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Dynamical zeta functions in the nonorientable case

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Abstract

We use a simple argument to extend the microlocal proofs of meromorphicity of dynamical zeta functions to the nonorientable case. In the special case of geodesic flow on a connected non-orientable negatively curved closed surface, we compute the order of vanishing of the zeta function at the zero point to be the first Betti number of the surface.

Keywords: Ruelle zeta function, microlocal analysis, dynamical systems, nonorientable

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1. Background

In this note we use a simple geometric argument to extend the results of Dyatlov and Zworski [5, 6] and of Dyatlov and Guillarmou [3, 4] to Axiom A flows with nonorientable stable and unstable bundles. It is classically known that on a closed manifold there are countably many closed orbits of such flows, and therefore one can define the *Ruelle zeta function*

$$\zeta_{\mathrm{R}}(\lambda) = \prod_{\gamma^{\sharp}} \left(1 - \mathrm{e}^{\mathrm{i}\lambda T^{\sharp}_{\gamma}}\right),$$

where the product is taken over all primitive closed geodesics $\gamma^{\sharp}$ with corresponding periods $T^{\sharp}_{\gamma}$. Note that by [3, lemma 1.17] and [4, section 3], this product converges for $\mathrm{Im}(\lambda) \gg 1$ large enough. The meromorphic continuation of ζ_{R} to all of $\mathbb{C}$ was conjectured by Smale [13], and proved by Fried [8] under analyticity assumptions. The case of smooth Anosov flows was first answered by Giulietti, Liverani and Policott [9] and then with microlocal methods by Dyatlov

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and Zworski [5] for manifolds with orientable stable and unstable bundles, and was extended to Axiom A flows by Dyatlov and Guillarmou [3, 4] under the same orientability assumptions. In [9, appendix B], the authors also outlined ideas for removing the orientability assumptions.

We remove the orientability assumption and give a full proof for Axiom A flows. Specifically, we shall show

Theorem 1. *If $(\phi_t)_{t\in\mathbb{R}}$ is an Axiom A flow on a closed manifold, the Ruelle zeta function ζ_{R} extends to a meromorphic function on $\mathbb{C}$.*

The definition of an Axiom A flow is given as definition 1.3.

We then restrict to the case of contact Anosov flow on a three-manifold, and study the order of vanishing of ζ_{R} at $\lambda = 0$. An important example is when $M = S^*\Sigma$, the cosphere bundle of a connected negatively curved closed surface Σ, and $(\phi_t)_{t\in\mathbb{R}}$ is geodesic flow [1]. This problem was treated in [6] in the case where the stable bundle is orientable, and it was shown that the order of vanishing is $b_1(M) - 2$, where $b_1(M)$ is the first Betti number of M.

We shall show that for nonorientable stable bundle, the analogous result is the following:

Theorem 2. *Let $(\phi_t)_{t\in\mathbb{R}}$ be the Reeb flow on a connected contact closed three-manifold. If $(\phi_t)_{t\in\mathbb{R}}$ is Anosov with nonorientable stable bundle E_s, the Ruelle zeta function has vanishing order at $\lambda = 0$ equal to $b_1(\mathscr{O}(E_s))$, the dimension of the first de Rham cohomology with coefficients in the orientation line bundle of E_s.*

The orientation line bundle is reviewed in definition 1.5.

In the special case of the geodesic flow on $M = S^*\Sigma$ with Σ nonorientable, the vanishing order at $\lambda = 0$ is given by $b_1(\Sigma)$, as is shown in proposition 3.10. This is in contrast to the orientable case, in which it is $b_1(\Sigma) - 2$.

More precisely, let $\chi'(\Sigma)$ be the derived Euler characteristic of Σ, i.e.,

$$\chi'(\Sigma) = \sum_{i=0}^{2} (-1)^i i b_i(\Sigma) = \begin{cases} -b_1(\Sigma) + 2, & \text{if } \Sigma \text{ is orientable,} \\ -b_1(\Sigma), & \text{otherwise.} \end{cases}$$

Corollary 3. *If $(\phi_t)_{t\in\mathbb{R}}$ is the geodesic flow on the cosphere bundle of a connected negatively curved closed surface (orientable or not), the Ruelle zeta function has vanishing order at $\lambda = 0$ equal to $-\chi'(\Sigma)$.*

1.1. Axiom A flows

Let M be a compact manifold without boundary of dimension n, and let $(\phi_t)_{t\in\mathbb{R}}$ be a flow on M generated by the vector field $V \in C^\infty(M; TM)$.

Definition 1.1. A ϕ_t-invariant set $K \subseteq M$ is called hyperbolic for the flow $(\phi_t)_{t\in\mathbb{R}}$ if V does not vanish on K and for each $x \in K$ the tangent space T_xM can be written as the direct sum

$$T_xM = E_0(x) \oplus E_s(x) \oplus E_u(x)$$

where $E_0(x) =$ span$(V(x))$, E_s, E_u are continuous ϕ_t-invariant vector bundles on K, and for some Riemannian metric $|\cdot|$, there are $C, \theta > 0$ such that for all $t > 0$,

$$\begin{aligned} |\mathrm{d}\phi_t(x)v|_{\phi_t(x)} &\leqslant Ce^{-\theta t}|v|_x \qquad v \in E_s(x) \\ |\mathrm{d}\phi_{-t}(x)w|_{\phi_{-t}(x)} &\leqslant Ce^{-\theta t}|w|_x \qquad w \in E_u(x). \end{aligned} \tag{1}$$

In the important case where all of M is hyperbolic, we call $(\phi_t)_{t\in\mathbb{R}}$ an *Anosov flow*.

There is an analogous notion of hyperbolicity at fixed points.

Definition 1.2. A fixed point $x \in M$, i.e., $V(x) = 0$, is called *hyperbolic* if the differential $DV(x)$ has no eigenvalues with vanishing real part.

