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Homework 3

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Problem 1 :

1. Here is a derivation of $\vdash (X \to Y) \to ((Y \to Z) \to (X \to Z))$:

$$\text{(}\to_I\text{)}\ \dfrac{\text{(}\to_I\text{)}\ \dfrac{\text{(}\to_I\text{)}\ \dfrac{\text{(}\to_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{Y \to Z \vdash Y \to Z} \qquad \text{(}\to_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{X \to Y \vdash X \to Y} \qquad \text{(Ax)}\ \dfrac{}{X \vdash X}}{\{X \to Y, X\} \vdash Y}}{\{X \to Y, Y \to Z, X\} \vdash Z}}{\{X \to Y, Y \to Z\} \vdash X \to Z}}{X \to Y \vdash (Y \to Z) \to (X \to Z)}}{\vdash (X \to Y) \to ((Y \to Z) \to (X \to Z))}$$

2. Here is a derivation of $\{X \to Y, X \to Z\} \vdash X \to (Y \wedge Z)$:

$$\text{(}\to_I\text{)}\ \dfrac{\text{(}\wedge_I\text{)}\ \dfrac{\text{(}\to_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{X \to Y \vdash X \to Y} \qquad \text{(Ax)}\ \dfrac{}{X \vdash X}}{\{X \to Y, X\} \vdash Y} \qquad \text{(}\to_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{X \to Z \vdash X \to Z} \qquad \text{(Ax)}\ \dfrac{}{X \vdash X}}{\{X \to Z, X\} \vdash Z}}{\{X \to Y, X \to Z, X\} \vdash Y \wedge Z}}{\{X \to Y, X \to Z\} \vdash X \to (Y \wedge Z)}$$

3. Assuming we have a derivation of $\Gamma \cup \{A\} \vdash \neg A$, here is a derivation of $\Gamma \vdash \neg A$:

$$\text{(}\vee_E\text{)}\ \dfrac{\begin{array}{c}\vdots \\ \Gamma \cup \{A\} \vdash \neg A\end{array} \qquad \text{(Ax)}\ \dfrac{}{\neg A \vdash \neg A} \qquad \text{(ExMid)}\ \dfrac{}{\vdash A \vee \neg A}}{\Gamma \vdash \neg A}$$

4. Assuming we have a derivation of $\varphi \vdash \psi$, here is a derivation of $\neg\psi \vdash \neg\varphi$:

$$\text{(}\vee_E\text{)}\ \dfrac{\text{(}\neg_E\text{)}\ \dfrac{\begin{array}{c}\vdots \\ \varphi \vdash \psi\end{array} \qquad \text{(Ax)}\ \dfrac{}{\neg\psi \vdash \neg\psi}}{\{\neg\psi, \varphi\} \vdash \neg\varphi} \qquad \text{(Ax)}\ \dfrac{}{\neg\varphi \vdash \neg\varphi} \qquad \text{(ExMid)}\ \dfrac{}{\vdash \varphi \vee \neg\varphi}}{\neg\psi \vdash \neg\varphi}$$

5. Assuming we have a derivation of $\neg\psi \vdash \neg\varphi$, here is a derivation of $\varphi \vdash \psi$:

$$\text{(}\vee_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{\psi \vdash \psi} \qquad \text{(}\neg_E\text{)}\ \dfrac{\text{(Ax)}\ \dfrac{}{\varphi \vdash \varphi} \qquad \begin{array}{c}\vdots \\ \neg\psi \vdash \neg\varphi\end{array}}{\{\varphi, \neg\psi\} \vdash \psi} \qquad \text{(ExMid)}\ \dfrac{}{\vdash \psi \vee \neg\psi}}{\varphi \vdash \psi}$$

Problem 2 :

1. Applying the $(\iff_I)$ rule, we get that $\Gamma \vdash \psi \iff \theta$. By the completeness theorem, $\Gamma \vDash \psi \iff \theta$. In particular, if $\delta \in \{0,1\}^P$ satisfies Γ, then $\psi_\delta = \theta_\delta$. Hence if $\varphi = \theta$, then $\varphi_{\psi/\theta} = \psi$ and we do have $\varphi_\delta = \theta_\delta = \psi_\delta = (\varphi_{\psi/\theta})_\delta$. We now prove the general case (i.e. $\varphi \neq \theta$) by induction on φ.
   - If $\varphi = X$ (and $\varphi \neq \theta$), $\varphi_{\psi/\theta} = X = \varphi$ and we do have $\varphi_\delta = (\varphi_{\psi/\theta})_\delta$.
   - If $\varphi = \neg\chi$ (and $\varphi \neq \theta$), then $\varphi_{\psi/\theta} = \neg(\chi_{\psi/\theta})$. By induction $(\chi_{\psi/\theta})_\delta = \chi_\delta$ and hence $(\varphi_{\psi/\theta})_\delta = f_\neg((\chi_{\psi/\theta})_\delta) = f_\neg(\chi_\delta) = \varphi_\delta$.

