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Solutions to Homework 5

Due October 8th

Problem 1 :

1. Let us first prove that for any quantifier free formula $\varphi(x_1,\ldots,x_n)$ and $a_1\ldots,a_n \in N$, we have $\mathcal{N} \vDash \varphi(a_1,\ldots,a_n)$ if and only if $\mathcal{M} \vDash \varphi(a_1,\ldots,a_n)$. In fact, we first have to show that for any term $t(x_1,\ldots,x_n)$, $t^{\mathcal{N}}(a_1,\ldots,a_n) = t^{\mathcal{M}}(a_1,\ldots,a_n)$, but that is an easy proof by induction: If $t = x_i$, then $t^{\mathcal{M}}(a_1,\ldots,a_n) = a_i = t^{\mathcal{N}}(a_1,\ldots,a_n)$, if $t = c$, $t^{\mathcal{M}}(a_1,\ldots,a_n) = c^{\mathcal{M}} = c^{\mathcal{M}} = t^{\mathcal{M}}(a_1,\ldots,a_n)$ and if $t = ft_1\ldots t_k$, then $t^{\mathcal{M}}(a_1,\ldots,a_n) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1,\ldots,a_n),\ldots,t_k^{\mathcal{M}}(a_1,\ldots,a_n)) = f^{\mathcal{N}}(t_1^{\mathcal{N}}(a_1,\ldots,a_n),\ldots,t_k^{\mathcal{N}}(a_1,\ldots,a_n)) = t^{\mathcal{N}}(a_1,\ldots,a_n)$.

   The result for φ is also proved by induction on φ. If $\varphi = Rt_1\ldots t_k$, then $\varphi^{\mathcal{M}}(a_1,\ldots,a_n) = 1$ if and only if $(t_1^{\mathcal{M}}(a_1,\ldots,a_n),\ldots,t_k^{\mathcal{M}}(a_1,\ldots,a_n)) \in R^{\mathcal{M}}$. But, by definition of a substructure, $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^k$ and hence, that last statement is equivalent to $(t_1^{\mathcal{N}}(a_1,\ldots,a_n),\ldots,t_k^{\mathcal{N}}(a_1,\ldots,a_n)) \in R^{\mathcal{N}}$, which is, by definition, equivalent to $\varphi^{\mathcal{N}}(a_1,\ldots,a_n) = 1$. If $\varphi = \neg\psi$, then $\varphi^{\mathcal{M}}(a_1,\ldots,a_n) = f_\neg(\psi^{\mathcal{M}}(a_1,\ldots,a_n)) = f_\neg(\psi^{\mathcal{N}}(a_1,\ldots,a_n)) = \varphi^{\mathcal{N}}(a_1,\ldots,a_n)$, and similarly for binary operators.

   Let us now assume that $\varphi = \forall x_1 \forall x_n\,\psi(x_1,\ldots,x_n)$. To prove that $\mathcal{N} \vDash \varphi$, se have to show that for all tuple $(a_1,\ldots,a_n) \in N$, $\mathcal{N} \vDash \psi(a_1,\ldots,a_n)$, i.e. $\mathcal{M} \vDash \psi(a_1,\ldots,a_n)$. But because $\mathcal{M} \vDash \forall x_1 \forall x_n\,\psi(x_1,\ldots,x_n)$, we do have $\mathcal{M} \vDash \psi(a_1,\ldots,a_n)$.

2. To show that $\mathcal{M} \vDash \exists x_1,\ldots,\exists x_n\,\psi(x_1,\ldots,x_n)$, we have to find a tuple $(a_1,\ldots,a_n) \in M$ such that $\mathcal{M} \vDash \psi(a_1,\ldots,a_n)$. But, by hypothesis, there exists $(a_1,\ldots,a_n) \in N \subseteq M$ such that $\mathcal{N} \vDash \psi(a_1,\ldots,a_n)$ and hence, because ψ is quantifier free, $\mathcal{M} \vDash \psi(a_1,\ldots,a_n)$.

