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Solutions to Midterm

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Problem 1 :

1. Here is the truth table:

X	Y	$\neg X$	$\neg Y$	$Y \to X$	$\neg Y \leftrightarrow [Y \Rightarrow X]$	$\neg X \to [\neg Y \leftrightarrow [Y \Rightarrow X]]$
0	0	1	1	1	1	1
0	1	1	0	0	1	1
1	0	0	1	1	1	1
1	1	0	0	1	0	1

2. Let us first derive $\neg Y \vdash Y \to X$:

$$(\to_I)\;\frac{(\neg_E)\;\dfrac{(\mathrm{Ax})\;\dfrac{}{Y \vdash Y} \qquad (\mathrm{Ax})\;\dfrac{}{\neg Y \vdash \neg Y}}{\{\neg Y, Y\} \vdash X}}{\neg Y \vdash Y \to X}$$

Now, let us derive $\{\neg X, Y \to X\} \vdash \neg Y$:

$$(\vee_E)\;\frac{(\neg_E)\;\dfrac{(\to_E)\;\dfrac{\dfrac{}{Y \vdash Y} \qquad \dfrac{}{Y \to X \vdash Y \to X}}{\{Y \to X, Y\} \vdash X} \qquad \dfrac{}{\neg X \vdash \neg X}}{\{\neg X, Y \to X, Y\} \vdash \neg Y} \qquad \dfrac{}{\neg Y \vdash \neg Y} \qquad \dfrac{}{\vdash Y \vee \neg Y}}{\{\neg X, Y \to X\} \vdash \neg Y}$$

And now, let us put these two derivations toguether:

$$(\to_I)\;\frac{(\leftrightarrow_I)\;\dfrac{\begin{matrix}\vdots \\ \neg Y \vdash Y \to X\end{matrix} \qquad \begin{matrix}\vdots \\ \{\neg X, Y \to X\} \vdash \neg Y\end{matrix}}{\neg X \vdash \neg Y \leftrightarrow [Y \Rightarrow X]}}{\vdash \neg X \to [\neg Y \leftrightarrow [Y \Rightarrow X]]}$$

Problem 2 :
Let $\varphi \in \Gamma$. By hypothesis, $\Delta \vDash \varphi$. By the compactness theorem, there exists $\Delta_\varphi \subseteq \Delta$ finite, such that $\Delta_\varphi \vDash \varphi$. Let $\Delta_0 = \bigcup_{\varphi \in \Gamma} \Delta_\varphi$. This is indeed a finite subset of Δ. Because $\Delta_0 \subseteq \Delta$, we obviously have that $\Delta \vDash \Delta_0$ (I do not think I ever used this notation before, but it obviously means that Δ implies all formulas in Δ_0). By construction, $\Delta_0 \subseteq\vDash \Gamma$ and hence, because $\Gamma \vDash \Delta$, we also have $\Delta_0 \vDash \Delta$.

Problem 3 :

1. Let us first assume that $q = 0$ then $b = q \cdot a$ if and only if $b = 0$ and $\psi_0(x,y) := \forall z\, z + y = y$ works. If we assume that $q = n/m > 0$ where n and m a positive integers, then $b = qa$ if and only if $n \cdot b = m \cdot a$ and $\psi_q(x,y) := \sum_{i=1}^{m} x = \sum_{i=1}^{n} y$ has the desired property. Finally, if $q = -n/m < 0$, then $b = q \cdot a$ if and only if $n \cdot b + m \cdot a = 0$ and $\psi_q(x,y) := \forall z\, \sum_{i=1}^{m} x + \sum_{i=1}^{n} y + z = z$ has the required propery.

2. Let $f : \mathcal{M} \to \mathcal{M}$ be an automorphism. Let $q = f(1)$. For all $r \in \mathbb{Q}$, $\mathcal{M} \vDash \varphi_r(1, r)$. Because f is an automorphism, we also have $\mathcal{M} \vDash \varphi_r(f(1), f(r))$, i.e. $f(r) = r \cdot f(1) = r \cdot q$. So all automorphims of $\mathcal{M}$ are of this form.

   Moreover, if q is not zero, for all a, $b \in \mathbb{Q}$, $q \cdot a \in \mathbb{Q}$, $q \cdot (a+b) = q \cdot a + q \cdot b$ and $q \cdot a = q \cdot b$ if and only if $a = b$. Hence the map $x \mapsto q \cdot x$ is a monomorphism $\mathbb{Q} \to \mathbb{Q}$. Moreover $q \cdot (a \cdot q^{-1}) = a$ and it is therefore also surjective, i.e. it is an automorphism[1].

3. Let $q \in \mathbb{Q}$ be such that there exists a formula $\varphi_q(x)$ such that $\mathcal{M} \vDash \varphi(a)$ if and onl if $a = q$. Let f be the automorphism of $\mathcal{M}$ sending x to $2 \cdot x$. Then $\mathcal{M} \vDash \varphi_q(f(q))$ and hence $2 \cdot q = f(q) = q$. But that is only possible if $q = 0$.

4. Let us assume that θ exists. Then $\mathcal{M} \vDash \theta(1, 1, 1)$ and $\mathcal{M} \vDash \theta(f(1), f(1), f(1))$, where f is as above. Then we should have $4 = 2 \cdot 2 = f(1) \cdot f(1) = f(1) = 2$, which is absurd. So θ does not exist.

5. An automorphism of $\mathcal{N}$ is, in particular, an automorphism of $\mathcal{M}$, and is there fore of the form $x \mapsto q \cdot x$ for some $q \in \mathbb{Q} \setminus \{0\}$. But because $\mathcal{N}$ contains a constant interpreted as 1, if $x \mapsto q \cdot x$ is an automorphism of $\mathcal{N}$, then $q = q \cdot 1 = 1$ and the automorphism is the identity (which is indeed an automorphism of $\mathcal{N}$).

[1] I agree that because of not formulating the question the way I indented to initially, I technically did not ask that second part. Nevertheless, if you wanted to use that $x \mapsto q \cdot x$ was an automorphism later on, you needed to prove it at some point...