Final

December 13th

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 2 class material questions (in Problem 1) and 3 independent problems. You don't have to do them in any particular order.
- These exams will be scanned, so using a pen and writing clearly will make it much easier for me to grade your exams.

Problem 1 :
To answer those questions, you are not allowed to use results we proved in class, only the definitions.

1. Let $f : G \to H$ be a bijective group homomorphism. Show that $f^{-1} : H \to G$ is a group homomorphism.
2. Let R be a principal ideal domain and $I \subsetneq R$ be prime. Show that I is maximal.

Problem 2 :
Let G be a group of order 8.

1. Assume G is Abelian. Show that $G \simeq \mathbb{Z}/8\mathbb{Z}$ or $G \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. Assume that G is not Abelian. Show that $|\mathrm{Z}(G)| = 2$.
3. Assume that G is not Abelian and let $x \in G$ be order 4. Show that $\langle x \rangle \cap \mathrm{Z}(G) \neq \{1\}$ and hence $x^2 \in \mathrm{Z}(G)$.
4. Assume that G is not Abelian. Let $x \in G$ be order 4 and $y \in G \smallsetminus \langle x \rangle$ be order 2. Show that $xy = yx^{-1}$.
5. Assume that G is not Abelian and that there exists $x \in G$ of order 4 and $y \in G \smallsetminus \langle x \rangle$ of order 2. Show that $G \simeq \mathrm{D}_8$.

Problem 3 :
Let X be some set and $R = \mathbb{R}^X$ with coordinatewise addition and multiplication — i.e. $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$. We admit that R is a ring under these operations.

1. Let $f \in R$. Show that the following are equivalent:
   a) there exists $a \in X$ such that $f(a) = 0$;
   b) f is a zero divisor;
   c) f is not a unit.
2. Let $f_1, \ldots, f_n \in R$. Assume that for all $a \in X$, there exists $i \leqslant n$ such that $f_i(a) \neq 0$. Show that $\sum_{i \leqslant n} f_i^2 \in R^\star$. Conclude that $(f_1, \ldots, f_n) = R$.
3. For all $a \in X$, define $I_a := \{f \in R : f(a) = 0\}$. Show that I_a is a maximal ideal of R.

4. Let $I \subseteq R$ be a finitely generated maximal ideal. Show that there exists $a \in X$ such that $I = I_a$.

Problem 4 :

Let F be a field, $\overline{F}$ it algebraic closure and $P \in F[X]$ be a non-constant polynomial. Assume that P only has simple root in $\overline{F}$.

1. Show that if $Q \in F[X]$ is such that Q^2 divides P, then Q is a constant polynomial.
2. Show that there exist $k \in \mathbb{Z}_{>0}$ and finite extensions K_i of F, for $i = 1 \dots k$ such that $F[X]/(P)$ is isomorphic to $\prod_{i=1}^{k} F_i$.
3. Show that $\overline{F}[X]/(P)$ is isomorphic to $\prod_{i=1}^{\deg(P)} \overline{F}$.