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Solutions to Homework 6()

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Problem 1 :

1. By the $(\forall_3)$ axiom, $\vdash \forall x \neg\varphi \rightarrow \neg\varphi$ holds. Moreover $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$ is a tautology, so, replacing A by $\forall x \neg\varphi$ and B by φ, applying the tautology rule and the Modus Ponens rule, we obtain that $\vdash \varphi \rightarrow \neg\forall \neg\varphi$ holds.

   But $\vdash \exists x \varphi \leftrightarrow \neg\forall x \neg\varphi$ holds by the $(\mathrm{Def}_\exists)$ rule, and by the tautology $(A \leftrightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (B \rightarrow A))$ applied to $A = \exists x \varphi$, $B = \varphi$ and $C = \neg\forall x \neg\varphi$, and two applications of Modus Ponens, we get that $\vdash \varphi \rightarrow \exists x \varphi$ holds.

   I am sorry I am not writing the deductions trees anymore, but they are too large (in particular because the tautology rule)...

2. Instead of providing a derivation, let us use completness. We have to show that $\vDash (\forall x(\varphi \rightarrow \psi)) \rightarrow ((\exists x\varphi) \rightarrow \psi)$. Let $\mathcal{M}$ be some $\mathcal{L}$-structure and $\delta \in M^V$. We want to show that $(\forall x(\varphi \rightarrow \psi)) \rightarrow ((\exists x\varphi) \rightarrow \psi)^{\mathcal{M}}_{\delta} = 1$. Let us assume that $(\forall x(\varphi \rightarrow \psi))^{\mathcal{M}}_{\delta} = 1$ and $(\exists x\varphi)^{\mathcal{M}}_{\delta} = 1$, we have to show that $\psi^{\mathcal{M}}_{\delta} = 1$. By the semantic of $\exists$, we can find δ' such that $\delta'(y) = \delta(y)$ whenever $y \neq x$ such that $\varphi^{\mathcal{M}}_{\delta'} = 1$. Moreover, by the semantic of $\forall$, we have $(\varphi \rightarrow \psi)^{\mathcal{M}}_{\delta'} = 1$ and hence $\psi^{\mathcal{M}}_{\delta'} = 1$. But x is not free in ψ and hence $\psi^{\mathcal{M}}_{\delta} = \psi^{\mathcal{M}}_{\delta'} = 1$.

   We can also provide a derivation, but some very tedious book keeping is involved. Let me sketch such a proof (in particular, I will not detail all the tautologies involed, especially when then concern transitivity of $\rightarrow$, or contraposition). First let us prove that $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow (\forall x\,(\neg\psi \rightarrow \varphi))$ holds. By (Taut), $\vdash (\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$ holds. By $(\forall_3)$, $\vdash (\forall x\,\varphi \rightarrow \psi) \rightarrow \varphi \rightarrow \psi$ holds and hence by tautology and Modus Ponens, $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow (\neg\psi \rightarrow \neg\varphi)$ holds. By (Gen), $\vdash \forall x\,((\forall x\,(\varphi \rightarrow \psi)) \rightarrow (\neg\psi \rightarrow \neg\varphi))$ holds and by $(\forall_1)$ and Modus Ponens, $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow (\forall x\,\neg\psi \rightarrow \neg\varphi))$ holds.

   By $(\forall_1)$, $\vdash (\forall x\,\neg\psi \rightarrow \neg\varphi) \rightarrow (\neg\psi \rightarrow (\forall x\,\neg\varphi))$ holds and by tautology and Modus Ponens (and the paragraph above), $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow (\neg\psi \rightarrow (\forall x\,\neg\varphi))$ holds. By tautology and Modus Ponens $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow ((\neg\forall x\,\neg\varphi) \rightarrow \psi)$ holds. Finally, using $(\mathrm{Def}_\exists)$, (Taut) and Modus Ponens, one gets that $\vdash (\forall x\,(\varphi \rightarrow \psi)) \rightarrow ((\exists x\,\varphi) \rightarrow \psi)$ holds.

3. We have to show that $\vdash' (\exists x\,\varphi) \rightarrow (\neg\forall x\,\neg\varphi)$ and $\vdash' (\neg\forall x\,\neg\varphi) \rightarrow (\exists x\,\varphi)$ both hold (and then we can conclude y some straightforward use of tautologies dans Modus Ponens). Let us first prove that $\vdash' (\exists x\,\varphi) \rightarrow (\neg\forall x\,\neg\varphi)$ holds. By $(\forall_3)$, $\vdash' \forall x\,\neg\varphi \rightarrow \neg\varphi$ holds. By tautologies (contraposition) and Modus ponens, $\vdash' \varphi \rightarrow \neg\forall x\,\neg\varphi$ holds. By (Gen), $\vdash' \forall x\,(\varphi \rightarrow \neg\forall x\,\neg\varphi)$ holds, and by $\exists_2$ and Modus Ponens, $\vdash' (\exists x\,\varphi) \rightarrow (\neg\forall x\,\neg\varphi)$ holds.

   Now let us prove that$\vdash' (\neg\forall x\,\neg\varphi) \rightarrow (\exists x\,\varphi)$ holds. By $\exists_1$, $\vdash' \varphi \rightarrow \exists x\,\varphi$ holds. By tautology and Modus Ponens, $\vdash' (\neg\exists x\,\varphi) \rightarrow \neg\varphi$ holds. By (Gen), $\vdash' \forall x, (\neg\exists x\,\varphi) \rightarrow \neg\varphi$ holds and by $\forall_1$ and Modus Ponens, $\vdash' (\neg\exists x\,\varphi) \rightarrow (\forall x\,\neg\varphi)$ holds. Finally, by tautology and Modus Ponens, $\vdash' (\neg\forall x\,\neg\varphi) \rightarrow (\exists x\,\varphi)$ also holds.

4. In question 1, we have proved that the two new rules $(\exists_1)$ and $(\exists_2)$ can be derived using the usual rules. It follows that the set T of (Γ, φ) suc that $\Gamma \vdash \varphi$ is closed under these two new rules (and is, by definition closed under the old rules), so, by minimality $T' \subseteq T$. Using question 2, we get that T' is closed unde the $(\mathrm{Def}_\exists)$ rule and hence, by minimality of T, we get that $T \subseteq T'$, i.e. $T = T'$ and $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$.

Problem 2 :

1. Let $\varphi_n = \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \neg x_i = x_j$. Then $\mathcal{M} \vDash \varphi_n$ if and only if $|M| \geqslant n$. Let $T_\infty = \{\varphi_n : n \in \mathbb{N}\}$ and $T' = T \cup T_\infty$. Then the models of T' are indeed exactly the infinite models of T'.

2. By the compactness theorem, it suffices to show that any finite subset of T' is consistent. But any finite subset of T' is included in $T \cup \{\varphi_n : n \leqslant m\}$ for some m. By hypothesis, there exists $\mathcal{M} \vDash T$ such that $|M| \geqslant m$. Then $\mathcal{M} \vDash T \cup \{\varphi_n : n \leqslant m\}$ and that theory is indeed satisfiable.

3. By the compactness theorem $T' \vDash \varphi$ if and only if $T \cup \{\varphi_n : n \leqslant m\} \vDash \varphi$ for some m. But models of $T \cup \{\varphi_n : n \leqslant m\}$ are exactly the models of T of cardinality at least m. That concludes the proof.