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Solutions to homework 10

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Problem 1 :

1. Let $\Delta = \lambda x\,(x)x$. There are eight λ-terms that are reducts of $(\lambda x\,((I)x)x)\lambda x\,((I)x)x$:
   - $(\lambda x\,((I)x)x)\lambda x\,((I)x)x$ itself,
   - $(\Delta)\lambda x\,((I)x)x$,
   - $(\lambda x\,((I)x)x)\Delta$,
   - $(\Delta)\Delta$,
   - $((I)\lambda x\,((I)x)x)\lambda x\,((I)x)x$,
   - $((I)\Delta)\lambda x\,((I)x)x$,
   - $((I)\lambda x\,((I)x)x)\Delta$,
   - $((I)\Delta)\Delta$.
2. The sixteen reducts of $((W)I)WI$ are:
   - $((W)I)WI$ itself,
   - $(\lambda y\,((I)y)y)(W)I$,
   - $(\Delta)(W)I$,
   - $(\Delta)\lambda y\,((I)y)y$,
   - $((W)I)\lambda y\,((I)y)y$,
   - $((W)I)\Delta$,
   - $(\lambda y\,((I)y)y)\Delta$,
   - $(\Delta)\Delta$,
   - $(\lambda y\,((I)y)y)\lambda y\,((I)y)y$,
   - $((I)(W)I)WI$,
   - $((I)\lambda y\,((I)y)y)(W)I$,
   - $((I)\Delta)(W)I$,
   - $((I)\Delta)\lambda y\,((I)y)y$,
   - $((I)(W)I)\lambda y\,((I)y)y$,
   - $((I)(W)I)\Delta$,
   - $((I)\lambda y\,((I)y)y)\Delta$,
   - $((I)\Delta)\Delta$,
   - $((I)\lambda y\,((I)y)y)\lambda y\,((I)y)y$.

   And hopefully, I have not forgotten any.

Problem 2 :

1. We have:
$$\begin{array}{rcl} ((\overline{m})f)((\overline{n})f)x & \to_\beta & ((\overline{m})f)(\lambda x\, t_n)x \\ & \to_\beta & ((\overline{m})f)t_n \\ & \to_\beta & (\lambda x\, t_m)t_n \\ & \to_\beta & (t_m)_{t_n/x}. \end{array}$$
We now prove by induction on m that $(t_m)_{t_n/x} = t_{m+n}$. If $m = 0$, $(t_m)_{t_n/x} = t_n$. Otherwise $(t_{m+1})_{t_n/x} = (f)(t_m)_{t_n/x} = (f)t_{m+n} = t_{m+1+n}$.

2. Let $S = \lambda n \lambda f \lambda x\, ((n)f)(f)x$. We have:
$$\begin{array}{rcl} (S)\overline{n} & \to_\beta & \lambda f \lambda x\, ((\overline{n})f)(f)x \\ & \to_\beta & \lambda f \lambda x\, (\lambda x\, t_n)t_1 \\ & \to_\beta & \lambda f \lambda x\, (t_n)_{t_1/x} \\ & = & \lambda f \lambda x\, t_{n+1}. \end{array}$$
In fact, $S' = \lambda n \lambda f \lambda x\, (f)((n)f)x$ also works.

3. Let $A = \lambda m \lambda n \lambda f \lambda x\, ((m)f)((n)f)x$. We have:
$$\begin{array}{rcl} ((A)\overline{m})\overline{n} & \to_\beta & (\lambda n \lambda f \lambda x\, ((\overline{m})f)((n)f)x)\overline{n} \\ & \to_\beta & \lambda f \lambda x\, ((\overline{m})f)((\overline{n})f)x \\ & \to_\beta^\star & \lambda f \lambda x\, t_{m+n}. \end{array}$$
Note that $A' = \lambda m \lambda n\, ((m)S)n$ also works.

4. Let $M = \lambda m \lambda n \lambda f\, (m)(n)f$. We have:
$$\begin{array}{rcl} ((M)\overline{m})\overline{n} & \to_\beta & \lambda n \lambda f\, (\overline{m})(n)f \\ & \to_\beta & \lambda f\, (\overline{m})(\overline{n})f \\ & \to_\beta & \lambda f\, (\overline{m})\lambda x\, t_n \\ & \to_\beta & \lambda f \lambda x\, (t_m)_{\lambda x\, t_n/f} \end{array}$$
Let us now show, by induction on m, that $(t_m)_{\lambda x\, t_n/f} \to_\beta^\star t_{m\cdot n}$. If $m = 0$, then $(t_m)_{\lambda x\, t_n/f} = x = t_0$. Otherwise,
$$\begin{array}{rcl} (t_{m+1})_{\lambda x\, t_n/f} & = & ((f)t_m)_{\lambda x\, t_n/f} = (\lambda x\, t_n)(t_m)_{\lambda x\, t_n/f} \\ & \to_\beta^\star & (\lambda x\, t_n)t_{m\cdot n} \\ & \to_\beta & (t_n)_{t_{m\cdot n}/x} \\ & = & t_{n+m\cdot n} \\ & = & t_{(m+1)\cdot n}. \end{array}$$
Note that $M' = \lambda m \lambda n ((m)(A)n)\underline{0}$ also works.

