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Solutions to homework 5

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Problem 1 :
Let m, $n \in \mathbb{Z}_{>0}$ be such that $\gcd(m,n) = 1$. Let G be a group of order mn, $H \leqslant G$ such that $|H| = n$ and $K \trianglelefteq G$.

1. Show that there exists m_0 dividing m such that $|HK| = m_0 n$.

   Solution: Since $K \trianglelefteq G$, $HK \leqslant G$ is a group. It follows that $|HK|$ divides mn. Also, since $H \leqslant HK$, n divides $|HK|$ and there exists $m_0 \in \mathbb{Z}$ such that $|HK| = m_0 n$. By the previous remark $m_0 n | mn$ and hence $m_0 | m$.

2. Show that there exists n_0 dividing n such that $|K| = m_0 n_0$.

   Solution: Let $n_0 = |H \cap K|$. Since $H \cap K \leqslant H$, we have that n_0 divides $n = |H|$. Moreover, $m_0 n = |HK| = \frac{|H||K|}{|H \cap K|} = \frac{n|K|}{n_0}$ and hence $|K| = \frac{m_0 n n_0}{n} = m_0 n_0$.

3. Show that $H \cap K$ is maximal among subgroups of K whose order divides n.

   Solution: Let $L \leqslant K$ be such that $H \cap K \leqslant L$ and $l := |L|$ divides n. Since $H \cap K \leqslant L \leqslant K$, we have that n_0 divides l which in turn divides $m_0 n_0$. So $l = n_0 l_0$ for some $l_0 \in \mathbb{Z}$ and $n_0 l_0$ divides $m_0 n_0$. Thus l_0 divides m_0 which divides m. But since l divides n, l_0 also divides n and hence it divides $\gcd(m,n) = 1$. It follows that $l_0 = 1$ and $|L| = n_0 = |H \cap K|$. Since $H \cap K \leqslant L$, we must have $H \cap K = L$ and $H \cap K$ is maximal.

4. Show that $\gcd(|HK/K|, |G/HK|) = 1$.

   Solution: We have $|HK/K| = \frac{|HK|}{|K|} = \frac{m_0 n}{m_0 n_0} = \frac{n}{n_0}$. On the other hand $|G/HK| = \frac{|G|}{|HK|} = \frac{mn}{m_0 n} = \frac{m}{m_0}$. Since $\frac{n}{n_0}$ divides n and $\frac{m}{m_0}$ divides m, their gcd divides $\gcd(m,n) = 1$ and therefore it is equal to 1.

5. Assume that $|G| = p^\alpha r$ where $\gcd(p,r) = 1$, $|K| = p^\beta s$ where $\gcd(p,s) = 1$ and $|H| = p^\alpha$. Show that $|H \cap K| = p^\beta$ and $|HK/K| = p^{\alpha-\beta}$.

   Solution: Let us apply the previous computations with $m = s$ and $n = p^\alpha$. Then $m_0 n_0 = p^\beta s$ where m_0 divides r which is coprime to p and hence p^β. It follows that $n_0 = p^\beta l$ for some $l \in \mathbb{Z}$. Since n_0 divides p^α, $l = p^\gamma$ for some $\gamma \in \mathbb{Z}$. Then $m_0 n_0 = m_0 p^{\beta+\gamma} = p^\beta s$ and hence $m_0 p^\gamma = s$. But $\gcd(s,p) = 1$ so $\gamma = 0$. It follows that $|H \cap K| = n_0 = p^\beta$ and $|HK/K| = \frac{n}{n_0} = p^{\alpha-\beta}$.

Problem 2 :
Let G be a group, $N \trianglelefteq G$ and $H \leqslant G$. Assume $H \cap N = \{1\}$.

1. Show that the map $f : N \times H \to NH$ defined by $f((n,h)) = n \cdot h$ is a bijection.

   Solution: The map f is surjective by definition of $NH = \{n \cdot h : n \in N \text{ and } h \in h\}$. Let us now show it is injective. Pick $n_1, n_2 \in N$ and $h_1, h_2 \in H$ and assume $n_1 \cdot h_1 = f(n_1, h_1) = f(n_2, h_2) = n_2 \cdot h_2$. Then $n_2^{-1} \cdot n_1 = h_2 \cdot h_1^{-1} \in N \cap H = \{1\}$. It follows that $n_2^{-1} \cdot n_1 = 1 = h_2 \cdot h_1^{-1}$ and hence $n_1 = n_2$ and $h_1 = h_2$.

2. Show that f is a group isomorphism if and only if $H \leqslant \mathrm{C}_G(N)$. Here $N \times H$ is considered as a group with the usual coordinatewise group law.

   Solution: Let us assume that f is a group homomorphism and pick $n \in N$ and $h \in H$. We have $n \cdot h = f(n,h) = f((1,h)\cdot(n,1)) = f(1,h)\cdot f(n,1) = h \cdot n$, so every element of H commutes with every element of N, i.e. $H \subseteq \mathrm{C}_G(N)$. Since both are subgroups of G, we do have $H \leqslant \mathrm{C}_G(N)$.

   Conversely, assume $H \leqslant \mathrm{C}_G(N)$ and pick $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Then $f((n_1,h_1)\cdot(n_2,h_2)) = f(n_1\cdot n_2, h_1\cdot h_2) = n_1\cdot n_2\cdot h_1\cdot h_2 = n_1\cdot h_1\cdot n_2\cdot h_2 = f(n_1,h_1)\cdot f(n_2,h_2)$.