A generalization of Anosov flows is the following, given first by Smale [13, II.5, definition 5.1]:

Definition 1.3. The flow $(\phi_t)_{t\in\mathbb{R}}$ is called *Axiom A* if

(a) All fixed points of $(\phi_t)_{t\in\mathbb{R}}$ are hyperbolic,
(b) The closure $\mathcal{K}$ of the union of all closed orbits of $(\phi_t)_{t\in\mathbb{R}}$ is hyperbolic,
(c) The nonwandering set ([4, definition 2.2]) of $(\phi_t)_{t\in\mathbb{R}}$ is the disjoint union of the set of fixed points and $\mathcal{K}$.

We now recall the definition of a locally maximal set, given in [4, definition 2.4].

Definition 1.4. A compact ϕ_t-invariant set $K \subseteq M$ is called locally maximal for $(\phi_t)_{t\in\mathbb{R}}$ if there is a neighbourhood V of K such that

$$K = \bigcap_{t\in\mathbb{R}} \phi_t(V).$$

We may then state the key proposition, which generalises [4, proposition 3.1] to the case where E_s or E_u is not necessarily orientable on $\mathcal{K}$.

Proposition 1.4. *Let $K \subseteq M$ be a locally maximal hyperbolic set for $(\phi_t)_{t\in\mathbb{R}}$, and let ζ_K be defined as the Ruelle zeta function where we only take the product over trajectories in K. Then ζ_K has a continuation to a meromorphic function on all of $\mathbb{C}$.*

Theorem 1 follows from proposition 1.4, as we may remark that by [13, II.5, Theorem 5.2] we can write $\mathcal{K} = K_1 \sqcup \cdots \sqcup K_N$ with K_j basic hyperbolic.[3] Then the product

$$\zeta_R(\lambda) = \prod_{j=1}^{N} \zeta_{K_j}(\lambda),$$

which *a priori* holds for $\mathrm{Im}(\lambda) \gg 1$, gives that ζ_R also has a meromorphic continuation to all of $\mathbb{C}$.

The goal of section 2 is to prove proposition 1.4.

1.2. The orientation bundle

To fix notation we recall the definition of transition functions of a vector bundle. Given a continuous real vector bundle E of rank k over a manifold M with projection map π, let $U_\alpha, U_\beta \subseteq M$ be two small open sets with nonempty intersection, and let $\psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^n$, $\psi_\beta : \pi^{-1}U_\beta \to U_\beta \times \mathbb{R}^n$ be local trivializations. Then the map $\psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ is of the form

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$$

where $\tau_{\alpha\beta} \in C^0(U_\alpha \cap U_\beta, \mathrm{GL}_k(\mathbb{R}))$ is called a *transition function*. If the local trivializations can be chosen such that $\tau_{\alpha\beta}$ are smooth, then E is a smooth vector bundle. Similarly, if $\tau_{\alpha\beta}$ can be chosen to be locally constant functions, then E is a flat vector bundle.

[3] These are locally maximal hyperbolic by definition (see [4, definition 2.5]).

Furthermore, suppose we are given an open cover $(U_\alpha)_{\alpha\in A}$ of M together with a set of continuous (resp. smooth, resp. locally constant) $\mathrm{GL}_k(\mathbb{R})$-valued functions $(\tau_{\alpha\beta})_{\substack{\alpha,\beta\in A\\ U_\alpha\cap U_\beta\neq\emptyset}}$ with $\tau_{\alpha\alpha}=I$ on U_α. Then there exists a continuous (resp. smooth, resp. flat) vector bundle E with transition functions $\tau_{\alpha\beta}$, provided the following *triple product property* holds:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p)=I$$

for any $p\in U_\alpha\cap U_\beta\cap U_\gamma$.

Definition 1.5. If E is a continuous (but not necessarily smooth) real vector bundle over M with transition functions $\tau_{\alpha\beta}$, the *orientation bundle* of E is a smooth flat line bundle $\mathscr{O}(E)$ with transition functions

$$\sigma_{\alpha\beta}(p)=\operatorname{sgn}\det(\tau_{\alpha\beta}(p))=\begin{cases}1 & \det(\tau_{\alpha\beta}(p))>0\\ -1 & \det(\tau_{\alpha\beta}(p))<0.\end{cases}$$

Recall that if $f:M\to M$ is a map, we say f *lifts* to a bundle map $F:E\to E$ if $\pi\circ F=f\circ\pi$.

Since $\mathscr{O}(E)$ is a flat vector bundle, using the associated flat connection, we can lift the flow $(\phi_t)_{t\in\mathbb{R}}$ to a flow $(\widetilde{\Phi}_t)_{t\in\mathbb{R}}$ on $\mathscr{O}(E)$. If the flow $(\phi_t)_{t\in\mathbb{R}}$ on M lifts to a flow $(\Phi_t)_{t\in\mathbb{R}}$ on E, if ψ,η are distinct trivializations of E near p, $\phi_t(p)$ respectively, and $\widetilde{\psi}$, $\widetilde{\eta}$ are trivializations of $\mathscr{O}(E)$ near p, $\phi_t(p)$ respectively, we have for $p\in M$ and $l\in\mathscr{O}(E)_p$:

$$\widetilde{\Phi}_t(l)=\widetilde{\eta}^{\,-1}\left(\phi_t(p),\operatorname{sgn}\left(\det\left(\eta\Phi_t\psi^{-1}\right)\big|_p\right)\operatorname{proj}_2\widetilde{\psi}(l)\right),\tag{2}$$

where proj_2 is the obvious projection to the second component.

1.3. Geodesic flows

Let Z be a negatively curved closed Riemannian manifold. Let $M=S^*Z$ be the cosphere bundle on Z. It is classical that the geodesic flow on M is Anosov [1].

Let $\pi:M\to Z$ be the canonical projection. For $x\in M$, we have a morphism of linear spaces

$$\pi_*:T_xM\to T_{\pi(x)}Z.\tag{3}$$

The following proposition is classical [1, section 22] and [12, proposition 6]. We include a proof for the sake of completeness.