- If $\varphi = [\varphi_1 \,\Box\, \varphi_2]$ (and $\varphi \neq \theta$), then $\varphi_{\psi/\theta} = [(\varphi_1)_{\psi/\theta} \,\Box\, (\varphi_2)_{\psi/\theta}]$ and by induction $((\varphi_i)_{\psi/\theta})_\delta = (\varphi_i)_\delta$ for $i = 1, 2$. It follows that $(\varphi_{\psi/\theta})_\delta = f_\Box(((\varphi_1)_{\psi/\theta})_\delta, ((\varphi_2)_{\psi/\theta})_\delta) = f_\Box((\varphi_1)_\delta, (\varphi_2)_\delta) = \varphi_\delta$.

2. If $\Gamma \vdash \varphi$, then by completeness $\Gamma \vDash \varphi$. Let δ satisfy Γ, then we have $\varphi_\delta = 1$ and, by the previous question, we also have $(\varphi_{\psi/\theta})_\delta = 1$. We have just proved that $\Gamma \vDash \varphi_{\psi/\theta}$. By completeness, we have $\Gamma \vdash \varphi_{\psi/\theta}$.

Problem 3 :

1. Assume that $\Gamma \vDash \bot$ and let $\delta \in \{0,1\}^P$ satisfy Γ. Then $\bot_\delta = 0$ by definition, and $\bot_\delta = 1$ because $\Gamma \vDash \bot$, a contradiction. It follows that Γ there are no such δ and hence $\Gamma \vDash \varphi$ for any formula φ.

2. a) Here is a derivation of $\vdash \bot \leftrightarrow (\varphi \wedge \neg\varphi)$:

$$(\leftrightarrow_I)\,\dfrac{(\bot_E)\,\dfrac{(\mathrm{Ax})\,\dfrac{}{\bot \vdash \bot}}{\bot \vdash \varphi \wedge \neg\varphi} \qquad (\neg_E)\,\dfrac{(\wedge_{EL})\,\dfrac{(\mathrm{Ax})\,\dfrac{}{\varphi \wedge \neg\varphi \vdash \varphi \wedge \neg\varphi}}{\varphi \wedge \neg\varphi \vdash \varphi} \qquad (\wedge_{ER})\,\dfrac{(\mathrm{Ax})\,\dfrac{}{\varphi \wedge \neg\varphi \vdash \varphi \wedge \neg\varphi}}{\varphi \wedge \neg\varphi \vdash \neg\varphi}}{\varphi \wedge \neg\varphi \vdash \bot}}{\vdash \bot \leftrightarrow (\varphi \wedge \neg\varphi)}$$

   b) By the $(\leftrightarrow_I)$ rule, we have to show that $\neg\varphi \vdash \varphi \to \bot$ and $\varphi \to \bot \vdash \neg\varphi$ hold. Here is a derivation of $\neg\varphi \vdash \varphi \to \bot$:

$$(\to_I)\,\dfrac{(\neg_E)\,\dfrac{(\mathrm{Ax})\,\dfrac{}{\varphi \vdash \varphi} \qquad (\mathrm{Ax})\,\dfrac{}{\neg\varphi \vdash \neg\varphi}}{\{\neg\varphi, \varphi\} \vdash \bot}}{\neg\varphi \vdash \varphi \to \bot}$$

   And here is a derivation of $\varphi \to \bot \vdash \neg\varphi$:

$$(\vee_E)\,\dfrac{(\bot_E)\,\dfrac{(\to_E)\,\dfrac{(\mathrm{Ax})\,\dfrac{}{\varphi \to \bot \vdash \varphi \to \bot} \qquad (\mathrm{Ax})\,\dfrac{}{\varphi \vdash \varphi}}{\{\varphi \to \bot, \varphi\} \vdash \bot}}{\{\varphi \to \bot, \varphi\} \vdash \neg\varphi} \qquad (\mathrm{Ax})\,\dfrac{}{\neg\varphi \vdash \neg\varphi} \qquad (\mathrm{ExMid})\,\dfrac{}{\vdash \varphi \vee \neg\varphi}}{\varphi \to \bot \vdash \neg\varphi}$$