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Solutions to homework 6

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Problem 1 :
Let R be a unitary commutative ring such that $1 \neq 0$ and $S \subseteq R$ be closed under multiplication (i.e. $\forall x, y \in S$, $xy \in S$) and contain 1. We define the relation E on $R \times S$ by $(a,s)E(b,t)$ if and only if there exists $x \in S$ such that $xat = xbs$.

1. Show that E is an equivalence relation.

   Solution: Let a, b, $c \in R$ and s, t, $u \in S$. We have $1as = 1as$ so $(a,s)E(a,s)$ and E is reflexive. If $(a,s)E(b,t)$, then there exists $x \in S$ such that $xat = xbs$ and $xbs = xat$ so $(b,t)E(a,s)$ and E is symmetric. Finally, assume $(a,s)E(b,t)$ and $(b,t)E(c,u)$, then there exists x and $y \in S$ such that $xat = xbs$ and $ybu = yct$. We then have $(xyt)au = yxbsu = (xyt)cs$ where $xyt \in S$, so $(a,s)E(c,u)$ and E is transitive.

2. Let R_S denote the set $(R \times S)/E$ (it is the set of E-classes). If $(a,s) \in R \times S$, we denote by $\overline{(a,s)} \in R_S$ the E-class of (a,s). Show that the map $(\overline{(a,s)}, \overline{(b,t)}) \mapsto \overline{(ab,st)}$ is well defined. We denote this map $\star$.

   Solution: Assume $(a_1,s_1)E(a_2,s_2)$ and $(b_1,t_1)E(b_2,t_2)$, then there exists x and $y \in S$ such that $xa_1s_2 = xa_2s_1$ and $yb_1t_2 = yb_2t_1$. It follows that $(xy)(a_1b_1)(s_2t_2) = (xy)(a_2b_2)(s_1t_1)$ and $\star$ is well defined.

3. Show that the map $(\overline{(a,s)}, \overline{(b,t)}) \mapsto \overline{(at+bs,st)}$ is well defined. We denote this map $\square$.

   Solution: Assume $(a_1,s_1)E(a_2,s_2)$ and $(b_1,t_1)E(b_2,t_2)$, then there exists x and $y \in S$ such that $xa_1s_2 = xa_2s_1$ and $yb_1t_2 = yb_2t_1$. It follows that $xy(a_1t_1+b_1s_1)t_2s_2 = xya_1t_1s_2t_2 + xyb_1s_1t_2s_2 = xya_2s_1t_1t_2 + xyb_2t_1s_1s_2 = xy(a_2t_2+b_2s_2)s_1t_1$, so $\square$ is well-defined.

4. Show that $(R_S, \square, \star)$ is a unitary commutative ring.

   Solution: Let us start by showing associativity of $\square$. We have

$$
\begin{aligned}
(\overline{(a,s)} \square \overline{(b,t)}) \square \overline{(c,u)} &= \overline{(at+bs,st)} \square \overline{(c,u)} \\
&= \overline{((at+bs)u+cst, stu)} \\
&= \overline{(atu+bsu+cst, stu)} \\
&= \overline{(atu+(bu+ct)s, stu)} \\
&= \overline{(a,s)} \square \overline{(bu+ct,tu)} \\
&= \overline{(a,s)} \square (\overline{(b,t)} \square \overline{(c,u)}).
\end{aligned}
$$

   We have commutativity of $\square$: $\overline{(a,s)} \square \overline{(b,t)} = \overline{(at+bs,st)} = \overline{(bs+at,ts)} = \overline{(b,t)} \square \overline{(a,s)}$; $\overline{(0,1)}$ is the additive identity: $\overline{(a,s)} \square \overline{(0,1)} = \overline{(a \cdot 1 + 0 \cdot s, s \cdot 1)} = \overline{(a,s)}$ and we have an additive inverse: $\overline{(a,s)} \square \overline{(-a,s)} = \overline{(as-as,s^2)} = \overline{(0,s^2)} = \overline{(0,1)}$. The last equality comes from the fact that $1 \cdot 0 \cdot 1 = 0 = 1 \cdot 0 \cdot s^2$.

Let us now show associativity of $\star$: $(\overline{(a,s)} \star \overline{(b,t)}) \star \overline{(c,u)} = \overline{(ab,st)} \star \overline{(c,u)} = \overline{(abc,stu)} = \overline{(a,s)} \star \overline{(bc,tu)} = \overline{(a,s)} \star (\overline{(b,t)} \star \overline{(c,u)})$. Also $\star$ is commutative: $\overline{(a,s)} \star \overline{(b,t)} = \overline{(ab,st)} = \overline{(ba,ts)} = \overline{(b,t)} \star \overline{(a,s)}$ and $\overline{(1,1)}$ is the multiplicative identity: $\overline{(a,s)} \star \overline{(1,1)} = \overline{(a \cdot 1, s \cdot 1)} = \overline{(a,s)}$.

Finally $\star$ distributes over $\square$:

$$\begin{aligned}
\overline{(a,s)} \star (\overline{(b,t)} \square \overline{(c,u)}) &= \overline{(a,s)} \star \overline{(bu+ct,tu)} \\
&= \overline{(a(bu+ct),stu)} \\
&= \overline{(abu+act,stu)} \\
&= \overline{(absu+acst,stsu)} \\
&= \overline{(ab,st)} \square \overline{(ac,su)} \\
&= (\overline{(a,s)} \star \overline{(b,t)}) \square (\overline{(a,s)} \star \overline{(c,u)}).
\end{aligned}$$

The fourth equality comes from the fact that $1 \cdot (absu+acst)stu = 1 \cdot (abu+act)s^2tu$.

