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Homework 6

Due October 29th

Problem 1 :
We want to replace the rule ($\mathrm{Def}_\exists$) (which says that $\Gamma \vdash (\exists x\varphi) \leftrightarrow (\neg\forall x\neg\varphi)$ always holds) by two new rules. The first one, ($\exists_1$) , states that $\Gamma \vdash \varphi \to \exists x\varphi$ always holds and the other one, ($\exists_2$), states that $\Gamma \vdash (\forall x(\varphi \to \psi)) \to ((\exists x\varphi) \to \psi)$ holds whenever $x \notin \mathrm{fvar}(\psi)$.

1. Let φ be a formula. Provide a deriation of $\vdash \varphi \to \exists\varphi$.
2. Let φ and ψ be formulas and x be a variable. Assume that $x \notin \mathrm{fvar}(\psi)$. Prove that $\vdash (\forall x(\varphi \to \psi)) \to ((\exists x\varphi) \to \psi)$ holds.
3. Let T' be the smallest set of pairs (Γ, φ) closed under the rules (MP), (Gen), (Ax), (Taut), ($\forall_1$), ($\forall_2$), ($\forall_3$), ($\exists_1$) and ($\exists_2$) (i.e. all of the usual except ($\mathrm{Def}_\exists$), plus the two new ones). We will write $\Gamma \vdash' \varphi$ if $(\Gamma, \varphi) \in T'$. Show that $\vdash' (\exists\varphi) \leftrightarrow (\neg\forall x\neg\varphi)$, i.e. show that the rule ($\mathrm{Def}_\exists$) can be derived from all the other rules.
4. Show that $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash' \varphi$.

Problem 2 :
Let C be a set of finite $\mathcal{L}$-structures such that for all $n \in \mathbb{N}$, there is some $\mathcal{M} \in C$ such that $|M| \geqslant n$. Let $T = \{\varphi : \varphi \text{ is a sentence and for all } M \in C,\ M \vDash \varphi\}$.

1. Give a theory T' such that the models of T' are exactly the infinite models of T.
2. Show that T' has a model (i.e. there exists an infinite model of T).
3. Show that $T' \vDash \varphi$ if and only if there exists some $n \in \mathbb{N}$ such that for all $\mathcal{M} \in C$ of cardinality greater than n, $\mathcal{M} \vDash \varphi$.