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Midterm

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Recall that these are the rules for deduction in propositional logic:

$$(\text{Ax})\ \frac{}{\Gamma \cup \{\varphi\} \vdash \varphi}$$

$$(\to_I)\ \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi} \qquad (\to_E)\ \frac{\Gamma_1 \vdash \varphi \to \psi \qquad \Gamma_2 \vdash \varphi}{\Gamma_1 \cup \Gamma_2 \vdash \psi}$$

$$(\wedge_I)\ \frac{\Gamma_1 \vdash \varphi_1 \qquad \Gamma_2 \vdash \varphi_2}{\Gamma_1 \cup \Gamma_2 \vdash \varphi_1 \wedge \varphi_2} \qquad (\wedge_{EL})\ \frac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_1} \qquad (\wedge_{ER})\ \frac{\Gamma \vdash \varphi_1 \wedge \varphi_2}{\Gamma \vdash \varphi_2}$$

$$(\vee_{IL})\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad (\vee_{IR})\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi}$$

$$(\vee_E)\ \frac{\Gamma_1 \cup \{\varphi_1\} \vdash \psi \qquad \Gamma_2 \cup \{\varphi_2\} \vdash \psi \qquad \Gamma_3 \vdash \varphi_1 \vee \varphi_2}{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vdash \psi}$$

$$(\leftrightarrow_I)\ \frac{\Gamma_1 \cup \{\varphi\} \vdash \psi \qquad \Gamma_2 \cup \{\psi\} \vdash \varphi}{\Gamma_1 \cup \Gamma_2 \vdash \varphi_1 \leftrightarrow \varphi_2} \qquad (\leftrightarrow_{EL})\ \frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \varphi \to \psi} \qquad (\leftrightarrow_{ER})\ \frac{\Gamma \vdash \varphi \leftrightarrow \psi}{\Gamma \vdash \psi \to \varphi}$$

$$(\neg_E)\ \frac{\Gamma_1 \vdash \varphi \qquad \Gamma_2 \vdash \neg\varphi}{\Gamma_1 \cup \Gamma_2 \vdash \psi} \qquad (\text{ExMid})\ \frac{}{\Gamma \vdash \varphi \vee \neg\varphi}$$

Problem 1 :
Let φ be the formula $(\neg X \to (\neg Y \leftrightarrow (Y \to X)))$.

1. Show, using a truth table, that φ is a tautology.
2. Show, providing a derivation, that $\vdash \varphi$ holds.

Problem 2 :
Let Γ be a finite set of propositional formulas and Δ is a set of propositional formulas that is logically equivalent to Γ (i.e. for all $\varphi \in \Gamma$, $\Delta \vDash \varphi$ and for all $\psi \in \Delta$, $\Gamma \vDash \psi$). Show that there exists $\Delta_0 \subseteq \Delta$ finite such that Δ is logically equivalent to Δ_0.

Problem 3 :
Let $\mathcal{L} = \{+\}$ and $\mathcal{M}$ be the $\mathcal{L}$-structure whose underlying set is $\mathbb{Q}$ and $+$ is interpreted as the addition.

1. Show that for all $q \in \mathbb{Q}$, there is a formula $\psi_q(x, y)$ such that $\mathcal{M} \vDash \psi_q(a, b)$ holds if and only if $b = q \cdot a$.
2. Show that all automorphisms of $\mathcal{M}$ are of the form $x \mapsto q \cdot x$ for $q \in \mathbb{Q} \setminus \{0\}$.
3. Show that for all $q \in \mathbb{Q}$, if there exist a formula $\varphi_q(x)$ such that $\mathcal{M} \vDash \varphi_q(a)$ if and only if $a = q$, then $q = 0$.
4. Show that there does not exists a formula $\theta(x, y, z)$ such that $\mathcal{M} \vDash \theta(a, b, c)$ if and only if $c = a \cdot b$.
5. Let $\mathcal{N}$ be the enrichement of $\mathcal{M}$ to the language $\{+, c\}$ where c is a constant interpreted as 1 in $\mathcal{M}$. Show that the only automorphism of $\mathcal{N}$ is the identity.