5. Show that the map $a \mapsto \overline{(a,1)}$ is a unitary ring homomorphism from $\varphi : R \to R_S$.

   Solution: Let us show it respects addition: $\varphi(a+b) = \overline{(a+b,1)} = \overline{(a \cdot 1 + b \cdot 1, 1^2)} = \overline{(a,1)} \square \overline{(b,1)}$; multiplication: $\varphi(ab) = \overline{(ab,1)} = \overline{(ab,1^2)} = \overline{(a,1)} \star \overline{(b,1)}$ and multiplicative identity: $\varphi(1) = \overline{(1,1)}$.

6. Show that if S contains 0 then R_S is the trivial ring.

   Solution: Let $a \in R$ and $s \in S$, we have $0 \cdot a \cdot 1 = 1 \cdot 0 \cdot s$, it follows that $\overline{(a,b)} = \overline{(0,1)}$ and that S contains only one element.

7. Show that φ is not injective if and only if S contains a zero-divisor.

   Solution: Let us first assume that S contains a zero divisor s. So there exists $y \in R$ such that $sy = 0$. Now $s \cdot y \cdot 1 = 0 = s \cdot 0 \cdot 1$. So $\varphi(y) = \overline{(y,1)} = \overline{(0,1)} = \varphi(0)$. So φ is not injective.

   Conversely, assume φ is not injective, then there exists $y \in \ker(\varphi) \smallsetminus \{0\}$, i.e. $\varphi(y) = \overline{(y,1)} = \overline{(0,1)}$. Hence there exists s, S such that $s \cdot y = s \cdot y \cdot 1 = s \cdot 0 \cdot 1 = 0$. So $s \in S$ is a zero divisor.

8. Show that $R \smallsetminus \{0\}$ is closed under multipication if and only if R is an integral domain.

   Solution: Let us assume that R is an integral domain and let $s,t \in R \smallsetminus \{0\}$. Then because neither s or t are zero divisors, $st \neq 0$ and $st \in R \smallsetminus \{0\}$.

   Conversely, if $R \smallsetminus \{0\}$ is closed under multiplication and $s,t \in R \smallsetminus \{0\}$, then $st \neq 0$, so s is not a zero divisor and the only zero divisor in R is 0: R is an integral domain (we already know it is unitary, commutative and non trivial).

9. Assume that R is an integral domain. Show that $R_{(R \smallsetminus \{0\})}$ is a field.

   Solution: Let $\overline{(a,s)} \in R_{R \smallsetminus \{0\}}$. If $a = 0$, as we saw above, $\overline{(a,s)} = \overline{(0,1)}$. If follows that if $\overline{(a,s)} \neq \overline{(0,1)}$, we must have $a \neq 0$ and hence $\overline{(s,a)} \in R_{R \smallsetminus \{0\}}$. We have $\overline{(a,s)} \star \overline{(s,a)} = \overline{(as,as)} = \overline{(1,1)}$. The last equality comes from the fact that $1 \cdot as \cdot 1 = 1 \cdot 1 \cdot as$. So every non zero element in $R_{R \smallsetminus \{0\}}$ is invertible, i.e. it is a field.

10. Show that $\mathbb{Z}_{(\mathbb{Z}\setminus\{0\})}$ is isomorphic (as a unitary ring) to $\mathbb{Q}$.

Solution: Let $\varphi(\overline{(m,n)}) = mn^{-1}$. Let us show that φ is well defined. If $\overline{(m,n)} = \overline{(p,q)}$, then there exists $s \in \mathbb{Z}\setminus\{0\}$ such that $smq = snp$. But $\mathbb{Z}$ is an integral domain and $s \neq 0$ so we have $mq = np$ and thus $mn^{-1} = pq^{-1}$.

Let us now show that φ is a unitary ring homomorphism. We have $\varphi(\overline{(m,n)} \,\Box\, \overline{(p,q)}) = \varphi(\overline{(mq+pn,nq)}) = (mq+pn)(nq)^{-1} = mqq^{-1}n^{-1} + pnq^{-1}n^{-1} = mn^{-1} + pq^{-1} = \varphi(\overline{(m,n)}) + \varphi(\overline{(p,q)})$; also $\varphi(\overline{(m,n)} \star \overline{(p,q)}) = \varphi(\overline{(mp,nq)}) = (mp)(nq)^{-1} = mn^{-1}pq^{-1} = \varphi(\overline{(m,n)})\varphi(\overline{(p,q)})$ and $\varphi(\overline{1,1}) = 1\cdot 1^{-1} = 1$.

If $mn^{-1} = \varphi(\overline{(m,n)}) = 0$ then, because $n \neq 0$, we have $m = 0$ and $\overline{(m,n)} = \overline{(0,1)}$. So $\ker(\varphi) = \{\overline{(0,1)}\}$ and φ is injective. Finally, pick any $q \in \mathbb{Q}$. We have $q = mn^{-1} = \varphi(\overline{(m,n)})$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{Z}\setminus\{0\}$ so φ is surjective.