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Homework 3

Due September 24th

Problem 1 (Derivations) **:**
Prove (by providing a derivation and not by using the completeness theorem) that the following hold:

1. $\vdash (X \to Y) \to ((Y \to Z) \to (X \to Z))$;
2. $\{X \to Y, X \to Z\} \vdash X \to (Y \wedge Z)$.

Let $\Gamma \subseteq F$, φ and $\psi \in F$. Prove the following (without using the completness theorem):

3. If $\Gamma \cup \{A\} \vdash \neg A$ then $\Gamma \vdash \neg A$;
4. If $\varphi \vdash \psi$ holds, then $\neg\psi \vdash \neg\varphi$ holds;
5. If $\neg\psi \vdash \neg\varphi$ holds then $\varphi \vdash \psi$ holds.

Problem 2 (Substitutions) **:**
Let $\Gamma \subseteq F$, φ, ψ and $\theta \in F$ be such that $\Gamma \cup \{\psi\} \vdash \theta$, $\Gamma \cup \{\theta\} \vdash \psi$.

1. Let δ satisfy Γ, then $(\varphi_{\psi/\theta})_\delta = (\varphi)_\delta$.
2. Assume $\Gamma \vdash \varphi$ then $\Gamma \vdash \varphi_{\psi/\theta}$.

Problem 3 :
We want to add a new symbol $\perp$ to our logic (for the always false formula). So now formulas are word over the alphabet $P \wedge \{\neg, \wedge, \vee, \to, \leftrightarrow, \perp\}$ and $\perp$ is a formula (and we still close under the same thing as before). For example, $[\perp \to X] \wedge Y$ is now a formula. We expand the notion of semantics to these new formulas by defining $\perp_\delta = 0$ for every assignement δ (and interpretation of more complicated formulas is defined by induction as usual). We also a new deduction rule (i.e. the set of valid deductions $\Gamma \vdash \varphi$ is closed under all the rules we add before plus this new one):

$$(\perp_E)\ \frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi}$$

where $\Gamma \subseteq F$ and $\varphi \in F$.

1. Show that the new rule is sound (i.e. if its premise hold for $\models$ then its conclusion also holds for $\models$).
2. Let $\varphi \in F$ (φ might contain $\perp$), show that the following hold:
   a) $\vdash \perp \leftrightarrow (\varphi \wedge \neg\varphi)$;
   b) $\vdash \neg\varphi \leftrightarrow (\varphi \to \perp)$.