spectra of compact locally symmetric manifolds of negative curvature", *Invent. Math.* **52**:1 (1979), 27–93. MR Zbl

[Dyatlov and Zworski 2016] S. Dyatlov and M. Zworski, "Dynamical zeta functions for Anosov flows via microlocal analysis", *Ann. Sci. Éc. Norm. Supér.* (4) **49**:3 (2016), 543–577. MR Zbl

[Dyatlov and Zworski 2017] S. Dyatlov and M. Zworski, "Ruelle zeta function at zero for surfaces", *Invent. Math.* (online publication April 2017).

[Fried 1986] D. Fried, "Analytic torsion and closed geodesics on hyperbolic manifolds", *Invent. Math.* **84**:3 (1986), 523–540. MR Zbl

[Fried 1987] D. Fried, "Lefschetz formulas for flows", pp. 19–69 in *The Lefschetz centennial conference, Part III* (Mexico City, 1984), Contemp. Math. **58**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Gel'fand et al. 1969] I. M. Gel'fand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders Co., Philadelphia, PA, 1969. MR

[Giulietti et al. 2013] P. Giulietti, C. Liverani, and M. Pollicott, "Anosov flows and dynamical zeta functions", *Ann. of Math.* (2) **178**:2 (2013), 687–773. MR Zbl

[Hecht and Schmid 1983] H. Hecht and W. Schmid, "Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules", *Acta Math.* **151**:1-2 (1983), 49–151. MR Zbl

[Helgason 1978] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics **80**, Academic Press, New York, 1978. MR Zbl

[Hirzebruch 1966] F. Hirzebruch, *Topological methods in algebraic geometry*, Die Grundlehren der Mathematischen Wissenschaften **131**, Springer, 1966. MR Zbl

[Hitchin 1974] N. Hitchin, "Harmonic spinors", *Advances in Math.* **14** (1974), 1–55. MR Zbl

[Juhl 2001] A. Juhl, *Cohomological theory of dynamical zeta functions*, Progress in Mathematics **194**, Birkhäuser, Basel, 2001. MR Zbl

[Knapp 1986] A. W. Knapp, *Representation theory of semisimple groups: an overview based on examples*, Princeton Mathematical Series **36**, Princeton University Press, 1986. MR Zbl

[Knapp 2002] A. W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics **140**, Birkhäuser, Boston, 2002. MR Zbl

[Kobayashi and Nomizu 1963] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. I*, Interscience Publishers, New York, 1963. MR Zbl

[Kostant 1976] B. Kostant, "On Macdonald's η-function formula, the Laplacian and generalized exponents", *Advances in Math.* **20**:2 (1976), 179–212. MR Zbl

[Li and Yau 1986] P. Li and S.-T. Yau, "On the parabolic kernel of the Schrödinger operator", *Acta Math.* **156**:3-4 (1986), 153–201. MR Zbl

[Ma 2017] X. Ma, "Geometric hypoelliptic Laplacian and orbital integrals (after Bismut, Lebeau and Shen)", lecture given at Séminaire Bourbaki, Exp. No. 1130, March 11 2017, available at http://www.bourbaki.ens.fr/TEXTES/1130.pdf. Video available at https://www.youtube.com/watch?v=dCDwN-HqcJw.

[Ma and Marinescu 2015] X. Ma and G. Marinescu, "Exponential estimate for the asymptotics of Bergman kernels", *Math. Ann.* **362**:3-4 (2015), 1327–1347. MR Zbl

[Matsushima 1967] Y. Matsushima, "A formula for the Betti numbers of compact locally symmetric Riemannian manifolds", *J. Differential Geometry* **1** (1967), 99–109. MR Zbl

[Milnor 1968a] J. Milnor, "A note on curvature and fundamental group", *J. Differential Geometry* **2** (1968), 1–7. MR Zbl

[Milnor 1968b] J. W. Milnor, "Infinite cyclic coverings", pp. 115–133 in *Conference on the Topology of Manifolds* (Lansing, MI, 1967), Prindle, Weber & Schmidt, Boston, 1968. MR Zbl

[Moscovici and Stanton 1991] H. Moscovici and R. J. Stanton, "R-torsion and zeta functions for locally symmetric manifolds", *Invent. Math.* **105**:1 (1991), 185–216. MR Zbl

[Müller 1978] W. Müller, "Analytic torsion and R-torsion of Riemannian manifolds", *Adv. in Math.* **28**:3 (1978), 233–305. MR Zbl

[Quillen 1985] D. Quillen, "Superconnections and the Chern character", *Topology* **24**:1 (1985), 89–95. MR Zbl

[Ray and Singer 1971] D. B. Ray and I. M. Singer, "R-torsion and the Laplacian on Riemannian manifolds", *Advances in Math.* **7** (1971), 145–210. MR Zbl

[Reidemeister 1935] K. Reidemeister, "Homotopieringe und Linsenräume", *Abh. Math. Sem. Univ. Hamburg* **11**:1 (1935), 102–109. MR Zbl

[Salamanca-Riba 1999] S. A. Salamanca-Riba, "On the unitary dual of real reductive Lie groups and the $A_{\mathfrak{g}}(\lambda)$ modules: the strongly regular case", *Duke Math. J.* **96**:3 (1999), 521–546. MR Zbl

[Satake 1957] I. Satake, "The Gauss–Bonnet theorem for V-manifolds", *J. Math. Soc. Japan* **9** (1957), 464–492. MR Zbl

[Seeley 1967] R. T. Seeley, "Complex powers of an elliptic operator", pp. 288–307 in *Singular integrals* (Chicago, IL, 1966), Amer. Math. Soc., Providence, RI, 1967. MR Zbl

[Selberg 1960] A. Selberg, "On discontinuous groups in higher-dimensional symmetric spaces", pp. 147–164 in *Contributions to function theory* (Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960. MR Zbl

[Shen 2016] S. Shen, "Analytic torsion, dynamical zeta functions and orbital integrals", *C. R. Math. Acad. Sci. Paris* **354**:4 (2016), 433–436. MR

[Varadarajan 1977] V. S. Varadarajan, *Harmonic analysis on real reductive groups*, Lecture Notes in Mathematics **576**, Springer, 1977. MR Zbl

[Vogan 1984] D. A. Vogan, Jr., "Unitarizability of certain series of representations", *Ann. of Math.* (2) **120**:1 (1984), 141–187. MR Zbl

[Vogan 2008] D. A. Vogan, Jr., "Unitary representations and complex analysis", pp. 259–344 in *Representation theory and complex analysis*, Lecture Notes in Math. **1931**, Springer, 2008. MR Zbl

[Vogan and Zuckerman 1984] D. A. Vogan, Jr. and G. J. Zuckerman, "Unitary representations with nonzero cohomology", *Compositio Math.* **53**:1 (1984), 51–90. MR Zbl

[Voros 1987] A. Voros, "Spectral functions, special functions and the Selberg zeta function", *Comm. Math. Phys.* **110**:3 (1987), 439–465. MR Zbl

[Wallach 1988] N. R. Wallach, *Real reductive groups, I*, Pure and Applied Mathematics **132**, Academic Press, Boston, 1988. MR Zbl

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