Final

December 12th

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 5 class material question (in Problem 1) and 4 independent problems. You don't have to do them in any particular order.
- Remember that using pens and writing clearly improve the readability after scanning.
- Remember to give all 14 pages back. It will make it easier to check that nothing was lost.

Problem 1 :

1. Show that any group of prime cardinal is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

2. Let R and S be rings and $f : R \to S$ be a ring homomorphism. Let $S_0 \leqslant S$ be a subring. Show that $f^{-1}(S_0) \subseteq R$ is a subring of R.

3. Let R be a commutative ring. Show that maximal ideals of R are prime.

4. Let R be a UFD. Show that the greater common divisor of any two $a, b \in R \setminus (R^\star \cup 0)$ exists.

5. Let F be a field. Show that any ideal in $F[X]$ is principal (the only result from class that you can use is the long division).

Problem 2 :
Let $(G, \cdot)$ be a group. It is said to be divisible if for all $x \in G$ and $n \in \mathbb{Z}_{>0}$, there exists $y \in G$ such that $y^n = x$.

1. Let F be characteristic zero field. Show that $(F, +)$ is a divisible group (beware of additive notation versus multiplicative notation).

2. Let F be an algebraically closed fields. Show that $(F^\star, \cdot)$ is a divisible group.

3. Let G be a divisible group and $H \trianglelefteq G$ be a normal subgroup. Show that G/H is divisible.

Problem 3 :
Let R be a unique factorization domain, $n \in \mathbb{Z}_{>1}$ and $a \in R$.

1. Let $b, c \in R$ be such that $ac^n = b^n$. Let $p \in R$ be irreducible such that $p|c$. Show that p divides b.

2. Let $q \in \text{Frac}(R)$ be such that $q^n = a$. Show that $q \in R$.

Problem 4 :
Let F be a field, $d \in \mathbb{Z}_{>0}$ and $a_0, \dots, a_d \in F$.

1. Let $P, Q \in F[X]$ have degree at most d. Assume that, for all $0 \leqslant i \leqslant d$, $P(a_i) = Q(a_i)$. Show that $P = Q$.

2. Show that there exists $P_i \in F[X]$ of degree at most d such that $P_i(a_i) = 1$ and $P_i(a_j) = 0$ if $j \neq i$.

3. For all $b_0, \dots, b_d \in F$, show that there exists a unique polynomial of degree at most d such that, for all $0 \leqslant i \leqslant d$, $P_i(a_i) = b_i$.

Problem 5 :
Let R be a ring, an element $e \in R$ is said to be idempotent if $e^2 = e$.

1. Let $e \in R$ be idempotent. Show that $1 - e$ is also idempotent.

2. Let $e \in R$ be idempotent. Show that $R \cong R/(e) \times R/(1 - e)$.

3. Let $e_1, \dots e_n \in R$ be idempotent. Assume that $\sum_{i=1}^n e_i = 1$ and that $e_i e_j = 0$ whenever $i \neq j$. Let $I_i := (e_j : j \neq i)$. Show that $R \cong \prod_{i=1}^n R/I_i$.

4. Let R_1 and R_2 be two rings. Find $e_1 \in R_1 \times R_2$ and $e_2 \in R_1 \times R_2$ such that e_1 and e_2 are idempotent, $e_1 + e_2 = 1$ and $R_1 \times R_2/(e_i) \cong R_i$ for $i = 1, 2$.