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Solutions to the midterm

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Problem 1 :
Let $\mathcal{L}$ be the language with one sort X and one predicate symbol $E \subseteq X^2$.

1. Write a theory T such that in models of T, E is an equivalence relation with exactly one class of size n for every $n \in \mathbb{Z}_{>0}$ (and possibly infinite classes).

   Solution: Let $\theta_{\geqslant n}(x) := \exists x_1 \dots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_i xEx_i$ and $\theta_{=n}(x) := \theta_{\geqslant n}(x) \wedge \neg\theta_{\geqslant n+1}(x)$. The formula $\theta_{=n}(x)$ states that the class of x has size n. We define:

$$\begin{aligned} T \quad := \quad & \{\forall x\, xEx, \forall x \forall y\, xEy \to yEx, \forall x \forall y \forall z\, (xEy \wedge yEz) \to xEz\} \\ & \cup \{\exists x\, \theta_{=n}(x) \wedge (\forall y\, \theta_{=n}(y) \to xEy) : n \in \mathbb{Z}_{>0}\} \end{aligned}$$

2. Show that T does not eliminate quantifiers.

   Solution: Let $M := \coprod_{n>0} \{n\} \times \{0, \dots, n-1\}$. Let E be the equivalence class that holds when the first coordinate is equal. Then $(M, E) \vDash T$. But note that we have an embedding $f : M \to M$ sending (i, j) to $(i+1, j)$. This embedding is not elementary as $M \vDash \theta_{=1}((1,0)))$ but $M \nvDash \theta_{=1}(f((1,0)))$.

3. Let $\mathcal{L}^\star := \mathcal{L} \cup \{c_{n,i} : n \in \mathbb{Z}_{>0} \text{ and } 0 \leqslant i < n\}$. Write a theory $T^\star$ whose models are models of T in which the class with n elements is $\{c_{n,0}, \dots, c_{n,n-1}\}$.

   Solution: We define $T^\star := T \cup \{\theta_{=n}(c_{n,i}) : n \in \mathbb{Z}_{>0} \text{ and } 0 \leqslant i < n\}$. Note that in models of $T^\star$, we know n elements in the class with n elements, so we know them all.

4. Show that $T^\star$ eliminates quantifiers.

   Solution: We have to show that if $M, N \vDash T^\star$, f is a partial $\mathcal{L}^\star$-embedding from M to N with domain A and $a \in M$, if N is $|A|^+$-saturated, then f can be extended to a. Note that f has a unique extension to $A \cup \{c_{n,i}^M : i \in \mathbb{Z}_{>0} \text{ and } 0 \leqslant i < n\}$ so we may assume that all the interpretations of the constants are in A. We can also assume that $a \notin A$ otherwise f is already defined at a. Let us first assume that there exists $b \in A$ such that aEb. Since every element in one of the finite class is named by a constant, the class of $f(b)$ is infinite. It follows that $\pi(x) := \{xEf(b)\} \cup \{x \neq f(a) : a \in A\}$ is a partial type and, since N is $|A|^+$-saturated, we can find $d \in N \smallsetminus f(A)$ such that $dEf(b)$. It is now easy to check that we can extend f by sending a to d.

   If there is no $b \in A$ such that aEb, then, by $|A|^+$-saturation, we find $d \in N$ such that the E-class of d does not intersect $f(A)$ (consider the type $\pi(x) := \{\neg xEf(a) : a \in A\}$ which is consistent because E has infinitely many classes). It is now easy to show that f can be extended by sending a to d.

5. Show that T has a prime model.

   Solution: Let $M^\star$ be the structure M as in Question 2 where $c_{n,i}$ is interpreted by (n, i). Then $M^\star \vDash T^\star$. Moreover, $M^\star$ can be embedded in every model of $T^\star$ by sending (n, i) to the interpretation of $c_{n,i}$. Since $T^\star$ eliminates quantifiers, the embedding is elementary and $M^\star$ is a prime model of $T^\star$. Now if $N \vDash T$, N can be enriched into a model of $T^\star$ and therefore $M^\star$ embeds elementarily (as an $\mathcal{L}^\star$-structure) in $N^\star$. But it follows immediately that this embedding is also an elementary $\mathcal{L}$-embedding.

6. Show that T has saturated models in all cardinality $\kappa \geqslant \aleph_0$.

   Solution: Let M_κ be the model $M \sqcup \kappa^2$ where the equivalence relation is given by equality if the first coordinate. Then $M_\kappa \vDash T$. Now, let $A \subseteq M_\kappa$ be such that $|A| < \kappa$ and p be a complete $\mathcal{L}$-type over A. If p contains $x = a$ then p then the only possible realization of p is a and since p is a type (in particular it is satisfiable in some elementary extension), it must be realized by a. If p contains xEa for some $a \in A$ (but $x \neq a$ for all $a \in A$), then the E-class of A in M_κ has size κ which is strictly larger than $|A|$ so we can find $b \in M_\kappa \smallsetminus A$ such that bEa. It is easy to see by quantifier elimination that any two points (in an elementary extension of M_κ) verifying bEa and $b \notin A$ have the same type over A. For example, the map sending fixing A and sending b_1 to b_2 is a partial embedding which is elementary by quantifier elimination. It follows that b realizes p.

   Finally, if p contains $\neg xEa$ for all $a \in A$ then, since $|A| < \kappa$, there must be a class in M_κ that does not contain any point of A. As above, any b is this class is a realization of p.

