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Solutions to the final exam

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Problem 1 (Atomless Boolean algebras) **:**

1. Let $a \in A \setminus \{0\}$. Because a is not an atom, there exists $c \in A$ such that $0 < c < a$. Let $c_1 = c$ and $c_2 = a \cap c^{\mathbf{c}}$, then $c_1 \cup c_2 = c \cup (a \cap c^{\mathbf{c}}) = (c \cup a) \cap (c \cup c^{\mathbf{c}}) = a \cap 1 = a$ and $c_1 \cap c_2 = c \cap a(\cap c^{\mathbf{c}}) = a \cap 0 = 0$.

2. We proceed by indcution on $|w|$. If $|w| = 0$, then we must have $\varepsilon_w = 1$. Otherwise, let us assume that $|w| = n$ and ε_w has been built. If $\varepsilon_w \cap a_n \neq 0$ and $\varepsilon_w \cap a_n^{\mathbf{c}} \neq 0$, then we must have $\varepsilon_{w1} = \varepsilon_w \cap a_k$ and $\varepsilon_{w0} = \varepsilon_w \cap a_k^{\mathbf{c}}$. Let us check these two values work. They are, by hypothesis, non zero, $\varepsilon_{w0} \cap \varepsilon_{w1} = (\varepsilon_w \cap a_k^{\mathbf{c}}) \cap (\varepsilon_w \cap a_k) = \varepsilon_W \cap 0 = 0$ and $\varepsilon_{w0} \cup \varepsilon_{w1} = (\varepsilon_w \cap a_k^{\mathbf{c}}) \cup (\varepsilon_w \cap a_k) = \varepsilon_w \cdot a_k^{\mathbf{c}} + \varepsilon_w \cdot a_k + \varepsilon_w \cdot a_k^{\mathbf{c}} \cdot \varepsilon_w \cdot a_k = \varepsilon_w \cdot (1 + a_k) + \varepsilon_w \cdot a_k = \varepsilon_w$.

   Finally, if one of them is zero, by question 1 we find c_0 and $c_1 \neq 0$ such that $c_0 \cup c_1 = \varepsilon_w$ and $c_0 \cap c_1 = 0$, So we can set $\varepsilon_{w0} = c_0$ and $\varepsilon_{w1} = c_1$.

3. For all $n \leqslant k$, we prove, by induction on $n - k$, that $\varepsilon_{w_k(f)} \leqslant \varepsilon_{w_n(f)}$. If $n = k$, this is obvious. Otherwise, by construction we have both $\varepsilon_{w1} \leqslant \varepsilon_w$ and $\varepsilon_{w0} \leqslant \varepsilon_w$ so $\varepsilon_{w_{n+1}(f)} \leqslant \varepsilon_{w_n(f)} \leqslant \varepsilon_{w_k(f)}$, by induction.

   Let us now assume that $a \cap \varepsilon_{w_k(f)} = 0$, the for all $n \geqslant k$, $a \cap \varepsilon_{w_n(f)} \leqslant a \cap \varepsilon_{w_k(f)} = 0$ and so $a \cap \varepsilon_{w_n(f)} = 0$. If $a \cap \varepsilon_{w_k(f)} \neq 0$, for all $n \leqslant k$, $0 < a \cap \varepsilon_{w_k(f)} \leqslant \varepsilon_{w_n(f)}$.

4. Let us first show that one of the statements must hold. Let us assume that there exists $k \in \mathbb{N}$ such that $\varepsilon_{w_k(f)} \cap a = 0$, the by the previous question, for all $n \geqslant k$, $\varepsilon_{w_n(f)} \cap a = 0$. If we also had $\varepsilon_{w_n(f)} \cap a^{\mathbf{c}} = 0$, by the same computation as above, we would have $\varepsilon_{w_n(f)} = (\varepsilon_{w_n(f)} \cap a) \cup (\varepsilon_{w_n(f)} \cap a^{\mathbf{c}}) = 0$, a contradiction. It follows that for all $n \geqslant k$, $\varepsilon_{w_n(f)} \cap a^{\mathbf{c}} \neq 0$. By the previous question, the equality also holds if $n \leqslant k$.