Proposition 1.5. *The morphism π_* induces an isomorphism of continuous vector bundles on M,*

$$E_s\oplus E_0\simeq\pi^*(TZ).\tag{4}$$

Proof. Since both sides of [4] have the same dimension, it is enough to show that $\pi_*|_{E_s\oplus E_0}$ is injective. We will show this using Jacobi fields. It is convenient to work on the sphere bundle $M'=SZ$. We identify M' with M via the Riemannian metric on Z.

We follow [7, section II.H]. Let $\mathcal{M}$ be the total space of TZ. Denote still by $\pi:\mathcal{M}\to Z$ the obvious projection. Let $T^V\mathcal{M}\subset T\mathcal{M}$ be the vertical subbundle of $T\mathcal{M}$. The Levi-Civita connection on TZ induces a horizontal subbundle $T^H\mathcal{M}\subset T\mathcal{M}$ of $T\mathcal{M}$, so that

$$T\mathcal{M}=T^V\mathcal{M}\oplus T^H\mathcal{M}.\tag{5}$$

Since $T^V\mathcal{M}\simeq\pi^*(TZ)$ and $T^H\mathcal{M}\simeq\pi^*(TZ)$, by [5], we can identify the smooth vector bundles,

$$T\mathcal{M}=\pi^*(TZ\oplus TZ).$$

For $x = (z, v) \in \mathcal{M}$, let γ_x be the unique geodesic on Z such that $(\gamma_x(0), \dot{\gamma}_x(0)) = (z, v)$. For $w \in T_x\mathcal{M}$, let $J_{x,w} \in C^\infty(\gamma_x, TZ|_{\gamma_x})$ be the unique Jacobi field along γ_x such that $\left(J_{x,w}(0), \dot{J}_{x,w}(0)\right) = w$, where $\dot{J}_{x,w}$ is the covariant derivation of $J_{x,w}$ in the direction $\dot{\gamma}_x$. Recall that a Jacobi field J is called stable, if there is $C > 0$ such that for all $t \geqslant 0$,

$$|J(t)| \leqslant C.$$

By [7, proposition VI.A], given $x \in \mathcal{M}$, for any $Y_1 \in T_zZ$, there exists one and only one stable Jacobi field J along γ_x such that $J(0) = Y_1$.

For $x = (z, v) \in M'$, we have

$$T_xM' = \{(Y_1, Y_2) \in T_zZ \oplus T_zZ : \langle Y_2, v\rangle = 0\}.$$

The morphism π_* in [3] is just

$$w \in T_xM' \to J_{x,w}(0) \in T_zZ.$$

By [7, proposition VI.B], $w \in E_s(x) \oplus E_0(x)$ if and only if the Jacobi fields $J_{x,w}$ is stable. By the uniqueness of stable Jacobi fields, we see that $\pi_*|_{E_s\oplus E_0}$ is injective. □

Since E_0 is a trivial line bundle, our proposition implies immediately:

Corollary 1.6. *We have the isomorphism of smooth flat line bundles*

$$\mathscr{O}(E_s) \simeq \pi^*(\mathscr{O}(TZ)).$$

2. Proof of proposition 1.4

We use the notation of [5]. If $0 \leqslant k \leqslant n-1$, let $\mathcal{E}_0^k \subset \Lambda^k(T^*M)$ denote the subbundle of k-forms ω such that $\iota_V\omega = 0$, where ι denotes interior multiplication.

Let $\widetilde{\mathcal{E}}_0^k = \mathcal{E}_0^k \otimes \mathscr{O}(E_s)$. We consider the pullback ϕ_{-t}^* on sections of $\widetilde{\mathcal{E}}_0^k$. Note that the flow $(\phi_t)_{t\in\mathbb{R}}$ lifts to a flow $(\Phi_t)_{t\in\mathbb{R}}$ on $\mathcal{E}_0^k$. Indeed, for $p \in M$, $\omega \in \mathcal{E}_{0,p}^k$, $\Phi_t\omega \in \mathcal{E}_{0,\phi_t(p)}^k$ is defined for $v_1, \ldots, v_k \in T_{\phi_t(p)}M$ by

$$\Phi_t\omega(v_1, \ldots, v_k) = \omega\left((\mathrm{d}\phi_t|_p)^{-1}v_1, \ldots, (\mathrm{d}\phi_t|_p)^{-1}v_k\right). \tag{6}$$

Note that from the above formula, it is easy to check that $\iota_V\Phi_t\omega = 0$. Recall also that the flow $(\phi_t)_{t\in\mathbb{R}}$ lifts to a flow $\widetilde{\Phi}_t$ on $\mathscr{O}(E_s)$ (see [2]). For a section s in $\widetilde{\mathcal{E}}_0^k$, we have

$$\phi_{-t}^*s\,(p) = \left(\Phi_t \otimes \widetilde{\Phi}_t\right)(s(\phi_{-t}(p))). \tag{7}$$

Let $\chi \in C^\infty(M)$ be a smooth function whose support is contained in a small neighbourhood of K such that $\chi(x) = 1$ for all $x \in K$. We now invoke the Guillemin trace formula (see [11, pp 501–502], [5, appendix B], [3, (4.6)]) which says that the flat trace $\mathrm{tr}^\flat\chi\phi_{-t}^*\chi|_{C^\infty\left(M;\,\widetilde{\mathcal{E}}_0^k\right)}$ is a distribution on $(0, \infty)$ given by

$$\mathrm{tr}^\flat\chi\phi_{-t}^*\chi|_{C^\infty\left(M;\widetilde{\mathcal{E}}_0^k\right)} = \sum_{\gamma\subset K} \frac{T_\gamma^\sharp\ \mathrm{tr}^{\widetilde{\mathcal{E}}_{0,y}^k}\left(\Phi_{T_\gamma} \otimes \widetilde{\Phi}_{T_\gamma}\right)}{|\det(I - \mathcal{P}_\gamma)|}\delta_{t-T_\gamma}, \tag{8}$$

where the sum is taken over all the periodic trajectories γ in K with period T_γ and primitive period $T_\gamma^\sharp$, y is any point on γ, and $\mathcal{P}_\gamma = \mathrm{d}\phi_{-T_\gamma}|_{(E_s\oplus E_u)_y}$ is the linearized Poincaré map at y.

