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Midterm

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To do a later question in a problem, you can always assume a previous question even if you have not answered it.

Problem 1 :
Let T be a theory with infinite models in a language $\mathcal{L}$ with one sort. Assume that:

- T eliminates quantifiers;
- For all $A \subseteq M \vDash T$, $\mathrm{acl}(A) = A$.
- For all $\mathcal{L}$-formula $\varphi(x, y)$, there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that for all $M \vDash T$ and $a \in M^y$, if $|\varphi(M, a)| \geqslant k$, then $\varphi(M, a)$ is infinite. We say that T eliminates $\exists^\infty$.

Let $M \vDash T$ and let $<$ be a total order on M. We say that $<$ is generic if for all M-definable infinite $X \subseteq M$ and $a < b \in M \cup \{+\infty, -\infty\}$, $X \cap (a, b) \neq \varnothing$, where (a, b) is the open interval between a and b.

1. Show that if T is strongly minimal and only has infinite models, then it eliminates $\exists^\infty$.
2. Let $M \vDash T$ be infinite and $<$ be a total order on M. Show that there exists $N \succcurlyeq M$ with a generic order extending $<$.
3. Let $\mathcal{L}_<$ be $\mathcal{L}$ with a new binary symbol $<$. Show that there exists an $\mathcal{L}_<$-theory $T_<$ such that models of $T_<$ are exactly the models of T where $<$ is generic (with respect to the $\mathcal{L}$-structure of M).
4. Show that $T_<$ eliminates quantifiers.
5. Show that $T_<$ is complete.
6. Let $M \vDash T_<$ and $A \subseteq M$. Show that $\mathrm{acl}(A) = A$ (here, the algebraic closure is understood in M as an $\mathcal{L}_<$-structure).
7. Assume $\mathcal{L}$ is countable. Show that $T_<$ is ω-categorical if and only if T is ω-categorical.

Problem 2 :
Let T be a theory, $\kappa > |\mathcal{L}|$, $M \vDash T$ be κ-saturated, and $X \subseteq M^x$ be $\varnothing$-definable. Consider the following statements.

(i) Any M-definable set $Y \subseteq X^n$ is X-definable.

(ii) For all $a \in M^z$, there exists $C \subseteq X$ such that $|C| < \kappa$ and for all $b \in M^z$, $\mathrm{tp}^M(a/C) = \mathrm{tp}^M(b/C)$ implies $\mathrm{tp}^M(a/X) = \mathrm{tp}^M(b/X)$.

(iii) For all $a, b \in M^z$, $\mathrm{tp}^{M^{\mathrm{eq}}}(a/C) = \mathrm{tp}^{M^{\mathrm{eq}}}(b/C)$ implies $\mathrm{tp}^M(a/X) = \mathrm{tp}^M(b/X)$, where $C = \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X)$.

1. Show that (i) implies (iii).

2. Assume that (i) does not hold. Show that there exists $a \in M^x$ and a formula $\varphi(x, y)$ such that, for any $C \subseteq X$ with $|C| < \kappa$, there exists $b_1, b_2 \in X^n$ with $\mathrm{tp}(b_1/C) = \mathrm{tp}(b_2/C)$, $M \vDash \varphi(a, b_1)$ and $M \vDash \neg\varphi(a, b_2)$.

3. Show that (i), (ii) and (iii) are equivalent.

4. Assume (i). Let $N \succcurlyeq M$, $a \in N^z$ and $X(N) := \psi(N)$ for any formula ψ such that $X = \psi(M)$. Show that $\mathrm{tp}(a/X)$ is realized in M if and only if $\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X(N)) \subseteq M^{\mathrm{eq}}$.

5. Assume (i) and M is saturated (and $|M| > |\mathcal{L}|$). Let $a, b \in M^z$ be such that $\mathrm{tp}(a/X) = \mathrm{tp}(b/X)$. Show that there exists $\sigma \in \mathrm{Aut}(M/X) = \{\sigma \in \mathrm{Aut}(M) : \sigma|_X = \mathrm{id}\}$ such that $\sigma(a) = b$.