   Let us now show that those two cases cannot happen at the same time. Let k be such that $a = a_k$. If $\varepsilon_{w_{k-1}(f)} \cap a \neq 0$ and $\varepsilon_{w_{k-1}(f)} \cap a^{\mathbf{c}} \neq 0$, by construction (bercause $|w_{k-1}(f)| = k$), we have $\varepsilon_{w_{k-1}(f)1} = \varepsilon_{w_{k-1}(f)} \cap a$ and $\varepsilon_{w_{k-1}(f)0} = \varepsilon_{w_{k-1}(f)} \cap a^{\mathbf{c}}$. If $f(k) = 1$, then $\varepsilon_{w_k(f)} = \varepsilon_{w_{k-1}(f)} \cap a$ and $\varepsilon_{w_k(f)} \cap a^{\mathbf{c}} = 0$. If $f(k) = 0$, then $\varepsilon_{w_k(f)} = \varepsilon_{w_{k-1}(f)} \cap a^{\mathbf{c}}$ and $\varepsilon_{w_k(f)} \cap a = 0$.

   It follows that either one of the statements is false for some $n < k$ or one of the statements is false for $n = k$. Hence both cannot be true for all n.

5. For all $n \in \mathbb{N}$, $\varepsilon_{w_n(f)} \cap 1 = \varepsilon_{w_n(f)} \neq 0$ so $1 \in U_f$, and $\varepsilon_{w_n(f)} \cap 0 = 0$ so $0 \notin U_f$. Let a and $b \in U_f$, then, for all n, $\varepsilon_{w_n(f)} \cap a \neq 0$ and $\varepsilon_{w_n(f)} \cap b \neq 0$. By question 4, there exists n and $m \in \mathbb{N}$ such that $\varepsilon_{w_n(f)} \cap a^{\mathbf{c}} = 0 = \varepsilon_{w_m(f)} \cap b^{\mathbf{c}}$. By question 3, taking the minimal one, we may assume that $m = n$. We have $\varepsilon_{w_n(f)} \cap (a \cap b)^{\mathbf{c}} = (\varepsilon_{w_n(f)} \cap a^{\mathbf{c}}) \cup (\varepsilon_{w_n(f)} \cap b^{\mathbf{c}}) = 0$ and hence, by question 4 again, for all $n \in \mathbb{N}$, $\varepsilon_{w_n(f)} \cap (a \cap b) \neq 0$ and $a \cap b \in U_f$.

   Finally, let $a \in A$. By question 4, either for all $n \in \mathbb{N}$, $\varepsilon_{w_k(f)} \cap a \neq 0$ and $a \in U_f$ or for all $n \in \mathbb{N}$, $\varepsilon_{w_k(f)} \cap a^{\mathbf{c}} \neq 0$ and $a^{\mathbf{c}} \in U_f$. So U_f is an ultrafilter.

6. Let us proceed by induction on n. As $1 = \varepsilon_\varnothing \in U$, the case $n = 0$ is taken care of. Let us assume that w_n is built. We have $\varepsilon_{w_n} \in U$ and $\varepsilon_{w_n0} \cup \varepsilon_{w_n1} = \varepsilon_{w_n}$. If $\varepsilon_{w_n0} \notin U$ and $\varepsilon_{w_n1} \notin U$, then $\varepsilon_{w_n0}^{\mathbf{c}} \in U$ and $\varepsilon_{w_n1}^{\mathbf{c}} \notin U$ and hence $\varepsilon_{w_n}^{\mathbf{c}} = \varepsilon_{w_n0} \cup \varepsilon_{w_n1}^{\mathbf{c}} = \varepsilon_{w_n0}^{\mathbf{c}} \cap \varepsilon_{w_n1}^{\mathbf{c}} \in U$, a contradiction. So ther exists $i \in \{0, 1\}$, such that $\varepsilon_{w_ni} \in U$. Let $w_{n+1} = w_n i$.

7. As we have seen in question 5, h is indeed a map from $\{0,1\}^{\mathbb{N}} \to \mathcal{S}(A)$. Let us show that it is injective. Let $f, g : \mathbb{N} \to \{0,1\}$ be distinct and let k be the minimal value at which they differ. We may assume that $f(n) = 1$. For all $k \in \mathbb{N}$, $\varepsilon_{w_n(f)} \cap \varepsilon_{w_k(f)} = \varepsilon_{w_{\min(n,k)}(f)} \neq 0$ so $\varepsilon_{w_n(f)} \in U_f$, but $\varepsilon_{w_n(f)} \cap \varepsilon_{w_n(g)} = \varepsilon_{w_{n-1}(f)1} \cap \varepsilon_{w_{n-1}(f)0} = 0$ so $\varepsilon_{w_n(g)} \notin U_f$. By symmetry, $\varepsilon_{w_n(g)} \in U_g$, so $U_f \notin U_g$.

