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Homework 3

Due September 18th

Problem 1 :
Let G be a finite group.

1. For any $a \in G$, let $f_a(x) = a \cdot x$. Show that $a \mapsto f_a$ is an injective group homomorphism from G into S_G.
2. Show that every finite group is isomorphic to a subgroup of $S_{\mathbb{Z}_{>0}}$.

Problem 2 :
Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. Assume that for all $x \in G$, $\sigma(x) = x$ implies $x = 1$ and that $\sigma^2 = 1$ (in this equation, the product and identity are considered in the group $\mathrm{Aut}(G)$).

1. Show that the map $f : G \to G$ defined by $f(x) = x^{-1}\sigma(x)$ is a bijection.
2. Show that for all $x \in G$, $\sigma(x) = x^{-1}$.
3. Show that G is Abelian.

Problem 3 :
If G is an Abelian group, let $\mathrm{tor}(G) := \{x \in G : |x| < \infty\}$. It is called the torsion group of G. For all $n \in \mathbb{Z}_{>0}$, let $Z_n := \{e^{\frac{2ik\pi}{n}} : k \in \mathbb{Z}\} \subseteq \mathbb{C}$. Let $Z := \bigcup_n Z_n$.

1. Show that $\mathrm{tor}(G) \leqslant G$.
2. Show that $\mathrm{tor}(\mathbb{C}^\star) = Z$.
3. Pick some k dividing n. Show that the unique subgroup of Z_n of order k is Z_k.
4. Show that $Z_n \leqslant Z_m$ if and only if $n|m$.
5. Show that there does not exists $a_1, \ldots, a_k \in Z$ such that $Z = \langle a_1, \ldots, a_k \rangle$