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Homework 2

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Problem 1 :

1. Let $a_n = |F_n|$. Then $a_0 = |F_0| = |P| = 6$ and $a_{n+1} = |F_{n+1}| = |F_n| + |\{\neg\varphi : \varphi \in F_n\}| \cup |\{[\varphi_1 \square \varphi_2] : \varphi_i \in F_n \text{ and } \square \in \{\wedge, \vee \rightarrow, \leftrightarrow\}\}| = 2a_n + 4a_n^2 \leqslant 5a_n^2$.

   Let us show, by induction on n that $a_n \leqslant 5^{2^n-1}6^{2^n}$. We have $a_0 = 6 = 5^{2^0-1}6^{2^0}$ and if we assume $a_n \leqslant 5^{2^n-1}6^{2^n}$ then $a_{n+1} \leqslant 5(5^{2^n-1}6^{2^n})^2 = 5^{1+2(2^n-1)}6^{2\cdot 2^n} = 5^{2^{n+1}-1}6^{2^{n+1}}$.

2. There are $2^{2^6} = 2^{64}$ functions from $\{0,1\}^P$ to $\{0,1\}$ and $5^{2^3-1}6^{2^3} \leqslant 30^{2^3} \leqslant 32^8 = 2^{40}$ formulas of height at most 3. So there are less than 2^{40} possible interpretations of formulas of height at most 3 hence. But we know that every function from $\{0,1\}^P$ to $\{0,1\}$ is a possible interpretation. It follows that some functions f can only interpret formulas of height at least 4.

Problem 2 :

1. We have to show that $\vee$ can be interpreted using $\rightarrow$ and $\neg$. But $X \vee Y$ is logically equivalent to $\neg X \rightarrow Y$. Hence $\{\rightarrow, \neg\}$ is complete.

   We proved in classe that $\{\neg\}$ is not complete so there only remains to show that $\{\rightarrow\}$ is not complete. But $X \rightarrow X$ is logically equivalent to X so the only function $\{0,1\} \rightarrow \{0,1\}$ that is the interpretation of a formula containing only $\rightarrow$ is the identity. Thus $\{\rightarrow\}$ is not complete.

2. Let us prove by induction on formulas in two variables X and Y containing only $\leftrightarrow$ and $\neg$ that the interpretation of such a formula takes an even number of times the value 1.

   The formulas X and Y take the value 1 half the time (i.e when X, respectively Y, is true), that is 2 times out of 4.

   If the formula φ has this property then because there are 4 possible assignements, the interpretation of φ also takes the value 0 an even number of time. Hence the interpretation of $\neg\varphi$ takes the value 1 an even number of times too.

   Let φ_1 and φ_2 be two formulas whose interpretations take the value 1 an even number of times. Let $a_{\epsilon_1,\epsilon_2}$ be the number of times φ_1 takes value ϵ_1 while φ_2 takes value ϵ_2. Then, by hypothesis $a_{1,1} + a_{1,0}$ (the number of times φ_1 is evaluated to 1) is even and so is $a_{0,0} + a_{1,0}$ (the number of times φ_2 is evaluated to 0). It follows that $a_{1,1} + 2a_{1,0} + a_{0,0}$ is even and hence so is $a_{1,1} + a_{0,0}$ which the number of times $\varphi_1 \leftrightarrow \varphi_2$ takes the value 1.

   But there are functions from $\{0,1\}^2 \rightarrow \{0,1\}$ that take the value 1 an odd number of times. So $\{\neg, \leftrightarrow\}$ cannot be complete.

3. The previous question yields such a set of connectives. There are exactly 8 binary connectives whose interpretation takes value 1 an even number of times. They are the interpretation of the following functions with variables in X and Y:

   - X;
   - Y;
   - $\neg X$;
   - $\neg Y$;
   - $X \leftrightarrow Y$;
   - $\neg X \leftrightarrow Y$;
   - $X \leftrightarrow \neg Y$;
   - $\neg X \leftrightarrow \neg Y$.

Because the set $\{\neg, \leftrightarrow\}$ is not complete, this set of 8 connectives cannot be either.

There are other ways of building such a set of connectives, let me give two others (and I am sure you can come up with many more).

Consider the 8 binary connectives that send $(1, 1)$ to 1. Then one can show by an easy induction that any composition of these connectives also send $(1, \ldots, 1)$ to 1. In particular these formulas cannot interpret $\neg$. Similarly one could the 8 binary connectives that send $(0, 0)$ to 0.