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Homework 7

Due November 5th

Problem 1 :
Recall that a universal sentence is a sentence of the form $\forall x_1 \dots \forall x_n \varphi$ where φ si quantifier free. Let T be an $\mathcal{L}$-theory and let $T_\forall = \{\varphi : T \vDash \varphi$ and φ is a universal sentence$\}$.

1. Let $\mathcal{M} \vDash T_\forall$. Show that the $\mathcal{L}(M)$-theory $\Delta(\mathcal{M}) \cup T$ is consistent

   Hint: Recall that if $T \vDash \varphi_{c_1/x_1,\dots,c_n/x_n}$ where the c_i are constants that do not appear in T, then $T \vDash \forall x_1 \dots \forall x_n \varphi$.

2. Show that $\mathcal{M} \vDash T_\forall$ if and only if there exists $\mathcal{N} \vDash T$ and an embedding $f : \mathcal{M} \to \mathcal{N}$.

3. Let T and T' be two theories, show that the following are equivalent:
   a) $T_\forall \subseteq T'_\forall$;
   b) Every model of T' can be embedded in a model of T.

4. Show that T is stable under substructure (i.e. if $\mathcal{N} \vDash T$ and $f : \mathcal{M} \to \mathcal{N}$ is an embedding, then $\mathcal{M} \vDash T$) if and only if T is equivalent to $T_\forall$.

Problem 2 :
Recall that an order $(X, \leqslant)$ is said to be total if for all $x, y \in X$, $x \leqslant y$ or $x \geqslant y$.

1. Let $(X, \leqslant)$ be a finite total order and $(Y, \leqslant)$ be an infinite total order. Show that X can be embedded in Y.

2. Let $\mathcal{L} = \{\leqslant\}$ and T be a consistent $\mathcal{L}$-theory containing the theory of infinite total orders. Let $(X, \leqslant)$ be a total order, show that $\Delta(X) \cup T$ is consistent.

3. Show that $T_\forall$ is equivalent to the theory of total orders.