Problem 2 :
Let T be a complete $\mathcal{L}$-theory with one sort X and no function symbols or constants. Assume that T eliminates quantifiers. Let $\mathcal{L}_f$ be the language $\mathcal{L}$ with a new sort Y and function symbol $f : X \to Y$.

1. Write a theory T_f such that in models of T:
   - Y is infinite and f is surjective;
   - For all $a \in Y$, $f^{-1}(a)$ is a model of T;
   - For all $\mathcal{L}$-predicate $R(x_1, \dots, x_n)$ and tuple $x_1, \dots, x_n \in X$, if $R(x_1, \dots, x_n)$ holds then for all i, j, $f(x_i) = f(x_j)$.

   Solution: For all $\mathcal{L}$-formula φ we define its realization to $f^{-1}(y)$, φ_y, by induction. If $\varphi = Rx_1 \dots x_n$, $\varphi_y := \bigwedge_i f(x_i) = y \wedge \varphi$, $(\neg\varphi)_y := \neg(\varphi_y)$, $(\varphi \wedge \psi)_y := (\varphi_y) \wedge (\psi_y)$ and $(\exists x\, \varphi)_y := \exists x\, f(x) = y \wedge \varphi_y$. Note that φ_y holds in an $\mathcal{L}_f$-structure if an only if φ holds in $f^{-1}(y)$. We define:

$$\begin{aligned} T_f \quad := \quad & \{\exists y\, y = y \wedge \forall y \exists x\, f(x) = y\} \\ & \cup \{\forall y\, \varphi_y : \varphi \in T\} \\ & \cup \{\forall x_1 \dots \forall x_n\, Rx_1 \dots x_n \to \bigwedge_i f(x_i) = f(x_1) : R \in \mathcal{L}\} \end{aligned}$$

2. Show that T_f eliminates quantifiers.

   Solution: Let $M, N \vDash T_f$, g a partial embedding from M into N whose domain is A and $a \in M$. Assuming that N is $|A|^+$-saturated, we have to extend g to A. We may assume that A is closed under f.

   Let us first assume that $a \in Y(M) \smallsetminus A$. Then $f^{-1}(a) \cap A = \varnothing$. Pick any $b \in Y(N) \smallsetminus g(A)$ (such a b exists by saturation). Then g can be extended by sending a to b.

   If $a \in X(M)$, by the previous case we may assume that $c := f(a) \in A$. We have that $f^{-1}(c)$, $f^{-1}(g(c)) \vDash T$ and since T is complete, we have $f^{-1}(c) \equiv f^{-1}(g(c))$. Let g_c be the restriction of g to $f^{-1}(c)$. The map g_c is a partial embedding from $f^{-1}(c)$ into

$f^{-1}(g(c))$. By quantifier elimination, g_c is a partial elementary embedding. When the domain of g_c is empty, we are using the fact that T is complete. Note that, since N is $|A|^+$-saturated, so is $f^{-1}(g(c))$. It follows that g_c can be extended to a. This extension is also an extension of g since predicates of $\mathcal{L}$ are always false when applied to points in distinct fibers.

3. Let $M \vDash T_f$, show that all $\mathcal{L}_f(M)$-definable subsets of $Y(M)$ are finite or cofinite.

   Solution: Atomic $\mathcal{L}(M)$-formulas with a free variable in Y are (equivalent to one) of the form $y = a$ where $a \in M$, whose interpretation is finite, $y = y$ whose interpretation is cofinite and $a = b$ whose interpretation is either finite or cofinite depending on whether it is true or false. It follows that all quantifier free formulas with a free variable in Y define finite or cofinite sets. By quantifier elimination, all definable subsets of Y are finite of cofinite.

4. Let $\kappa \geqslant \aleph_0$ be a cardinal. Show that if T is κ-stable, then T_f is κ-stable.

   Solution: Let $M \vDash T_f$ be of size κ. We have to show that $|\mathcal{S}_y(M) \cup \mathcal{S}_x(M)| = \kappa$ for y (resp. x) a variable in the sort Y (resp. X). Let us start with types in the variable y. Since all the $\mathcal{L}_f(M)$-definable subsets of Y are finite or cofinite, the partial type $\{x \neq a : a \in Y(M)\}$ is complete. It follows that the only types in $\mathcal{S}_y(M)$ are those containing $x = a$ for some $a \in M$ and the completion of $\{x \neq a : a \in Y(M)\}$. There are κ of those.

   Now, pick $p \in \mathcal{S}_x(M)$. Let us first consider the case where p contains $f(x) = a$ for some $a \in Y(M)$. Since the relations are not defined outside of a given fiber. Any atomic formula $\theta(y, c)$ is not in p if c contains elements outside of $f^{-1}(a) \cap M$. It follows (by quantifier elimination), that p is completely determined by the $\mathcal{L}(f^{-1}(a) \cap M)$-formulas it contains, i.e. an $\mathcal{L}$-type in $\mathcal{S}_x(f^{-1}(a) \cap M)$, of which there are at most κ by κ-stability of T.

   Finall if p contains $f(x) \neq a$ for all $a \in Y(M)$, p is completely determined by the $\mathcal{L}$-formulas it contains, i.e. an $\mathcal{L}$-type in $\mathcal{S}_x(T)$, of which there are, once more, at most κ by κ-stability of T.