3. Let c be a constant symbol. Because $\mathcal{M}_i \leqslant_{\mathcal{L}} \mathcal{M}_j$ whenever $i \leqslant j$, it follows that $c^{\mathcal{M}_i}$ does not depend on i. Let $c^{\mathcal{M}} = c^{\mathcal{M}_i}$ for any/all i. Similalry, if f is an n-ary function symbol and $a_1,\ldots,a_n \in M_i$ then for all $j \geqslant i$, $f^{\mathcal{M}_j}(a_1,\ldots,a_n)$ does not depend on j and let $f^{\mathcal{M}}(a_1,\ldots,a_n) = f^{\mathcal{M}_i}(a_1,\ldots,a_n)$ for any/all i such that $(a_1,\ldots,a_n) \in M_i$. Finally, let $R^{\mathcal{M}} = \bigcup_i R^{\mathcal{M}_i}$. Note that if $(a_1,\ldots,a_n) \in M_i$ and $j \geqslant i$, then $(a_1,\ldots,a_n) \in R^{\mathcal{M}_j}$ if and only if $(a_1,\ldots,a_n) \in R^{\mathcal{M}_i}$. It immediately follows that $(a_1,\ldots,a_n) \in R^{\mathcal{M}}$ if and only if $(a_1,\ldots,a_n) \in R^{\mathcal{M}_i}$ for all/any i such that $(a_1,\ldots,a_n) \in R^{\mathcal{M}_i}$.

   It follows from the definition of the structure $\mathcal{M}$ that, for all i, $\mathcal{M}_i \leqslant_{\mathcal{L}} \mathcal{M}$.

4. Let $(a_1,\ldots,a_n) \in M$, then there is some i_0 such that $(a_1,\ldots,a_n) \in M_{i_0}$. Because $M_{i_0} \vDash \forall x_1\ldots\forall x_n \exists y_1\ldots\exists y_m\psi$, there exists $(b_1,\ldots b_m) \in M_{i_0} \subseteq M$ such that $M_{i_0} \vDash \varphi(a_1,\ldots,a_n,b_1,\ldots,b_m)$, i.e. $M \vDash \varphi(a_1,\ldots,a_n,b_1,\ldots,b_m)$ and hence, $M \vDash \forall x_1\ldots\forall x_n \exists y_1\ldots\exists y_m\psi$.

Problem 2 :

1. The formula $\forall y\, x \cdot y = x$ is a formula that defines the set $\{0\}$.

2. The formula $\forall y\, x \cdot y = y$ is a formula that defines the set $\{1\}$.

3. The forumla $\forall y \forall z\,(x = y \cdot z \rightarrow (y = x \vee z = x)) \wedge x \cdot x \neq x$ is a formula whose realisations are the prime numbers.

4. Because of the previous question, and the fact that any automorphism must preserve all definable sets, an automorphism σ of $\mathcal{M}$ must send 0 to 0, 1 to 1 and prime numbers to prime numbers (although σ may not fix each prime number). Let us prove that this are the only conditions. Let τ be a permutation of the set of prime numbers (i.e. a bijection between the set of prime numbers onto itself). We define σ_τ as follows: if $x = \prod_i p_i^{n_i}$, then $\sigma_\tau(x) = \prod_i \tau(p_i)^{n_i}$. Now, let $x = \prod_i p_i^{n_i}$ and $y = \prod_j p_j^{m_j}$. Setting some of the exponant to 0, we may assume that the same prime appear in the decomposition of x and y. Then $xy = \prod_i p_i^{n_i+m_i}$ and $\sigma_\tau(xy) = \prod_i \tau(p_i)^{n_i+m_i} = (\prod_i \tau(p_i)^{n_i})(\prod_i \tau(p_i)^{m_i}) = \sigma_\tau(x)\sigma_\tau(y)$.

   Moreover, because τ is injetive, by uniqueness of prime decompisition, σ_τ is also injective. And because τ is surjective, so is σ_τ.

Finally, if σ is an automorphism of $\mathcal{M}$, then by the discussion at the start of the question, σ induces a permutation τ_σ of the set of primes and because σ preserves multiplication, $\sigma = \sigma_{\tau_\sigma}$ and hence every automorphism of $\mathcal{M}$ is of the form σ_τ for some permutation τ of the set of primes.

5. Let n be any nonnegative integer which is not 0 or 1 and let p be a prime that appears in the prime decomposition of n. Consider τ to be a permutation of the set of primes sending p to some prime q which does not appear in the decomposition on n. Then $\sigma_\tau(n) \neq n$ and hence any formula satisfied by n is also satisfied by $\sigma_\tau(n)$ and hence no formula is satisfied by n and only n.