   Finally, let us prove that h is surjective. Let U be an ultrafilter on A. By the previous question there exists $(w_n)_{n\in\mathbb{N}}$ such that $\varepsilon_{w_n} \in U$. Let $f(n)$ be the $n+1$-th letter of w_k for $k > n$ (this is well defined because w_n is a prefix of w_k of length n). Then $w_n(f) = w_{n-1}$. We claim that $h(f) = U$.

   Let $a \in h(f)$, there exists n such that $\varepsilon_{w_n(f)} \cap a^{\mathrm{c}} = 0$ and hence $a^{\mathrm{c}} \notin U$, i.e. $a \in U$ as U is an ultrafilter. Conversely, if $a \in U$, then $\varepsilon_{w_n(f)} \cap a \in U$ cannot be zero and thus $a \in h(f)$.

Problem 2 (Model theory of $\mathbb{Z}$) **:**

1. Let c be a new constant and T be the $\mathcal{L}(c)$-theory:

$$T = \mathrm{Th}(\mathcal{Z}) \cup \{\exists x\, x \cdot \overline{n} = c : n \in \mathbb{Z} \smallsetminus \{0\}\}.$$

   Let $\mathcal{M}^\star \models T$, then the reduct of $\mathcal{M}^\star$ is elementarily equivalent to $\mathcal{Z}$ and if $a = c^{\mathcal{M}}$, then $\mathcal{M} \models \exists x\, x \cdot \overline{n} = c$ for all $n \in Zz \smallsetminus \{0\}$ and hence c is divisble by all $\overline{n}$ for $n \in Zz \smallsetminus \{0\}$. So it suffices to show that T is consistant. By compactness, it suffices to show that any finite $T_0 \subseteq T$ is consistent. Let T_0 be such a finite theory then $T_0 \subseteq \mathrm{Th}(\mathcal{Z}) \cup \{\exists x\, x \cdot \overline{n_i} = c : 0 < i < k\}$. Let $n = \max(n_i)!$, then, taking $c = n$, we make $\mathcal{Z}$ into a model of T_0. That concludes the proof.

2. We proceed by contraposition. Let us assume that $\mathcal{M} \models \exists x\, \neg\varphi(x)$. Then, because $\mathcal{M} \equiv \mathcal{Z}$, we also have $\mathcal{Z} \models \exists x\, \neg\varphi(x)$. In particular, there exists $n \in \mathbb{Z}$ such that $\mathcal{Z} \models \neg\varphi(n)$ and hence $\mathcal{Z} \models \neg\varphi(\overline{n})$. As $\mathcal{M} \equiv \mathcal{Z}$, we have that $\mathcal{M} \models \neg\varphi(\overline{n})$ for some $n \in \mathbb{Z}$.

3. Once again, we prove the contraposition. Let $\varphi(x)$ be such that $\mathcal{M} \models \varphi(a)$ if and only if $a = \overline{n}^{\mathcal{M}}$ for some $n \in \mathbb{Z}$. By the previous question, we have $\mathcal{M} \models \forall x\, \varphi(x)$ and hence every $a \in M$ is equal to some $\overline{n}^{\mathcal{M}}$. Let $f : \mathcal{Z} \to \mathcal{M}$ be defined by $f(n) = \overline{n}^{\mathcal{M}}$, it is easy to check that f is an embedding (because $\mathcal{M} \equiv \mathcal{Z}$) and we have just shown it is surjective. It follows that $\mathcal{M}$ is isomorphic to $\mathcal{Z}$.

Problem 3 (Axiomatizability of equivalence relations) **:**

1. Let T_E be the theory that contains:
   - $\forall x\, xEx$;
   - $\forall x \forall y\, xEy \to yEx$;
   - $\forall x \forall y \forall z\, (xEy \wedge yEz) \to xEz$.

   Then the models of T_E are exactly the $\mathcal{L}$-structures where E is an equivalence relation, so this class is finitely axiomatizable.