Note that as trace and determinant are invariant under conjugation, the right-hand side does not depend on y.

By [6], the trace of Φ_{T_γ} on $\mathcal{E}^k_{0,y}$ is just $\mathrm{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right)$. By [2], we may take trivializations ψ, $\widetilde{\psi}$ of E_s, $\mathscr{O}(E_s)$ in a neighbourhood of y and have the induced lifting on $\mathscr{O}(E_s)$ to be $\mathrm{sgn}\left(\det\left(\psi \mathrm{d}\phi_{T_\gamma}|_{E_{s,y}}\psi^{-1}\right)|\right)$. By definition we get this to be equal to

$$\mathrm{sgn}\left(\det \mathrm{d}\phi_{T_\gamma}|_{E_{s,y}}\right) = \mathrm{sgn}\left(\det \mathrm{d}\phi_{-T_\gamma}|_{E_{s,y}}\right) = \mathrm{sgn}\ \det\left(\mathcal{P}_\gamma|_{E_s}\right),$$

and as it is a map between one dimensional spaces, the trace is given by that expression as well. By the above consideration, we can rewrite [8] as

$$\mathrm{tr}^\flat \chi\phi^*_{-t}\chi|_{C^\infty\left(M;\widetilde{\mathcal{E}}^k_0\right)} = \sum_{\gamma\subset K} \frac{T^\sharp_\gamma\ \mathrm{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right)\mathrm{sgn}(\det \mathcal{P}_\gamma|_{E_s})}{|\det(I-\mathcal{P}_\gamma)|}\delta_{t-T_\gamma}. \tag{9}$$

Let us follow [4, section 3]. By [4, lemma 3.2], we may and we will assume that near K, $(\phi_t)_{t\in\mathbb{R}}$ is an open hyperbolic system in the sense of [3, assumptions (A1)–(A4)]. By [3, lemma 1.17], there is $C>0$ such that for all $t \geqslant 0$,

$$|\{\gamma\ \text{closed trajectory in}\ K : T_\gamma \leqslant t\}| \leqslant Ce^{Ct}. \tag{10}$$

For $\mathrm{Im}(\lambda) \gg 1$ big enough, set

$$\zeta_{K,k}(\lambda) = \exp\left(-\sum_{\gamma\subset K}\frac{T^\sharp_\gamma}{T_\gamma}\frac{\mathrm{tr}\left(\bigwedge^k \mathcal{P}_\gamma\right)\mathrm{sgn}(\det \mathcal{P}_\gamma|_{E_s})}{|\det(I-\mathcal{P}_\gamma)|}\mathrm{e}^{\mathrm{i}\lambda T_\gamma}\right). \tag{11}$$

Lemma 2.1. *For* $\mathrm{Im}(\lambda) \gg 1$ *big enough, we have*

$$\partial_\lambda \log \zeta_{K,k}(\lambda) = -i\int_0^\infty e^{\mathrm{i}\lambda t}\mathrm{tr}^\flat\chi\phi^*_{-t}\chi|_{C^\infty\left(M;\widetilde{\mathcal{E}}^k_0\right)}\mathrm{d}t. \tag{12}$$

The function $\zeta_{K,k}(\lambda)$ *has a holomorphic extension to* $\mathbb{C}$.

Proof. Let us first remark that by [9, 10], the right-hand side of [12] is well defined. Taking a logarithm and differentiating [11] and using Guillemin trace formula [9], we get [12]. The last part of the lemma follows from the arguments of [3, section 4]. □

Recall that for $\mathrm{Im}(\lambda) \gg 1$ big enough, we have

$$\zeta_K(\lambda) = \prod_{\gamma^\sharp\subset K}\left(1-\mathrm{e}^{\mathrm{i}\lambda T^\sharp_\lambda}\right) = \exp\left(-\sum_{\gamma\subset K}\frac{T^\sharp_\gamma}{T_\gamma}\mathrm{e}^{\mathrm{i}\lambda T_\gamma}\right). \tag{13}$$

Proposition 1.4 is a consequence of the following lemma. This lemma was stated in [2], but we restate and prove it for convenience.

Lemma 2.2. *The following identity of meromorphic functions on* $\mathbb{C}$ *holds,*

$$\zeta_K(\lambda) = \prod_{k=0}^{n-1}\left(\zeta_{K,k}(\lambda)\right)^{(-1)^{k+\dim E_s}}. \tag{14}$$

Proof. Following [5, (2.4) and (2.5)], since $\det(I-\mathcal{P}_\gamma)=\sum_{k=0}^{n-1}(-1)^k\mathrm{tr}\left(\bigwedge^k\mathcal{P}_\gamma\right)$, by [11, 13], it is enough to show

$$|\det(I-\mathcal{P}_\gamma)|=(-1)^{\dim E_s}\ \mathrm{sgn}\left(\det\mathcal{P}_\gamma|_{E_s}\right)\det(I-\mathcal{P}_\gamma). \tag{15}$$

Remark that

$$\begin{aligned}\det(I-\mathcal{P}_\gamma)&=\det(I-\mathcal{P}_\gamma|_{E_u})\det(I-\mathcal{P}_\gamma|_{E_s})\\&=(-1)^{\dim E_s}\ \det(I-\mathcal{P}_\gamma|_{E_u})\det(I-\mathcal{P}_\gamma^{-1}|_{E_s})\det(\mathcal{P}_\gamma|_{E_s}).\end{aligned} \tag{16}$$