6. Let τ be a permutation of the primes sending 2 to 3. Then $\sigma_\tau(2) = 3$ and $\sigma_\tau(4) = 9 \neq 3 + 3$. It follows that no formula $\varphi(x, y, z)$ can only be realised by tuples such that $z = x + y$.

Problem 3 :

1. Let $\varphi(x_1, \ldots, x_k)$ be a formula and f_φ be the function $(a_1, \ldots, a_k) \mapsto 1$ if $M \vDash \varphi(a_1, \ldots, a_k)$ and 0 otherwise. Then $\varphi(x_1, \ldots, x_k)$ and $\psi(x_1, \ldots, x_k)$ are equivalent, in $\mathcal{M}$, if and only if $f_\varphi = f_\psi$. But there are at most $b = 2^{|M|^k}$ such functions and hence, we can find up to b formulas with k free variables such that every possible f_φ (and hence every equivalence class in $\mathcal{M}$) is represented.

2. Let φ_n be the sentence $\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j$. Then $\mathcal{M} \vDash \varphi_n$ if and only if $|M| \geqslant n$. Thus $\mathcal{M} \vDash \varphi_j \wedge \neg \varphi_{j+1}$ and so does $\mathcal{N}$ by elementary equivalence.

3. Let φ be the sentence $\exists x_1 \ldots \exists x_j \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i \varphi_i^{\epsilon_i}(x_1 \ldots, x_j)$ where $\epsilon_i = 1$ if $\mathcal{M} \vDash \varphi_i(m_1, \ldots, m_j)$ and 0 otherwise, and as usual $\varphi^1 = \varphi 0$ and $\varphi^0 = \neg\varphi$. Because $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \vDash \varphi$, we also have $\mathcal{N} \vDash \varphi$. Let $n_1, \ldots, n_j$ be the elements whose existence is implied by φ. Then, they are distinct and there are j of them, so $N = \{n_1, \ldots, n_j\}$. Moreover $\mathcal{M} \vDash \varphi_i(m_1, \ldots, m_j)$ if and only if $\epsilon_i = 1$ if and only if $\mathcal{N} \vDash \varphi_i(n_1, \ldots, n_j)$.

4. Let $\sigma(m_i) = n_i$ with the notations as above. Let $\varphi(x_1, \ldots, x_k)$ be some formula and $a_1, \ldots, a_k \in M$, we want to show that $\mathcal{M} \vDash \varphi(a_1, \ldots, a_k)$ if and only if $\mathcal{N} \vDash \varphi(\sigma(a_1), \ldots, \sigma(a_k))$. First of all, renaming the a_i that appear twice and adding unused variables for those that do not appear, we may assume that the tuple $a_1, \ldots, a_k$ is in fact the tuple $m_1, \ldots, m_j$. By the first question, there also exists i such that $\mathcal{M} \vDash \forall x_1 \ldots \forall x_j\, (\varphi(x_1 \ldots, x_j) \leftrightarrow \varphi_i(x_1 \ldots, x_j))$. But this also holds in $\mathcal{N}$ and hence $\mathcal{M} \vDash \varphi(m_1, \ldots, m_j)$ if and only if $\mathcal{M} \vDash \varphi_i(m_1, \ldots, m_j)$ if and only if $\mathcal{N} \vDash \varphi_i(\sigma(m_1), \ldots, \sigma(m_j))$ if and only if $\mathcal{N} \vDash \varphi(\sigma(m_1), \ldots, \sigma(m_j))$.

   It is now easy to check that σ is an automorphism. Indeed, $\mathcal{M} \vDash a = c$ if and only if $\mathcal{N} \vDash \sigma(a) = c$, $\mathcal{M} \vDash f(a_1, \ldots, a_k) = a$ if and only if $\mathcal{N} \vDash f(\sigma(a_1), \ldots, \sigma(a_k)) = \sigma(a)$ and $\mathcal{M} \vDash R(a_1, \ldots, a_k)$ if and only if $\mathcal{N} \vDash R(\sigma(a_1), \ldots, \sigma(a_k))$.