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Homework 1

Due September 10th

Problem 1 (Tautologies) **:**
Prove that the following formulas are tautologies:

1. $[[A \to B] \wedge A] \to B]$;
2. $[A \to B] \vee [C \to A]$.

Prove that the following formulas are logically equivalent:

3. $[A \wedge B] \wedge C$ and $A \wedge [B \wedge C]$;
4. A and $A \vee [A \wedge B]$;
5. $[A \wedge B] \to C$ and $A \to [B \to C]$.

Problem 2 (Independent formulas) **:**
Let A and B be two sets of formulas. We say that A implies B if for all $\varphi \in B$, $A \vDash \varphi$. We say that A is equivalent to B is A implies B and B implies A. And finally, we say that A is logically independent if for every $\varphi \in A$, $A \smallsetminus \{\varphi\} \nvDash \varphi$.

1. Show that the following are equivalent:
   (a) A and B are logically equivalent;
   (b) for all formula $\varphi \in F$, $A \vDash \varphi$ if and only if $B \vDash \varphi$.
2. Show that if A is finite, then there exists an independent set $B \subseteq A$ which is logically equivalent to A.
3. Does the infinite set $\{\bigwedge_{i=0}^{n} X_i : n \in \mathbb{N}\}$ have an equivalent and independent subset (the X_i are propositional variables)?

Problem 3 (Totally ordered sets) **:**
Recall that a (strict) order on a set S is a binary relation $<$ on S such that:

- For all $s \in S$, we do not have $s < s$;
- For all $s, t, u \in S$, if $s < t$ and $t < u$ then $s < u$.

An order $(S, <)$ is said to be total if for every s and $t \in X$, if $s \neq t$, we have $s < t$ or $t < s$. Let $<_1$ and $<_2$ be two orders on S, we say that $<_2$ extends $<_1$ if for all s and $t \in S$, if $s <_1 t$ then $s <_2 t$.

Let $(S, <)$ be some ordered set and let $P = \{X_{s,t} : s, t \in S\}$. Let $\delta : P \to \{0, 1\}$ be an assignement. We define the relation $<_\delta$ on S by $s <_\delta t$ if and only if $\delta(X_{s,t}) = 1$.

(i) Find a set of formulas A (with variables in P) such that A is satisfied by δ if and only if $<_\delta$ is a total order extending $<$.

(ii) Show that there exists a total order $<'$ on S extending $<$ if and only if for every finite $S_0 \subseteq S$, the order $(S_0, <)$ can be extended to a total order on S_0.

In fact statement (ii) is always true and hence so every order can be extended to a total order.