As time is running in the negative direction, we have by [1] that the eigenvalues λ of $\mathcal{P}_\gamma|_{E_u}$ have $|\lambda|<1$, and the eigenvalues μ of $\mathcal{P}_\gamma^{-1}|_{E_s}$ have $|\mu|<1$. This gives any eigenvalues of $I-\mathcal{P}_\gamma|_{E_u}$ to be either $1-\lambda$ for $\lambda\in(-1,1)$ or conjugate pairs $1-\lambda, 1-\overline{\lambda}$ when λ is not real. In any case, we get by multiplying that

$$\det(I-\mathcal{P}_\gamma|_{E_u})>0$$

and similarly

$$\det(I-\mathcal{P}_\gamma^{-1}|_{E_s})>0.$$

Then taking signs in (16), we get (15). □

Remark. The key point of our argument is based on the smoothness of $\mathscr{O}(E_s)$. Thanks to this property, most of the analytic arguments in the proof of proposition 1.4 are reduced to [3]. In [2, section 2], Baladi and Tsujii used the orientation bundle in a different way for the flow with discrete time.

3. Vanishing order at zero on a contact three-manifold

In this section, we assume that M is a connected closed three-manifold with a contact form α, and that V is the associated Reeb vector field. We suppose also that the flow $(\phi_t)_{t\in\mathbb{R}}$ of V is Anosov. One such example would be when $M=S^*\Sigma$, the cosphere bundle of a connected closed surface Σ with negative (variable) curvature, and $(\phi_t)_{t\in\mathbb{R}}$ is geodesic flow.

The following result was proven in [6]:

Theorem 3.1. *If $(\phi_t)_{t\in\mathbb{R}}$ is a contact Anosov flow on a connected closed three-manifold with orientable E_u and E_s, the Ruelle zeta function has vanishing order at $\lambda=0$ equal to $b_1(M)-2$, where $b_1(M)$ denotes the first Betti number of M.*

The goal of this section is to determine the order of vanishing of ζ_R at 0 in the case that E_s, E_u are not orientable, and hence give a proof of theorem 2. We remark that since a contact manifold is orientable, orientability of E_s is equivalent to orientability of E_u.

3.1. The twisted cohomology

Let us recall some background and general facts on the twisted cohomology of a flat vector bundle. Let X be a closed manifold. Let F be a flat vector bundle on X with flat connection ∇. It induces a sheaf $\mathcal{F}$ on X defined by locally constant sections, i.e., if $U\subset X$ is an open set, then

$$\mathcal{F}(U)=\{s\in C^\infty(U;F|_U):\nabla s=0\}.$$

The twisted cohomology $H^\bullet(X;F)$ is defined by the cohomology of the sheaf $\mathcal{F}$ [10, section II.4.4]. They are the algebraic invariants which describe the rigidity properties of the global flat sections of F. Let $b_k(F)$ be the twisted Betti number

$$b_k(F) = \dim H^k(X;F).$$

If F is the trivial line bundle, we get the classical de Rham cohomology with real coefficients.

To evaluate $H^\bullet(X;F)$, one can use the twisted de Rham complex. Indeed, if we denote $F^k = \Lambda^k(T^*X) \otimes F$, the flat connection ∇ extends to an operator $d_k : C^\infty(X;F^k) \to C^\infty(X;F^{k+1})$ by Leibniz rule: if $\alpha \in C^\infty(X;\Lambda^k(T^*X))$ and $s \in C^\infty(X;F)$, we have

$$d_k(\alpha \cdot s) = \mathrm{d}\alpha \cdot s + (-1)^k \alpha \wedge \nabla s.$$

By the flatness of ∇, we have $d_{k+1}d_k = 0$, so that $(C^\infty(X;F^\bullet), d_\bullet)$ is a complex. By the de Rham isomorphism [10, théorème II.4.7.1], we have

$$H^k(X;F) = \ker\, d_k/\mathrm{Im}\, d_{k-1}. \tag{17}$$

As an analogue of [6, lemma 2.1], using the theory of elliptic operators, we can evaluate $H^\bullet(X;F)$ using the complex of twisted currents, or more generally twisted currents with wavefront conditions.

More precisely, let $\Gamma \subset T^*X$ be a closed cone. We denote by $\mathcal{D}'_\Gamma(X;F^k)$ the space of F^k-valued distributions whose wavefront set is contained in Γ (see [6, section 2.1]). By microlocality, we have

$$d_k :\ \mathcal{D}'_\Gamma\left(X;F^k\right) \to \mathcal{D}'_\Gamma\left(X;F^{k+1}\right).$$

For simplicity, we will write d sometimes.

Lemma 3.2. *If $u \in \mathcal{D}'_\Gamma\left(X;F^k\right)$ and $\mathrm{d}u \in C^\infty\left(X;F^{k+1}\right)$, then there exist $v \in C^\infty\left(X;F^k\right)$ and $w \in \mathcal{D}'_\Gamma\left(X;F^{k-1}\right)$ such that*

$$u = v + \mathrm{d}w.$$

In particular, if $u \in \mathcal{D}'_\Gamma(X;F)$ and $\mathrm{d}u \in C^\infty\left(X;F^1\right)$, then $u \in C^\infty(X;F)$.

Proof. Take a Riemannian metric on X and a Hermitian metric on F. Remark that these two metrics induce a fibrewise scalar product $\langle\cdot,\cdot\rangle$ on F^k. For $u, v \in C^\infty\left(X;F^k\right)$, we can define the L^2-product by

$$\langle u, v\rangle_{L^2(X;F^k)} = \int_X \langle u, v\rangle\, \mathrm{dvol}, \tag{18}$$

where dvol is a volume form. Let $\delta_{k+1} : C^\infty\left(X;F^{k+1}\right) \to C^\infty\left(X;F^k\right)$ be the formal adjoint of d with respect to the L^2-product (18). Define the twisted Hodge Laplacian by

$$\Delta_k = d_{k-1}\delta_k + \delta_{k+1}d_k : C^\infty\left(X;F^k\right) \to C^\infty\left(X;F^k\right).$$

Then Δ_k is an essentially self-adjoint second order elliptic differential operator. The remainder of the proof carries over identically from that of [6, lemma 2.1]. □

Remark that if F is the orientation bundle of certain vector bundle and $u \in C^\infty(X;F)$, then for $x \in X$, $|u(x)|^2$ is independent of the choice of trivializations. It defines a Hermitian metric on F.