3. Show that there exists a group homomorphism $\theta : H \to \mathrm{Aut}(N)$ such that for all $n \in N$ and $h \in H$, $h \cdot n = [\theta(h)](n)\cdot h$.

   Solution: Since $N \trianglelefteq G$, for all $n \in N$ and $h \in H$, $h\cdot n\cdot h^{-1} \in N$. So H acts on N by conjugation (the fact that it is indeed an action, is the same computation as always : $h_1\cdot h_2\cdot n\cdot h_2^{-1}\cdot h_1^{-1} = h_1\cdot h_2\cdot n(h_1\cdot h_2)^{-1}$). Let $\theta : H \to \mathrm{S}_N$ be the associated permutation representation. For all $n \in N$ and $h \in H$, we have $[\theta(h)](n) = h\cdot n\cdot h^{-1}$ and hence $h\cdot n = [\theta(h)](n)\cdot h$.

   Let us now check that for all $h \in H$, $\theta(h)$ is a group homomorphism. For all $n_1, n_2 \in N$, we have $[\theta(h)](n_1\cdot n_2) = h\cdot n_1\cdot n_2\cdot h^{-1} = h\cdot n_1\cdot h^{-1}\cdot h\cdot n_2\cdot h^{-1} = [\theta(h)](n_1)\cdot[\theta(h)](n_2)$. Since $\theta(h) \in S_N$, it follows that $\theta(h) \in \mathrm{Aut}(N)$. Since θ is already known to be a group homomorphism from H to S_N and $\mathrm{Aut}(N) \leqslant S_N$, we are done.

4. Let us define the operation on $N \times H$: $(n_1,h_1)\star(n_2,h_2) = (n_1\cdot[\theta(h_1)](n_2), h_1\cdot h_2)$. Show that $(N\times H, \star)$ is a group and that it is isomorphic to $(NH, \cdot)$.

   Solution: We have first to prove that $\star$ defines a group law on $N \times H$. Let us first prove associtivity. Pick $n_1, n_2, n_3 \in N$ and $h_1, h_2, h_3 \in H$. We have:

$$
\begin{aligned}
((n_1,h_1)\star(n_2,h_2))\star(n_3,h_3) &= (n_1\cdot[\theta(h_1)](n_2), h_1\cdot h_2)\star(n_3,h_3)\\
&= (n_1\cdot[\theta(h_1)](n_2)\cdot[\theta(h_1\cdot h_2)](n_3), h_1\cdot h_2\cdot h_3)\\
&= (n_1\cdot[\theta(h_1)](n_2)\cdot[\theta(h_1)]([\theta(h_2)](n_3)), h_1\cdot h_2\cdot h_3)\\
&= (n_1\cdot[\theta(h_1)](n_2\cdot[\theta(h_2)](n_3)), h_1\cdot h_2\cdot h_3)\\
&= (n_1,h_1)\star(n_2\cdot[\theta(h_2)](n_3), h_2\cdot h_3)\\
&= (n_1,h_1)\star((n_2,h_2)\star(n_3,h_3)).
\end{aligned}
$$

   Let us now prove that $1,1$ is the identity. Pick $n \in N$ and $h \in H$, we have $(n,h)\star(1,1) = (n\cdot[\theta(h)](1), h\cdot 1) = (n\cdot 1, h) = (n,h)$ and $(1,1)\star(n,h) = (1\cdot[\theta(1)](n), 1\cdot h) = (n,h)$. Finally, let us prove that $(\theta[h^{-1}](n^{-1}), h^{-1})$ is the inverse of n,h. We have:

$$
\begin{aligned}
(n,h)\star(\theta[h^{-1}](n^{-1}), h^{-1}) &= (n\cdot[\theta(h)](\theta[h^{-1}](n^{-1})), h\cdot h^{-1})\\
&= (n\cdot\theta[h\cdot h^{-1}](n^{-1}), 1)\\
&= (n\cdot n^{-1}, 1)\\
&= (1,1)
\end{aligned}
$$

   and

$$
\begin{aligned}
(\theta[h^{-1}](n^{-1}), h^{-1})\star(n,h) &= (\theta[h^{-1}](n^{-1})\cdot\theta[h^{-1}](n), h^{-1}\cdot h)\\
&= (\theta[h^{-1}](n^{-1}\cdot n), 1)\\
&= (\theta[h^{-1}](1), 1)\\
&= (1,1)
\end{aligned}
$$

Finally, let us check that f is a group homomorphism. Pick $n_1, n_2 \in N$ and $h_1, h_2 \in H$. We have:

$$\begin{aligned} f((n_1, h_1) \star (n_2, h_2)) &= f(n_1 \cdot [\theta(h_1)](n_2), h_1 \cdot h_2) \\ &= n_1 \cdot [\theta(h_1)](n_2) \cdot h_1 \cdot h_2 \\ &= n_1 \cdot h_1 \cdot n_2 \cdot h_2 \\ &= f(n_1, h_1) \cdot f(n_2, h_2). \end{aligned}$$

One other way of proceeding would be to first prove that $f((n_1, h_1) \star (n_2, h_2)) = f(n_1, h_1) \cdot f(n_2, h_2)$ and then use this, and the fact that f is bijective to prove that $(N \times H, \star)$ is a group. Some of the computations are a lot easier, associativity, for example.