5. Let $E = \lambda m \lambda n \lambda f \lambda x\, (((m)n)f)x$. We have:
$$\begin{array}{rcl} ((E)\overline{m})\overline{n} & \to_\beta & \lambda n \lambda f \lambda x\, (((\overline{m})n)f)x \\ & \to_\beta & \lambda f \lambda x\, (((\overline{m})\overline{n})f)x \\ & \to_\beta & \lambda f \lambda x\, ((\lambda x\, (t_m)_{\overline{n}/f})f)x \\ & \to_\beta & \lambda f \lambda x\, (((t_m)_{\overline{n}/f})_{f/x})x. \end{array}$$
If $m = 0$, $((t_0)_{\overline{n}/f})_{f/x} = x_{f/x} = f$ and we have the expected result. If $m \geqslant 1$, we show by induction on m that $((t_m)_{\overline{n}/f})_{f/x} \to_\beta^\star \lambda x\, t_{n^m}$. If $m = 1$,
$$\begin{array}{rcl} ((t_1)_{\overline{n}/f})_{f/x} & = & ((\overline{n})x)_{f/x} \\ & = & (\overline{x})f \\ & = & \lambda x\, t_n. \end{array}$$

Otherwise,

$$\begin{array}{rcl} ((t_{m+1})_{\overline{n}/f})_{f/x} & = & (((f)t_m)_{\overline{n}/f})_{f/x} \\ & = & (\overline{n})((t_m)_{\overline{n}/f})_{f/x} \\ & \to_\beta^\star & (\overline{n})\lambda x\, t_{n^m} \\ & \to_\beta & \lambda x\,(t_n)_{\lambda x\, t_{n^m}/f} \\ & \to_\beta^\star & \lambda x\, t_{n\cdot n^m} \\ & = & \lambda x\, t_{n^{m+1}}. \end{array}$$

It follows that:

$$\begin{array}{rcl} ((E)\overline{m})\overline{n} & \to_\beta^\star & \lambda f\lambda x\,(\lambda x\, t_{n^{m+1}})x \\ & \to_\beta & \lambda f\lambda x\, t_{n^{m+1}}. \end{array}$$

Note that $E' = \lambda m\lambda n((m)(M)n)\underline{1}$ also works.

Problem 3 :

1. Let us prove by induction on n that $\{f : A \to A, x : A\} \vdash t_n : A$ holds. If $n = 0$, then $t_0 = x$ and

$$(\text{Ax})\,\dfrac{}{\{f : A \to A, x : A\} \vdash x : A}$$

Let us now assume that $\{f : A \to A, x : A\} \vdash t_n : A$ holds. We have the following derivation:

$$(\to_E)\,\dfrac{\text{Ax}\,\dfrac{}{f : A \to A \vdash f : A \to A} \qquad \dfrac{\vdots}{\{f : A \to A, x : A\} \vdash t_n : A}}{\{f : A \to A, x : A\} \vdash (f)t_n : A}$$

We can now conclude by the followind derivation:

$$(\to_I)\,\dfrac{(\to_I)\,\dfrac{\dfrac{\vdots}{\{f : A \to A, x : A\} \vdash t_n : A}}{f : A \to A \vdash \lambda x\, t_n : A \to A}}{\vdash \lambda f\lambda x\, t_n : (A \to A) \to (A \to A)}$$

2. Let $\Gamma = \{x : A \to (B \to C), y : A \to B, z : A\}$. We have the following derivation:

$$(\to_I)\,\dfrac{(\to_I)\,\dfrac{(\to_I)\,\dfrac{(\to_E)\,\dfrac{(\to_E)\,\dfrac{\text{Ax}\,\dfrac{}{\Gamma \vdash x : A \to (B \to C)} \quad \text{Ax}\,\dfrac{}{\Gamma \vdash z : A}}{\Gamma \vdash (x)z : B \to C} \qquad (\to_E)\,\dfrac{\text{Ax}\,\dfrac{}{\Gamma \vdash y : A \to B} \quad \text{Ax}\,\dfrac{}{\Gamma \vdash z : A}}{\Gamma \vdash (y)z : B}}{\Gamma \vdash ((x)z)(y)z : C}}{\{x : A \to (B \to C), y : (A \to B)\} \vdash \lambda z\,((x)z)(y)z : (A \to C)}}{x : A \to (B \to C) \vdash \lambda y\lambda z\,((x)z)(y)z : (A \to B) \to (A \to C)}}{\vdash \lambda x\lambda y\lambda z\,((x)z)(y)z : (A \to (B \to C)) \to ((A \to B) \to (A \to C))}$$

3. Let us prove that $((T)K)K \vdash A \to A$ holds for any $A \in T$. First for any A and $B \in T$, we have the following derivation:

$$(\to_I)\,\dfrac{(\to_I)\,\dfrac{\text{Ax}\,\dfrac{}{\{x : A, y : B\} \vdash x : A}}{x : A \vdash \lambda y\, x : B \to A}}{\vdash K : A \to (B \to A)}$$

We also have the following derivations:

$$(\to_E)\ \frac{\begin{array}{c}\vdots\\ \vdash T : (A \to ((B \to A) \to A)) \to (A \to (B \to A)) \to A \to A\end{array} \qquad \begin{array}{c}\vdots\\ \vdash K : A \to ((B \to A) \to A)\end{array}}{\vdash (T)K \vdash (A \to (B \to A)) \to A \to A}$$

and

$$(\to_E)\ \frac{\begin{array}{c}\vdots\\ \vdash (T)K \vdash (A \to (B \to A)) \to A \to A\end{array} \qquad \begin{array}{c}\vdots\\ \vdash K : A \to (B \to A)\end{array}}{\vdash ((T)K)K \vdash A \to A}$$