3.2. Resonant state spaces

Let M be a connected three-dimensional closed manifold with a contact form $\alpha \in C^{\infty}(M, T^*M)$. Let V be the associated Reeb vector field. Then,

$$i_V\alpha = 1, \quad \iota_V \mathrm{d}\alpha = 0. \tag{19}$$

We assume that the flow $(\phi_t)_{t\in\mathbb{R}}$ associated to V is Anosov. Let $E_u^* \subset T^*M$ be the dual of E_s. We will apply the results of section 3.1 to the case where $(X, F, \Gamma) = (M, \mathcal{O}(E_s), E_u^*)$.

Since the flow $(\phi_t)_{t\in\mathbb{R}}$ is Anosov, we have $K = M$. For $0 \leqslant k \leqslant 2$, we write $\zeta_k = \zeta_{K,k}$. By (14), we have

$$\zeta_{\mathrm{R}}(\lambda) = \frac{\zeta_1(\lambda)}{\zeta_0(\lambda)\zeta_2(\lambda)}. \tag{20}$$

We consider the operator $P_k = -i\mathcal{L}_V$, where $\mathcal{L}_V$ (in a slight abuse of notation) denotes the natural action on sections of $\widetilde{\mathcal{E}_0^k}$, given by the Lie derivative on sections of $\mathcal{E}_0^k$ tensored with the flat connection on $\mathscr{O}(E_s)$. For $\mathrm{Im}\,\lambda \gg 1$ large enough, the integral $R_k(\lambda) = i\int_0^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}\phi_{-t}^*\mathrm{d}t$ converges and defines a bounded operator on the L^2-space; this is nothing more than the resolvent operator of P_k. Then by [6, section 2.3] we have that R_k extends meromorphically to the entire complex plane,

$$R_k(\lambda) : C^{\infty}\left(M; \widetilde{\mathcal{E}_0^k}\right) \to \mathcal{D}'\left(M; \widetilde{\mathcal{E}_0^k}\right).$$

More precisely, near $\lambda_0 \in \mathbb{C}$, we have

$$R_k(\lambda) = R_{k,H}(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(P_k - \lambda)^{j-1}\Pi_k}{(\lambda - \lambda_0)^j}$$

where $R_{k,H}$ is a holomorphic family defined near λ_0, $J(\lambda_0) \in \mathbb{N}$, and Π_k has rank $m_k(\lambda_0) < \infty$. By the arguments at the end of [5], we have that at λ_0, the function ζ_k has a zero of order $m_k(\lambda_0)$.

We define the space of *resonant states* at λ_0 to be

$$\mathrm{Res}_k(\lambda_0) = \left\{u \in \mathcal{D}'_{E_u^*}\left(M; \widetilde{\mathcal{E}_0^k}\right) : (P_k - \lambda_0)u = 0\right\}.$$

Then a special case of [6, lemma 2.2] gives the following:

Lemma 3.3. *Suppose P_k satisfies the semisimplicity condition:*

$$u \in \mathcal{D}'_{E_u^*}\left(M; \widetilde{\mathcal{E}_0^k}\right), \quad (P_k - \lambda_0)^2 u = 0 \implies (P_k - \lambda_0)u = 0.$$

Then $m_k(\lambda_0) = \dim \mathrm{Res}_k(\lambda_0)$.

Recall that we are trying to find the order at $\lambda = 0$ of ζ_{R}, which by (20) is simply

$$m_{\mathrm{R}}(0) = m_1(0) - m_0(0) - m_2(0). \tag{21}$$

We will compute each of these individually, by computing $\dim \mathrm{Res}_k(0)$ and checking that the semisimplicity condition in lemma 3.3 holds.

We begin with twisted 'zero-forms', which are just sections of the orientation bundle $\mathscr{O}(E_s)$.

Proposition 3.4. *If E_s is nonorientable, the space* $\mathrm{Res}_0(0)$ *is* $\{0\}$.

Proof. Suppose $u \in \mathrm{Res}_0(0)$, i.e.,

$$P_0 u = 0. \tag{22}$$

Since the flow $(\phi_t)_{t\in\mathbb{R}}$ preserves the contact volume form $\alpha \wedge \mathrm{d}\alpha$, $P_0 : C^\infty(M; \mathscr{O}(E_s)) \to C^\infty(M; \mathscr{O}(E_s))$ is a symmetric operator with respect to the L^2-product (18). By [6, lemma 2.3], $u \in C^\infty(M; \mathscr{O}(E_s))$. Using $\partial_t\left(\phi_{-t}^* u\right) = -\phi_{-t}^* \nabla_V u$ (where ∇ is the flat connection), we see that u is constant on the flow line: for all $t \in \mathbb{R}$,

$$u = \phi_{-t}^* u. \tag{23}$$

Let $(x, v) \in TM$. The pairing $\langle \mathrm{d}u(x), v\rangle$ is an element of $\mathscr{O}(E_s)_x$. By [7] and (23), we have

$$\langle \mathrm{d}u(x), v\rangle = \langle \phi_{-t}^*(\mathrm{d}u)(x), v\rangle = \widetilde{\Phi}_t\langle \mathrm{d}u(\phi_{-t}(x)), \mathrm{d}\phi_{-t}(x)v\rangle.$$

