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Homework 7

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Problem 1 :

1. Let us assume T is consistent otherwise $T_\forall$ is not consistent either and this is not a very hard question.

   We have to show that $\Delta(\mathcal{M}) \cup T$ is consistent. Let us assume it is not. Then, by compactness, there exists $T_0 \subseteq \Delta(\mathcal{M}) \cup T$ finite such that T_0 is inconsistent. Let $\varphi_i \in \Delta(\mathcal{M})$ for $0 < i \leqslant k$ be such that $T \cup \{\varphi_i : 0 < i \leqslant k\}$ is inconsistent. Let $\varphi = \bigwedge_i \varphi_i$, then $T \cup \{\varphi\}$ is inconsistent. By a lemma proved in class, this means that $T \vDash \neg\varphi$. But, because the only thing that were added in $\mathcal{L}(\mathcal{M})$ were the constants $\underline{a}$, for $a \in M$, there exists an $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ such that $\varphi = \psi(\underline{a}_1, \ldots, \underline{a}_n)$ for some $a_i \in M$. But now, by another lemma proved in class, because the $\underline{a}_i$ do not appear in T, $T \vDash \forall x_1 \ldots \forall x_n \neg\psi$ which is a universal sentence, i.e. $\forall x_1 \ldots \forall x_n \neg\psi \in T_\forall$ and hence $\mathcal{M} \vDash \forall x_n \neg\psi$, in particular $\mathcal{M} \vDash \neg\psi(a_1, \ldots, a_n)$. But because $\varphi = \psi(\underline{a}_1, \ldots, \underline{a}_n) \in \Delta(\mathcal{M})$, we also have $\mathcal{M}^\star \vDash \psi(\underline{a}_1, \ldots, \underline{a}_n)$, i.e. $\mathcal{M} \vDash \psi(a_1, \ldots, a_n)$, a contradiction.

2. Let us assume that $\mathcal{M} \vDash T_\forall$, then, by the previous question, there exists $\mathcal{N}^\star \vDash \Delta(\mathcal{M}) \cup T$. Let $\mathcal{N}$ denote the reduct of $\mathcal{N}^\star$ to $\mathcal{L}$. Then $\mathcal{N} \vDash T$ and, by a proposition proved in class, there exists an embedding $f : \mathcal{M} \to \mathcal{N}$.

   Conversely, assume that there exists an embedding $f : \mathcal{M} \to \mathcal{N}$ and $\mathcal{N} \vDash T$. Pick $\varphi \in T_\forall$. Because $\mathcal{N} \vDash T$, and $T \vDash \varphi$, $\mathcal{N} \vDash \varphi$. Because φ is a universal sentence $f(\mathcal{M}) \vDash \varphi$ (we saw that in class and in Homework 5) and, because f is an isomorphism between $\mathcal{M}$ and $f(\mathcal{M})$, we also have $\mathcal{M} \vDash \varphi$. We just have showed that $\mathcal{M} \vDash T_\forall$.

3. Let us assume that $T_\forall \subseteq T'_\forall$ and let $\mathcal{M} \vDash T'$. Then $\mathcal{M} \vDash T'_\forall$ and hence $M \vDash T_\forall$. It follows from question 2 that $\mathcal{M}$ can be embedded in a model of T.

   Conversely, assume that every model of T' can be embedded in a model of T. By question @, it follows that every model of T' is a model of $T_\forall$. It follows that for all $\varphi \in T_\forall$, $T' \vDash \varphi$ and $\varphi \in T'_\forall$.

4. Let us assume that T is stable under substructure. First of all, by definition we have $T \vDash T_\forall$, and thus we only have to show that every model of $T_\forall$ is a model of T. Let $\mathcal{M} \vDash T_\forall$, then, by question 2, there exists an embedding $f : \mathcal{M} \to \mathcal{N} \vDash T$. But because T is stable under substructure, we also have $\mathcal{M} \vDash T$.

   Conversely, assume that T is equivalent to $T_\forall$ and let $f : \mathcal{M} \to \mathcal{N} \vDash T$ be an embedding. Then, by question 2, $\mathcal{M} \vDash T_\forall$ and, by equivalence, $\mathcal{M} \vDash T$.

Problem 2 :

1. Let $X = \{x_1, \ldots, x_k\}$ such that $x_1 < x_2 < \cdots < x_k$. Let y be some point in Y. Let us first assume that $\{x \in Y : x \leqslant y\}$ is infinite. Then we can find $y_1 < y_2 < \cdots < y_k = y \in Y$ and the map $x_i \mapsto y_i$ is an embedding. If $\{x \in Y : x \leqslant y\}$ is finite, then $\{x \in Y : x \geqslant y\}$ is infinite and we can find $y = y_1 < y_2 < \cdots < y_k \in Y$. Then $x_i \mapsto y_i$ is also an embedding.

2. By the compactness theorem, the theory $\Delta(X) \cup T$ is consistent if and only if every finite $T_0 \subseteq \Delta(X) \cup T$ is consistent. Let $X_0 \subseteq X$ contain all the $a \in X$ such that $\underline{a}$ appears in T_0. Because T_0 is finite X_0 is finite and $T_0 \subseteq \Delta(X_0) \cup T$. So it suffices to find a model of $T_0 \subseteq \Delta(X_0) \cup T$, i.e. prove that X_0 can be embedded in a model of T. Let $Y \vDash T$, then Y is an infinite total order and so, by the previous question X_0 can be embedded in Y.

3. Let $T_\leqslant$ be the theory of total orders, i.e. $T_\leqslant = \{\forall x \forall y \forall z\, (x \leqslant y \wedge y \leqslant z) \rightarrow x \leqslant z, \forall x\, x \leqslant x, \forall x \forall y\, (x \leqslant y \wedge y \leqslant x) \rightarrow x = y, \forall x \forall y\, x \leqslant y \vee y \leqslant x\}$. Then $T_\leqslant \subseteq T$ is a universal theory and hence $T_\leqslant \subseteq T_\forall$. On the other hand, by the previous question, every model of $T_\leqslant$ can be embedded in a model of T and hence every model of $T_\leqslant$ is a model of $T_\forall$, i.e. $T_\leqslant \vDash T_\forall$ and thus $T_\leqslant$ and $T_\forall$ are equivalent.