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Final

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Problem 1 :
Let G be a group and let $\mathcal{L}_G$ be the language with one sort X and for each $g \in G$, a function $\lambda_g : X \to X$. Let T_G be the theory whose models are the infinite $\mathcal{L}_G$-structures where $(g, x) \mapsto \lambda_g(x)$ is a free action of G. Recall that an action of G is free if for all $x \in X$, $x = g \cdot x$ implies $g = 1$.

1. Show that T_G eliminates quantifiers, is complete and strongly minimal.
2. Let $A \subseteq M \vDash T_G$ and $a \in M^x$ a tuple. Show that $\mathrm{MR}(a/A)$ is the number of G-orbits containing points of a but no points of A.
3. Let $A \subseteq M \vDash T_G$ be infinite. Show that $T_G \cup \Delta(A)$ weakly eliminates imaginaries, where $\Delta(A)$ is the quantifier free diagram of A in M.
4. Let G be finite. Show that T_G is $\aleph_0$-categorical.
5. Still assuming G finite, show that T_G weakly eliminates imaginaries.

Problem 2 :
Let T be a complete, totally transcendental theory, $A \subseteq M \vDash T$, $p \in \mathcal{S}(A)$ and α an ordinal. We say that $\mathrm{U}(p) \geqslant \alpha$ if:

- $\alpha = 0$;
- $\alpha = \beta + 1$ and there are $A \subseteq B \subseteq N \succcurlyeq M$ and $q \in \mathcal{S}(B)$ a forking extension of p such that $\mathrm{U}(q) \geqslant \beta$;
- α is limit and for all $\beta < \alpha$, $\mathrm{U}(p) \geqslant \beta$.

We say that $\mathrm{U}(p) = \alpha$ if $\mathrm{U}(p) \geqslant \alpha$ but $\mathrm{U}(p) < \alpha + 1$. If $\mathrm{U}(p) \geqslant \alpha$, for every ordinal α, we say that $\mathrm{U}(p) = \infty$.

1. Show that, for all $p \in \mathcal{S}(A)$, $\mathrm{U}(p) \leqslant \mathrm{MR}(p) < \infty$.
2. Show that $\mathrm{U}(p) = 0$ if and only if p is an algebraic type.
3. Assume T strongly minimal, show that, for all $p \in \mathcal{S}(A)$, $\mathrm{U}(p) = \mathrm{MR}(p)$.
4. Let $a, b \in M$ be tuples, show that $\mathrm{U}(\mathrm{tp}(ab/A)) \geqslant \mathrm{U}(\mathrm{tp}(a/A))$.
5. Let $a, b \in M$ be tuples, assume that $b \in \mathrm{acl}(Aa)$, show that $\mathrm{U}(\mathrm{tp}(b/A)) \leqslant \mathrm{U}(\mathrm{tp}(a/A))$.

Problem 3 :
Let $A, A' \subseteq M \vDash T$ a complete theory.

- We say that two strongly minimal types $p, q \in \mathcal{S}(A)$ are almost orthogonal (and we write $p \perp^{\mathrm{a}} q$) if for all $a \vDash p$ (in some elementary extension of M), no tuple from $\mathrm{acl}(Aa)$ realizes q.

- We say that two strongly minimal types $p \in \mathcal{S}(A)$ and $q \in \mathcal{S}(A')$ are orthogonal, and we write $p \perp q$, if for all $A \cup A' \subseteq B \subseteq N \succcurlyeq M$, $p|_B \perp^{\mathrm{a}} q|_B$

1. Let $p, q \in \mathcal{S}(A)$ be strongly minimal types. Show that $p \perp^{\mathrm{a}} q$ if and only if $p(x) \cup q(y)$ has a unique completion over A.
2. Show that $\not\perp$ — non orthogonality — is an equivalence relation on the set $X := \bigcup_{A \subseteq M} \{p \in \mathcal{S}(A) : p \text{ strongly minimal}\}$.
3. Let $p \in \mathcal{S}(A)$ be strongly minimal. Show that p is isolated if and only for any strongly minimal $\varphi \in p$, $\varphi(M) \cap \operatorname{acl}(A)$ is finite. Conclude that for all $A \subseteq B \subseteq N \succcurlyeq M$, if $p|_B$ is isolated, then so is p.
4. Assume T is totally transcendental, $M \vDash T$ and $p, q \in \mathcal{S}(M)$ are strongly minimal. Show that there exists $N \succcurlyeq M$ realizing p and omitting q.
5. Assume $\mathcal{L}$ is countable and T is uncountably categorical. Show that there is a unique non orthogonality class of strongly minimal types.