If $v \in E_u(x)$, then sending $t \to \infty$ gives $\langle \mathrm{d}u(x), v\rangle = 0$ by [1]. Similarly, if $v \in E_s(x)$, then sending $t \to -\infty$ gives $\langle \mathrm{d}u(x), v\rangle = 0$. This shows that

$$\mathrm{d}u|_{E_s\oplus E_u} = 0. \tag{24}$$

By Cartan's formula and by (22), we have $\iota_V \mathrm{d}u = 0$, i.e.,

$$\mathrm{d}u|_{E_0} = 0. \tag{25}$$

By (24) and (25), we have $\mathrm{d}u = 0$. So $u \in H^0(M; \mathscr{O}(E_s))$. Since E_s is nonorientable, we have $H^0(M; \mathscr{O}(E_s)) = 0$, so $u = 0$ and $\mathrm{Res}_0(0)$ is trivial. □

Corollary 3.5. *If E_s is nonorientable, the multiplicity for zero-forms is $m_0(0) = 0$.*

Proof. If $P_0^2(u) = 0$, then $P_0 u \in \mathrm{Res}_0(0)$. By proposition 3.4, $P_0 u = 0$, so $u \in \mathrm{Res}_0(0)$. This shows semisimplicity, so by lemma 3.3 we see that $m_0(0) = \dim \mathrm{Res}_0(0) = 0$. □

Proposition 3.6. *If E_s is nonorientable, the space* $\mathrm{Res}_2(0)$ *is* $\{0\}$.

Proof. We claim that

$$\alpha\wedge : \mathcal{E}_0^2 \to \mathcal{E}^3 \tag{26}$$

is a bundle isomorphism. Indeed, using (19), it is easy to see that the inverse of (26) is given by ι_V. Tensoring with $\mathscr{O}(E_s)$, we get a bundle isomorphism

$$\alpha\wedge : \widetilde{\mathcal{E}_0^2} \to \widetilde{\mathcal{E}^3}. \tag{27}$$

Let $u \in \mathrm{Res}_2(0)$. Since $\mathcal{E}^3$ is generated by $\alpha \wedge \mathrm{d}\alpha$, by (27), there is $v \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$ such that $\alpha \wedge u = v\alpha \wedge \mathrm{d}\alpha$. Applying ι_V and using $\iota_V u = 0$, we have $u = v\mathrm{d}\alpha$. Then

$$0 = P_2(u) = (P_0 v)\mathrm{d}\alpha.$$

But this gives $P_0 v = 0$, so by proposition 3.4 we have $v = 0$. Therefore, $u = 0$. □

The following is then clear for the same reason as corollary 3.5.

Corollary 3.7. *If E_s is nonorientable, the multiplicity for two-forms is $m_2(0) = 0$.*

We now turn to the case of P_1 acting on the space of twisted one-form-valued distributions $\mathcal{D}'_{E_u^*}\left(M;\ \widetilde{\mathcal{E}_0^1}\right)$. We can now state the analogous proposition for one-forms:

Proposition 3.8. *If E_s is nonorientable, the space* $\mathrm{Res}_1(0)$ *has dimension* $b_1(\mathscr{O}(E_s))$.

Proof. The proof is analogous to that of [6, lemma 3.4], but slightly easier due to the holomorphy of the resolvent R_0 near 0. Let $u \in \mathrm{Res}_1(0)$. Then $\mathrm{d}u \in \mathrm{Res}_2(0)$ by proposition 3.6, so $\mathrm{d}u = 0$. By lemma 3.2 there is a $\phi \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$ such that

$$u - \mathrm{d}\phi \in C^\infty\left(M; \widetilde{\mathcal{E}^1}\right), \quad \mathrm{d}(u - \mathrm{d}\phi) = 0.$$

We shall show that the map:

$$\Theta : u \mapsto [u - \mathrm{d}\phi] \in H^1(M; \mathscr{O}(E_s))$$

is well-defined, linear and bijective, which is enough to prove the lemma.

Well-Definedness and linearity. Suppose there is another section $\psi \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$ with $u - \mathrm{d}\psi \in C^\infty\left(M;\ \widetilde{\mathcal{E}^1}\right)$. Then subtracting gives $\mathrm{d}(\phi - \psi) \in C^\infty\left(M;\ \widetilde{\mathcal{E}^1}\right)$, so $\phi - \psi \in C^\infty(M; \mathscr{O}(E_s))$ by lemma 3.2. This shows that the map Θ is well-defined. It is also easy to see that Θ is linear.

Injectivity. If $\Theta(u) = 0$, then $u - d\phi$ is exact, so without loss of generality we can assume that $u = d\phi$. Combining with $\iota_V u = 0$, we get $\phi \in \mathrm{Res}_0(0)$, so $\phi = 0$ by proposition 3.4. Therefore $u = 0$, and this shows Θ to be injective.

Surjectivity. Let $v \in C^\infty\left(M;\ \widetilde{\mathcal{E}^1}\right)$ with $\mathrm{d}v = 0$. Then as $m_0(0) = 0$, the resolvent R_0 is holomorphic near 0. Take $\phi = iR_0(0)\iota_V v \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$. Then $P_0\phi = i\iota_V v$. This rearranges to $\iota_V(v + \mathrm{d}\phi) = 0$, so $v + \mathrm{d}\phi \in \mathrm{Res}_1(0)$. This gives that Θ is surjective, and completes the proof of our proposition. □

Proposition 3.9. *If E_s is nonorientable, the multiplicity for one-forms is $m_1(0) = b_1(\mathscr{O}(E_s))$.*

Proof. By lemma 3.3, we must only check that the semisimplicity condition is satisfied. Take $u \in \mathcal{D}'_{E_u^*}\left(M;\ \widetilde{\mathcal{E}_0^1}\right)$ such that $(P_1)^2 u = 0$. Then $v = \iota_V \mathrm{d}u \in \mathrm{Res}_1(0)$. It is enough to show that $v = 0$.

