Midterm 1

February 13th

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 2 class material questions (in Problem 1) and 2 independent problems. You don't have to do them in any particular order.
- Remember that using a pen and writing clearly improves readability.

Problem 1 :

1. Let $f : G \to H$ be a group homomorphism and $H_0 \leqslant H$. Show that $f^{-1}(H_0) \leqslant G$.

2. Define what a cyclic group is and give an example of a cyclic group of every order (finite and infinite).

Problem 2 :

Let G be a group and $x, y \in G$, we define $[x, y] = x \cdot y \cdot x^{-1} \cdot y^{-1}$ and $[G] := \{[x, y] : x, y \in G\}$.

1. Let $f : G \to H$ be a group homomorphism and assume H is Abelian. Show that $[G] \subseteq \ker(f)$.

2. Show that G is Abelian if and only if $[G] = \{1\}$.

3. Show that, for all $n \geqslant 3$, $[D_{2n}] = \{r^{2i} : i \in \mathbb{Z}\}$.

Problem 3 :

Let n, $m \in \mathbb{Z}_{>0}$ be coprime, G be a group of order mn, $a \in G$ have order n and $b \in G$ have order m.

1. Show that $\langle a \rangle \cap \langle b \rangle = \{1\}$.

2. For all i_1, i_2, j_1 and $j_2 \in \mathbb{Z}$, show that $a^{i_1} b^{j_1} = a^{i_2} b^{j_2}$ if and only if $i_1 \equiv i_2 \mod n$ and $j_1 \equiv j_2 \mod m$.

3. Show that every elements of G is of the form $a^i b^j$ for some $i,\ j \in \mathbb{Z}$.