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Homework 8

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Problem 1 :

1. Let x, y be in A, t is their lower upper bound in $(A, \leqslant')$ if and only if for all $z \in A$ such that $x \leqslant' z$ and $y \leqslant' z$ then $t \leqslant' z$, i.e. if $z \leqslant x$ and $z \leqslant y$ then $z \leqslant t$. It follows that t exists and t in the upper lower bound of x and y in $(A, \leqslant)$. In other terms $x \cup' y = x \cap y$.

   Similarly, the upper lower bound of x and y in $(A, \leqslant')$ exists and is equal to $x \cup y$. Moreover 1 is the smallest element in $(A, \leqslant')$ and 0 is the greatest element in $(A, \leqslant')$. So $(A, \leqslant')$ is a lattice. Because $\cap$ and $\cup$ distribute over one another, $(A, \leqslant')$ is distributive.

   Finally, let $x \in A$, the completment (if it exists) of x in $(A, \leqslant')$ is y such that $x \cap' y = 0'$ and $x \cup' y = 1'$, i.e. $x \cup y = 1$ and $x \cap y = 0$. So $y = x^{\mathbf{c}}$ works.

2. Because $(x^{\mathbf{c}})^{\mathbf{c}} = x$, the map $x \mapsto x^{\mathbf{c}}$ is surjective. To prove that it is an isomorphism of Boolean algebras, we only have to show that $x \leqslant y$ if and only if $x^{\mathbf{c}} \leqslant' y^{\mathbf{c}}$, that is $y^{\mathbf{c}} \leqslant x^{\mathbf{c}}$.

   We have $y^{\mathbf{c}} \leqslant x^{\mathbf{c}}$ if and only if $y^{\mathbf{c}} x^{\mathbf{c}} = y^{\mathbf{c}}$ if and only if $(1+y)(1+x) = 1+y+x+yx = 1+y$ if and only if $yx = -x = x$ which is equivalent to $x \leqslant y$.

3. As seen before, we have $0' = 1$ and $1' = 0$. Because complementation is an isomorphism between $(A, \leqslant)$ and $(A, \leqslant')$ (and it is its own inverse), we have $x +' y = (x^{\mathbf{c}} + y^{\mathbf{c}})^{\mathbf{c}} = 1 + (1+x+1+y) = 1+x+y$ and $x \cdot' y = (x^{\mathbf{c}} \cdot y^{\mathbf{c}})^{\mathbf{c}} = 1+(1+x)(1+y) = 1+1+x+y+xy = x+y+xy$.

Problem 2 :

1. Let $Y \subseteq \mathcal{P}(E)$ and let $Z = \bigcup_{X_i n Y} X$. Then $Z \in \mathcal{P}(E)$ is an upper bound of Y and it is contained in all sets that contain all the $X \in Y$, i.e. Z is the lower upper bound of Y.

2. Let us first assume that B is complete (the converse is easily obtained by considering f^{-1} instead of f). Let $X \subseteq A$. We define $Y = f(X) \subseteq B$. Because B is complete, Y has a lower upper bound Z in B. Let $T = f^{-1}(Z)$.

   Then for all $a \in X$, $f(a) \in Y$ and hence $f(a) \leqslant Z = f(T)$, so $a \leqslant T$. Moreover if T' is an upper bound of X, $f(T')$ is an upper bound of Y and hence $f(T) \leqslant Z \leqslant f(T')$, so $T \leqslant T'$. It follows that T is the lower upper bound of Y

3. By the previous question an the previous problem, $(A, \leqslant)$ is complete if and only if $(A, \leqslant')$ is complete. But a lower upper bound in $(A, \leqslant')$ is an upper lower bound in $(A, \leqslant)$. Moreover, by the previous question, the upper lower bound of X in $(A, \leqslant)$ is the image by complement of the lower upper bound of $X^{\mathbf{c}}$ in $(A, \leqslant)$, so:

$$\bigcap_{x \in X} x = (\bigcup_{x \in X} x^{\mathbf{c}})^{\mathbf{c}}.$$

4. Let us assume that $a \nleqslant x$ for all $x \in X$. Then, because a is an atom, $a \leqslant x^{\mathbf{c}}$ and hence $a \leqslant \bigcap_{x \in X} x^{\mathbf{c}} = (\bigcup_{x \in X} x)^{\mathbf{c}}$. In particular, $a \nleqslant \bigcup_{x \in X} x$.

5. Let us define $h : A \to \mathcal{P}(\mathcal{A})$ by $h(x) = \{a \in \mathcal{A} : a \leqslant x\}$. Let us first show that h is surjective. Let $X \subseteq \mathcal{A}$ and $x = \bigcup_{a \in X} a$. For all $a \in X$, we have $a \leqslant x$, so $X \subseteq h(x)$. Furthermore, if $e \in \mathcal{A}$ and $e \leqslant x = \bigcup_{a \in X} a$, by the previous question $e \leqslant a$ for some $a \in X$, but because both e and a are atoms, we must have $a = e$. It follows that $h(x) = X$.

   Let us now assume that $x \leqslant y$. Then for all $a \in \mathcal{A}$, if $a \leqslant x$ then $a \leqslant y$, so $h(x) \subseteq h(y)$. Conversely, if $x \nleqslant y$ then $x(1+y) \neq 0$ and hence there is an atom $a \leqslant x(1+y)$. We have $a \leqslant x$ and hence $a \in h(x)$, but $a \leqslant 1 + x = x^{\mathbf{c}}$ so $a \nleqslant y$ and thus $a \notin h(y)$; so $h(x) \nsubseteq h(y)$.

   We have just showed that h is surjective and $x \leqslant y$ if and only if $h(x) \subseteq h(y)$, so h is indeed an isomorphism of Boolean algebras.