Recall that in the proof of proposition 3.8, we have seen that elements in $\mathrm{Res}_1(0)$ are closed. In particular,

$$\mathrm{d}v = 0. \tag{28}$$

Note that $\alpha \wedge \mathrm{d}u \in \mathcal{D}'_{E_u^*}(M;\ \widetilde{\mathcal{E}^3})$. We claim that

$$\alpha \wedge \mathrm{d}u = 0. \tag{29}$$

Indeed, there is some $a \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$ such that

$$\alpha \wedge \mathrm{d}u = a\,\alpha \wedge \mathrm{d}\alpha.$$

Since $\mathcal{L}_V(\alpha) = 0$, by (28), we have

$$(\mathcal{L}_V a)\alpha \wedge \mathrm{d}\alpha = \alpha \wedge \mathcal{L}_V(\mathrm{d}u) = \alpha \wedge \mathrm{d}\iota_V \mathrm{d}u = \alpha \wedge \mathrm{d}v = 0.$$

Then $\mathcal{L}_V a = 0$, so $a = 0$ by proposition 3.4. This gives (29).

Since $\alpha(V) = 1$, we have $(\alpha\wedge) \circ \iota_V + \iota_V \circ (\alpha\wedge) = \mathrm{id}$. By (29), we have

$$du = ((\alpha\wedge) \circ \iota_V + \iota_V \circ (\alpha\wedge))du = \alpha \wedge v. \tag{30}$$

By lemma 3.2 and by (28), there are $w \in C^\infty\left(M;\ \widetilde{\mathcal{E}^1}\right)$, $\phi \in \mathcal{D}'_{E_u^*}(M; \mathscr{O}(E_s))$ such that

$$v = w + d\phi, \quad dw = 0. \tag{31}$$

Then

$$\iota_V w = \iota_V(v - d\phi) = -\mathcal{L}_V \phi. \tag{32}$$

In particular, $\mathcal{L}_V \phi$ is smooth. We compute by Stokes' Theorem and by (30)–(32),

$$\begin{aligned} 0 &= \mathrm{Re} \int_M du \wedge \overline{w} = \mathrm{Re} \int_M \alpha \wedge d\phi \wedge \overline{w} = \mathrm{Re} \int_M \phi\overline{w} \wedge d\alpha \\ &= \mathrm{Re} \int_M \iota_V(\phi\overline{w})\, \alpha \wedge d\alpha = -\mathrm{Re} \int_M \phi(\overline{\mathcal{L}_V \phi})\alpha \wedge d\alpha \\ &= -\mathrm{Re}\, \langle \mathcal{L}_V \phi, \phi \rangle_{L^2(M; \mathscr{O}(E_s))}, \end{aligned}$$

where the fourth equality comes from the fact the

$$(\alpha\wedge) \circ \iota_V(\phi\overline{w} \wedge d\alpha) = ((\alpha\wedge) \circ \iota_V + \iota_V \circ (\alpha\wedge))(\phi\overline{w} \wedge d\alpha) = \phi\overline{w} \wedge d\alpha.$$

In the above formula, we use the fact that a product of two twisted forms is untwisted. By [6, lemma 2.3] we have $\phi \in C^\infty(M; \mathscr{O}(E_s))$, so $v \in C^\infty\left(M;\ \widetilde{\mathcal{E}_0^1}\right)$. Then by the same argument as in proposition 3.4 (see [6, lemma 3.5]) we have $v = 0$. □

Now theorem 2 is a consequence of (21), corollaries 3.5, 3.7, and proposition 3.9.

Let Σ be a connected negatively curved closed surface. Take $M = S^*\Sigma$. By corollary 1.6, we have

$$H^1(M; \mathscr{O}(E_s)) = H^1(M; \pi^* \mathscr{O}(T\Sigma)).$$

Proposition 3.10. *If Σ is a connected negatively curved closed surface (oriented or not), we have*

$$\dim H^1(M; \pi^* \mathscr{O}(T\Sigma)) = \dim H^1(\Sigma). \tag{33}$$

Proof. By the Gysin long exact sequence, we have the exact sequence

$$0 \longrightarrow\ H^1(\Sigma; \mathscr{O}(T\Sigma)) \xrightarrow{\pi^*} H^1(M; \pi^* \mathscr{O}(T\Sigma)) \xrightarrow{\pi_*} H^0(\Sigma) \xrightarrow{e\wedge} H^2(\Sigma; \mathscr{O}(T\Sigma)) \longrightarrow,$$

where π^* is the pullback, π_* is the integration along the fibre of $M \to \Sigma$, and $e \in H^2(\Sigma; \mathscr{O}(T\Sigma))$ is the Euler class of $T\Sigma$.

We claim that the last map

$$e\wedge : H^0(\Sigma) \to H^2(\Sigma; \mathscr{O}(T\Sigma))$$

in the Gysin exact sequence is an isomorphism. Indeed, since Σ is connected, we have $\dim H^0(\Sigma) = 1$, and by Poincaré duality, $\dim H^2(\Sigma; \mathscr{O}(T\Sigma)) = 1$. It is enough to show that $e \in H^2(\Sigma; \mathscr{O}(T\Sigma))$ is non zero, or equivalently $\int_\Sigma e \neq 0$. This is a consequence of the fact that

Σ has negative curvature, as $e = K\mu$ where μ is the Riemannian density and $K < 0$ is the Gauss curvature.

Therefore, we get an isomorphism

$$\pi^* : H^1(\Sigma; \mathscr{O}(T\Sigma)) \simeq H^1(M; \pi^*\mathscr{O}(T\Sigma)). \tag{34}$$

By Poincaré duality, we have

$$H^1(\Sigma; \mathscr{O}(T\Sigma)) \simeq \left(H^1(\Sigma)\right)^*. \tag{35}$$

By (34) and (35), we get (33). □

Now corollary 3 is a consequence of theorem 2 and proposition 3.10.

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