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Midterm

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To do a later question in a problem, you can always assume a previous question even if you have not answered it.

Problem 1 :
Let $\mathcal{L}$ be the language with one sort X and one predicate symbol $E \subseteq X^2$.

1. Write a theory T such that in models of T, E is an equivalence relation with exactly one class of size n for every $n \in \mathbb{Z}_{>0}$ (and possibly infinite classes).
2. Show that T does not eliminate quantifiers.
3. Let $\mathcal{L}^\star := \mathcal{L} \cup \{c_{n,i} : n \in \mathbb{Z}_{>0}$ and $0 \leqslant i < n\}$. Write a theory $T^\star$ whose models are models of T in which the class with n elements is $\{c_{n,0}, \ldots, c_{n,n-1}\}$.
4. Show that $T^\star$ eliminates quantifiers.
5. Show that T has a prime model.
6. Show that T has saturated models in all cardinality $\kappa \geqslant \aleph_0$.

Problem 2 :
Let T be a complete $\mathcal{L}$-theory with one sort X and no function symbols or constants. Assume that T eliminates quantifiers. Let $\mathcal{L}_f$ be the language $\mathcal{L}$ with a new sort Y and function symbol $f : X \to Y$.

1. Write a theory T_f such that in models of T:
   - Y is infinite and f is surjective;
   - For all $a \in Y$, $f^{-1}(a)$ is a model of T;
   - For all $\mathcal{L}$-predicate $R(x_1, \ldots, x_n)$ and tuple $x_1, \ldots, x_n \in X$, if $R(x_1, \ldots, x_n)$ holds then for all i, j, $f(x_i) = f(x_j)$.
2. Show that T_f eliminates quantifiers.
3. Let $M \vDash T_f$, show that all $\mathcal{L}_f(M)$-definable subsets of $Y(M)$ are finite or cofinite.
4. Let $\kappa \geqslant \aleph_0$ be a cardinal. Show that if T is κ-stable, then T_f is κ-stable.