Midterm 2

March 20th

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 2 class material questions (in Problem 1) and 2 independent problems. You don't have to do them in any particular order.
- Remember that using a pen and writing clearly improves readability.
- You may use, without proving it, that if $K \leqslant H \leqslant G$ and $[G:H]$ and $[H:K]$ are finite, then $[G:K] = [G:H][H:K]$.

Problem 1 :

1. Let G be acting on a set X, show that $x \sim y$ defined by $\exists g \in G\ g \star x = y$ is an equivalence relation on X.

2. State Cauchy's theorem.

Problem 2 :
Let G be a group and $H \leqslant G$ be a subgroup. For all $g, l \in G$, we define $g \star lH = glH$.

1. Show that $\star$ is a group action of G on G/H.

2. Let $K = \{g \in G : \forall l \in G,\ g \star lH = lH\}$. Show that $K \leqslant H$ and $K \trianglelefteq G$.

3. Let $n = [G : H]$ and $k = [H : K]$. Show that k divides $(n-1)!$.

4. Assume that $|G|$ is finite and n is the smallest prime dividing $|G|$. Show that $k = 1$. Conclude that $H \trianglelefteq G$.

Problem 3 :
Let G be a group and $K_i \trianglelefteq G$ for $i = 1, 2$ be such that $[G : K_i] < \infty$.

1. Show that $[K_1 : K_1 \cap K_2] < \infty$. Conclude that $[G : K_1 \cap K_2] < \infty$.