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Homework 5

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Problem 1 :

1. A sentence φ is said to be universal if it is of the form $\forall x_1 \dots \forall x_n\, \psi(x_1, \dots, x_n)$ where no quantifier appear in ψ. Let φ be universal, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{N}$ a substructure of $\mathcal{M}$. Show that if $\mathcal{M} \vDash \varphi$ then $\mathcal{N} \vDash \varphi$.
2. A sentence φ is said to be existential if it is of the form $\exists x_1 \dots \exists x_n\, \psi(x_1, \dots, x_n)$ where no quantifier appear in ψ. Let φ be existential, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{N}$ a substructure of $\mathcal{M}$. Show that if $\mathcal{N} \vDash \varphi$ then $\mathcal{M} \vDash \varphi$.
3. Let for all $i \in \mathbb{N}$, let $\mathcal{M}_i$ be an $\mathcal{L}$-structure such that for all $i < j$, $\mathcal{M}_i \leqslant_{\mathcal{L}} \mathcal{M}_j$. Show that $M = \bigcup_i M_i$ can be made into an $\mathcal{L}$-structure $\mathcal{M}$ such that for all i, $\mathcal{M}_i \leqslant_{\mathcal{L}} \mathcal{M}$.
4. Let $\varphi = \forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \psi(x_1, \dots, x_n, y_1, \dots, y_m)$, where ψ is quantifier free, be a sentence. Show that if for all i, $\mathcal{M}_i \vDash \varphi$, then $\mathcal{M} \vDash \varphi$.

Problem 2 :
Let $\mathcal{L} = \{\cdot\}$ (and the equality, but I don't mention it anymore) and let $\mathcal{M}$ be the $\mathcal{L}$-structure whose underlying set is $\mathbb{N}$ (the non negative integers) and where $\cdot$ is interpreted as the multiplication.

1. Show that there is a formula $\varphi_0(x)$ such that $\mathcal{M} \vDash \varphi_0(a)$ if and only if $a = 0$.
2. Show that there is a formula $\varphi_1(x)$ such that $\mathcal{M} \vDash \varphi_1(a)$ if and only if $a = 1$.
3. Show that there is a formula $\varphi_{\text{prime}}(x)$ such that $\mathcal{M} \vDash \varphi_{\text{prime}}(a)$ if and only if a is prime.
4. Find all automorphisms of $\mathcal{M}$.
5. Show that the only two elements of M such that there exist a formula φ_n such that $\mathcal{M} \vDash \varphi_n(a)$ if and only if $a = n$ are 0 and 1.
6. Show that there is no formula $\psi(x, y, z)$ such that $\mathcal{M} \vDash \psi(a, b, c)$ if and only if $c = a + b$.

Problem 3 :
Let $\mathcal{M}$ be a finite $\mathcal{L}$-structure.

1. Let k be a positive integer. Show that there exist $l \in \mathbb{N}$ and formulas $\varphi_i(x_1, \dots, x_k)$ for $i = 1, \dots, l$ such that any formula $\varphi(x_1, \dots, x_k)$ is equivalent, in $\mathcal{M}$, to one of the $\varphi_i(x_1, \dots, x_k)$.
2. Let $M = \{m_1, \dots, m_j\}$ and $\mathcal{N} \equiv \mathcal{M}$. Show that $|N| = j$ (remember that you have = in the language).
3. Show also that you can find $n_1, \dots, n_j$ such that $N = \{n_1, \dots, n_j\}$ and for all formulas φ_i of question 1 (for $k = j$) we have $\mathcal{M} \vDash \varphi_i(m_1, \dots, m_j)$ if and only if $\mathcal{N} \vDash \varphi_i(n_1, \dots, n_j)$.
4. Show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.