2. This is not axiomatizable either. Let $\psi_n = \exists x_1 \dots \exists x_n \bigwedge_{i\neq j} \neg x_i E x_j$. Let T be an $\mathcal{L}$-theory whose models are exactly the $\mathcal{L}$-structures where E is an equivalence relation with finitely many classes. Then $T' = T \cup \{\psi_n : n \in \mathbb{N}_{>0}\}$ is consistent. Indeed, by compactness, it suffices to show that any finite $T_0 \subseteq T'$ is consistent. But $T_0 \subseteq T \cup \{\psi_n : n \leqslant n_0\}$ for some $n_0 \in \mathbb{N}_{>0}$ and that theory is clearly consistent (take any equivalence relation with n_0-classes). So T' is consistent but a model of T' is a model of T with infinitely many classes, a contradiction.

3. This is not axiomatizable. Let c be a new constant and $\varphi_n = \exists x_1 \ldots \exists x_n \bigwedge_i cEx_i \wedge \bigwedge_{i\neq j} \neg x_i = x_j$. Let us assume T is an $\mathcal{L}$-theory whose models are exactly the $\mathcal{L}$-structures where E is an equivalence relation whose classes are finite. Then $T' = T \cup \{\varphi_n : n \in \mathbb{N}_{>0}\}$ is consistent. Indeed, by compactness, it suffices to prove that finite $T_0 \subseteq T'$ are consistent, but $T_0 \subseteq T \cup \{\varphi_n : n \leqslant n_0\}$ for some $n_0 \in \mathbb{N}_{>0}$ and that theory clearly has a model (take E to have only one class of cardinality n_0 and c to be any point in E). So T' is consistent, but a model of T' is a model of T so every class is finite and the class of c is infinite, a contradiction.

4. Let $T_{2,\infty}$ be the theory that consists of:

   - T_E;
   - $\exists x_1 \exists x_2 \neg x_1 E x_2 \wedge (\forall x_3\, (x_3 E x_1 \vee x_3 E x_2))$;
   - for all $n \in \mathbb{N}_{>0}$, the formula $\theta_n := \forall x \exists y_1 \ldots \exists y_n \bigwedge_j xEy_j \wedge \bigwedge_{i\neq j} y_i \neq y_j$.

   Then any model of $T_{2,\infty}$ is an equivalence relation with two equivalence classes that are both infinite. So this class is axiomatizable.

   However, it is not finitely axiomatisable. Indeed, let T' be some finite theory whose models are exactly the equivalence relation with two equivalence classes that are both infinite. Taking the conjonction of the formulas in T', we may assume $T' = \{\varphi\}$. We have $T_{2,\infty} \vDash \varphi$ and hence, by compactness, there is a finite $T_0 \subseteq T_{2,\infty}$ such that $T_0 \vDash \varphi$. Let n_0 be the largest n such that θ_n appears in T_0, then any equivalence relation with two classes of cardinality n_0 are models of T_0 and hence of φ, a contradiction.

5. Let $\chi_n(x) := \exists y_1 \exists y_n \bigwedge_j xEy_i \wedge \bigwedge_{i\neq j} y_i \neq y_j \wedge (\forall z\, (zEx \to \bigvee_j z = y_j))$ which states that the class of x has size exactly n. Let $\zeta_n = \exists x\, \chi_n(x) \wedge (\forall y\, (\chi_n(y) \to yEx))$ whcih states that there is exactly one class of cardinality n. Let $T_{\mathbb{N}}$ be the theory consisting of:

   - T_E;
   - For all $n \in \mathbb{N}_{>0}$, ζ_n.

   Then the models of $T_{\mathbb{N}}$ have exactly one class of each finite (non zero) cardinality, so this class is axiomatisable. It is not finitely axiomatisable. Indeed, if there exists a formula ψ such that the models of ψ have exactly one class of each finite (non zero) cardinality, then $T_{\mathbb{N}} \vDash \psi$ and hence there exists some finite $T_0 \subseteq T_{\mathbb{N}}$ such that $T_0 \vDash \psi$. Let n_0 be the largest n such that $\zeta_n \in T_0$, then equivalence relation with exactly one class of each cardinality smaller or equal to n_0 is a model of T_0 and thus of ψ, a contradiction.

Problem 4 (λ-calculus) **:**

1. If A is a propositional variable $(A \to A) \to A$ is not a tautology, so no λ-term can have type $(A \to A) \to A$ in the empty context.

