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Final

December 15th

To do a later question in a problem, you can always assume a previous question even if you have not answered it.

Problem 1 (Vaughtian Pairs) **:**

1. Let T_{RG} be the theory of the random graph. Show that T_{RG} has a Vaughtian pair.

   Hint: If $G \vDash T_{\mathrm{RG}}$ and $x \in G$, consider $G \smallsetminus \{x\}$.

2. Show that the theory of real closed fields does not have Vaughtian pairs.

   Hint: You can use the fact that if $R \leqslant S$ is a strict extension of ordered fields then $S \smallsetminus R$ is dense (for the order) in S.

Problem 2 :
Let K be an infinite field. Let $\mathcal{L} = \{V; 0 : V, + : V^2 \to V, (\lambda_k : V \to V)_{k \in K}, W : V\}$.

1. Write a theory T whose models are K-vector spaces (where $+$ is the addition, 0 is the neutral element and λ_k is scalar multiplication by $k \in K$) and such that W is a proper non trivial subspace.
2. Show that T eliminates quantifiers and is complete.
3. Assume that K is countable. Show that T is ω-stable.
4. Show that T is not κ-categorical for any infinite cardinal κ and give an example of a Vaughtian pair in T.
5. Show that W is strongly minimal in T.

Problem 3 :
Let T be the theory of discrete linear orders without endpoints in the language with one sort and a binary predicate for the order. Recall that T is complete and it eliminates quantifiers if one adds the successor function.

1. Let $M \vDash T$ and $X \subseteq M$ be $\mathcal{L}(M)$-definable. Assume there exists $a \in M$ such that for all $c \in X(M)$, $c > a$. Show that $X(M)$ has a minimal element.
2. Let $\mathcal{L}'$ be $\mathcal{L}$ with a new constant and T' be T in that new language (i.e. we don't say anything about the constant). Let $M \vDash T'$ and $X \subseteq M^n$ be $\mathcal{L}'(M)$-definable. Show that $X(M) \cap \ulcorner X \urcorner \neq \varnothing$.
3. Show that T' eliminates imaginaries.

Problem 4 :
Let $\mathcal{L}$ be the language with one sort X and one function symbol $f : X \to X$.

1. Write the theory T of $\mathcal{L}$-structures such that f is a bijection and that for all $n \in \mathbb{Z}_{>0}$, $f^{(n)}$, the n-th iterate of f, does not have fixed points.

2. Show that T has quantifier elimination and is complete.

3. Show that T is strongly minimal.

4. Show that for all $M \vDash T$ and $A \subseteq M$, $\mathrm{acl}(A) = \mathrm{dcl}(A) = \{f^n(a) : a \in A,\ n \in \mathbb{Z}\}$.

5. Let a_1 and $a_2 \in M \vDash T$ be such that $\{a_1, a_2\}$ is coded by some tuple b (i.e. $b \in \ulcorner\{a_1, a_2\}\urcorner \cap M$ and $\{a_1, a_2\}$ is $\mathcal{L}(b)$-definable). Show that there exists an $\mathcal{L}$-formula $\varphi(x, y)$ such that $M \vDash \varphi(a_1, a_2)$ and $M \vDash \neg\varphi(a_2, a_1)$.

6. Show that T weakly eliminates imaginaries but does not eliminate imaginaries.