Silvain Rideau
1091 Evans
silvain.rideau@berkeley.edu
www.normalesup.org/~srideau/eng

Homework 1

Due September 10th

Problem 1 (Tautologies) **:**
We have to do truth tables:

1.

A	B	$[A \to B]$	$[[A \to B] \wedge A]$	$[[A \to B] \wedge A] \to B]$
0	0	1	0	1
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1

2.

A	B	C	$[A \to B]$	$[C \to A]$	$[A \to B] \vee [C \to A]$
0	0	0	1	1	1
1	0	0	0	1	1
0	1	0	1	1	1
1	1	0	1	1	1
0	0	1	1	0	1
1	0	1	0	1	1
0	1	1	1	0	1
1	1	1	1	1	1

3.

A	B	C	$[A \wedge B]$	$[A \wedge B] \wedge C$	$[B \wedge C]$	$A \wedge [B \wedge C]$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
1	1	0	1	0	0	0
0	0	1	0	0	0	0
1	0	1	0	0	0	0
0	1	1	0	0	1	0
1	1	1	1	1	1	1

And we can check that column 5 and 7 are the same.

4.

A	B	$[A \wedge B]$	$A \vee [A \wedge B]$
0	0	0	0
1	0	0	1
0	1	0	0
1	1	1	1

And we can check that column 1 and 4 are the same.

5.

A	B	C	$[A \wedge B]$	$[A \wedge B] \to C$	$[B \to C]$	$A \to [B \to C]$
0	0	0	0	1	1	1
1	0	0	0	1	1	1
0	1	0	0	1	0	1
1	1	0	1	0	0	0
0	0	1	0	1	1	1
1	0	1	0	1	1	1
0	1	1	0	1	1	1
1	1	1	1	1	1	1

And we can check that column 5 and 7 are the same.

Problem 2 (Independent formulas) **:**

1. Let us prove both implications:

(a) ⇒ (b) Let us assume that A and B are logically equivalent. Let φ be a formula such that $A \vDash \varphi$ and $\delta \in \{0,1\}^P$ be an assignement. Let us assume that δ staisfies B. For any $\psi \in A$, we have $B \vDash \psi$ and hence δ satisfies ψ. We have just shown that δ satisfies A. It follows, because $A \vDash \varphi$, that δ satisfies φ. We conclude by symmetry that $B \vDash \varphi$ implies $A \vDash \varphi$.

(b) ⇒ (a) Let us assume that $A \vDash \varphi$ if and only if $B \vDash \varphi$. Now, let $\varphi \in A$, then $A \vDash \varphi$ and hence $B \vDash \varphi$. We conclude by symmetry.

2. We proceed by induction on the cardinality of A. Note that if $A = \varnothing$ then it is logically independent and logically equivalent to itself. Now assume we have proved the question for $|A| = n$ and let A be such that $|A| = n + 1$. If A is logically independent and logically equivalent to itself so $B = A$. Now, if A is not logically independent, there exists $\varphi \in A$ such that $A \setminus \varphi \vDash \varphi$. Then $A \setminus \varphi$ is logically equivalent to A and, by induction there exists $B \subseteq A \setminus \varphi$ such that B is logically independent and logically equivalent to $A \setminus \varphi$ and hence to A.

3. Let $A = \{\bigwedge_{i=0}^{n} X_i : n \in \mathbb{N}\}$ and let $B \subseteq A$ have at least two elements. Then B contains $\bigwedge_{i=0}^{n_1} X_i$ and $\bigwedge_{i=0}^{n_2} X_i$ for some $n_1 \leqslant n_2$. But $\bigwedge_{i=0}^{n_2} X_i \vDash \bigwedge_{i=0}^{n_1} X_i$ and so B cannot be independent. Hence if $B \subseteq A$ is independent, $|B| = 1$, i.e. $B = \bigwedge_{i=0}^{n} X_i$ for some n. But one can check that $\bigwedge_{i=0}^{n} X_i \nvDash \bigwedge_{i=0}^{n+1} X_i$. Indeed, consider δ defined by $\delta(X_i) = 1$ if $i \leqslant n$ and $\delta(X_{n+1}) = 0$, then δ satisfies $\bigwedge_{i=0}^{n} X_i$, but not $\bigwedge_{i=0}^{n+1} X_i$. It follows that B is not logically equivalent to A.

Problem 3 (Totally ordered sets) **:**

1. Let $A(S) = \{\neg X_{s,s} : s \in S\} \cup \{[X_{s,t} \wedge X_{t,u}] \to X_{s,u} : s,t,u \in S\} \cup \{X_{s,t} \vee X_{t,s} : s,t \in S \text{ distinct}\} \cup \{X_{s,t} : s,t \in S \text{ such that } s < t\}$. Let $\delta \in \{0,1\}^P$ satisfy $A(S)$, then for every $s \in S$, $(\neg X_{s,s})_\delta = 1$ and hence $\delta(X_{s,s}) = 0$. It follows that $s <_\delta s$ does not hold.

   Moreover, let s, t and $u \in S$ be such that $s <_\delta t$ and $t <_\delta u$. Then by definition of $<_\delta$, we have $\delta(X_{s,t}) = 1$ and $\delta(X_{t,u}) = 1$. It follows from the semantic of $\to$ that if $([X_{s,t} \wedge X_{t,u}] \to X_{s,u})_\delta = 1$, then $\delta(X_{s,u}) = 1$ and hence $s <_\delta u$. We have just proved that $<_\delta$ is an order.

   Now, let s and $t \in S$ be distinct, then $(X_{s,t} \vee X_{t,s})_\delta = 1$ and hence either $\delta(X_{s,t}) = 1$ (in which case $s <_\delta t$) or $\delta(X_{t,s}) = 1$ (in which case $t <_\delta s$). It follows that $<_\delta$ is a total order.

   Finally, let s, t in S be such that $s < t$, then $\delta(X_{s,t}) = 1$ and hence $s <_\delta t$, i.e. $<_\delta$ extends $<$.

2. If there is a total order on S extending $<$ then its restriction to any finite S_0 is a total order on S_0 extending $<$. So one implication is trivial. Let us prove the other one and let us assume that for every S_0, $(S_0, <)$ can be extended to a total order on S_0.

   By the previous question, there exists a total order on S extendeng $<$ if the set $A(S)$ is satisfiable. By compactness, $A(S)$ is satisfiable if anf only if every finite subset of $A(S)$ is. Let $A_0 \subseteq A(S)$ be finite and let $S_0 \subseteq S$ be the set of s such that $X_{s,t}$ or $X_{t,s}$ appear in A_0 for some $t \in S$. Then, because A_0 is finite, S_0 is finite too. By hypothesis, $(S_0, <)$ can be extended to a total order on S_0 and so by the previous question $A(S_0)$ is satisfiable. But it is easy to check that $A_0 \subseteq A(S_0)$ so A_0 is also satisfiable. That concludes the proof.