2. Let t be a normal λ-term such that $\vdash t : (A \to A) \to (A \to A)$ holds. Let us discuss what can be the last rule applied to get that statement. It cannot be (Ax) because the context is empty. And it cannot be $(\to_E)$ because t being normal, there would exists $x \in V$ and normal $t_1, \ldots t_n \in \Lambda$ such that $t = (\ldots((x)t_1)\ldots)t_n$. But the variable x would be free in t but not in the context (which is empty), a contradiction. So the last rule applied is $(\to_I)$, $t = \lambda f, u$ and $f : A \to A \vdash u : A \to A$ holds.

   Let us now assume that u does not star with a λ. Then $u = (\ldots((y)t_1)\ldots)t_n$ where $y \in V$ and $t_1, \ldots t_n \in \Lambda$ are normal. Because y is free in u and the only variable that appears in the context is f, we must have $y = f$ and the only possible rules to prove the statement $f : A \to A \vdash u : A \to A$ is n applications of $(\to_E)$. So there exists $A_i \in T$ such that $f : A \to A \vdash f : A_1 \to (A_2 \to \ldots (A_n \to (A \to A)) \ldots)$ which only holds (by the (Ax) rule) if $n = 0$. So $t = \lambda f\, f$.

Let us now assume that $u = \lambda x\, v$ where v is a normal λ-term. Then, $\{f : A \to A, x : A\} \vdash v : A$ holds. Let us prove by induction on v that the only normal λ-terms such that $\{f : A \to A, x : A\} \vdash v : A$ are the t_n where $t_0 = x$ and $t_{n+1} = (f)t_n$. Let us discuss what are the possibilities for the last rule applied to prove $\{f : A \to A, x : A\} \vdash v : A$. Because A is not of the form $B \to C$, it cannot be $(\to_I)$. So v does not start with a λ and $v = (\ldots((y)v_1)\ldots)v_n$ for some $y \in V$ and normal λ-terms v_i. But then the only possible rules to prove $\{f : A \to A, x : A\} \vdash v : A$ is n applications of $(\to_E)$ and we have types A_i such that $\{f : A \to A, x : A\} \vdash y : A_1 \to (A_2 \to \ldots (A_n \to A)\ldots)$.

The only two variables in the context are x and f so y must be one of them. If $y = x$, then $n = 0$ and $v = t_0$. If $y = f$, then $n = 1$, $A_1 = A$ and $\{f : A \to A, x : A\} \vdash v_i : A$ holds. By induction $v_1 = t_n$ for some n and $v = (f)v_1 = (f)t_n = t_{n+1}$.

So the only possibilities are $t = \lambda f\, f$ and $t = \lambda f \lambda x\, t_n$.

3. We proceed by induction on t. If $t = x$, then the only subterm of t is t itself and the statement is obvious. If $t = \lambda x\, u$ then the subterms of t are either t itself or substerms of u. As we have $\Gamma \vdash t : A$, $A = B \to C$ and $\Gamma \cup \{x : B\} \vdash u : C$ holds. By induction, the statement holds for any subterm of u and it clearly holds for t. If $t = (u)v$, then there is some type B such that $\Gamma \vdash u : B \to A$ and $\Gamma \vdash v : B$ both hold. The subterms of t are t itself, subterms of u and subterms of v and in all these three cases the statement holds (by induction in the last two cases).

4. We have:

$$\begin{aligned}(Y)t &\to_\beta (\lambda x\,(t)(x)x)\lambda x\,(t)(x)x \\ &\to_\beta (t)(\lambda x\,(t)(x)x)\lambda x\,(t)(x)x\end{aligned}$$

and

$$(t)(Y)t \to_\beta (t)(\lambda x\,(t)(x)x)\lambda x\,(t)(x)x$$

So these two terms are β-equivalent.

5. There is no such type and context. To prove that statemant, by (the converse of) question 2, it suffices to prove that there is no $A \in T$ and a context Γ such that $(x)x : A$ holds (it is a subterm of Y).

If such a context and type exist, there also exist B such that $\Gamma \vdash x : B \to A$ and $\gamma \vdash x : B$ hold. Therefore there is a type C such that $x : C \in \Gamma$ and $B \to A = C = B$ (so that we can apply the (Ax) rule). But there is no type A and B such that $A = B \to A$.