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Homework 1

Problem 1 :
Let $\mathfrak{U}$ be some an ultrafilter on some infinite set I. Show that the following are equivalent:

1. The filter $\mathfrak{U}$ is not principal;
2. The filter $\mathfrak{U}$ contains the Fréchet filter on I.

Problem 2 :
Let $\mathcal{L}$ be a countable language, $(M_i)_{i\in\mathbb{Z}_{\geqslant 0}}$ a countable family of $\mathcal{L}$-structures, $\mathfrak{U}$ a non principal ultrafilter on $\mathbb{Z}_{\geqslant 0}$, and $\Sigma(\overline{x}) = \{\varphi_j : j \in \mathbb{Z}_{\geqslant 0}\}$ a set of $\mathcal{L}$-formulas (whose free variables are in the tuple $\overline{x}$). We assume that for all finite $\Sigma_0 \subseteq \Sigma$, $\prod_{i\in\mathfrak{U}} M_i \vDash \exists\overline{x} \bigwedge_{\varphi\in\Sigma_0} \varphi(\overline{x})$.

1. For all $n \in \mathbb{Z}_{\geqslant 0}$, let $S(n) \coloneqq \{i \in \mathbb{Z}_{\geqslant 0} : M_i \vDash \exists\overline{x} \bigwedge_{j=0}^{n} \varphi_j(\overline{x})\}$. Show that the $S(n)$ form a decreasing chain of elements of $\mathfrak{U}$.
2. For all i, et $\overline{b}_i \in M_i^{\overline{x}}$ be defined as follows:
   - If $i \in S(0)$, let m be maximal such that $m \leqslant i$ and $i \in S(m)$. Then pick $\overline{b_i} \in M_i$ such that $M_i \vDash \bigwedge_j \varphi_{j=0}^{m}(\overline{b}_i)$;
   - Otherwise, pick any $\overline{b}_i \in M_i$.

   Show that for all $\varphi \in \Sigma$, we have $\prod_{i\to\mathfrak{U}} M_i \vDash \varphi([(\overline{b}_i)_i]_{\mathfrak{U}})$.
3. Let $M \coloneqq \mathbb{Q}^{\mathfrak{U}} = \prod_{i\to\mathfrak{U}} \mathbb{Q}$ with the natural $\mathcal{L}_{\text{or}}$-structure. We identify $\mathbb{Q}$ with its image under the diagonal embedding (i.e. the map $x \mapsto [(x)_i]_{\mathfrak{U}}$). Let $\mathcal{O} \coloneqq \{a \in M : \exists n \in \mathbb{Z}_{>0}\ -n < a < n\}$ and $\mathfrak{M} \coloneqq \{a \in M : \forall n \in \mathbb{Z}_{>0}\ -\frac{1}{n} < a < \frac{1}{n}\}$. Show that $\mathcal{O}$ is a ring and that $\mathfrak{M}$ is a maximal ideal.
4. Show that $\mathcal{O}/\mathfrak{M}$ is isomorphic to $\mathbb{R}$.

Problem 3 :
Prove whether each of the following classes is elementary (i.e. the set of models of a theory), finitely axiomatisable (i.e. the set of models of a fintie theory) or none of those.

1. Infinite sets in some language $\mathcal{L}$ with one sort.
2. Finite sets in some language $\mathcal{L}$ with one sort.
3. Fields in $\mathcal{L}_{\text{rg}} \coloneqq \{\mathbf{K}; 0 : \mathbf{K}, 1 : \mathbf{K}, - : \mathbf{K} \to \mathbf{K}, + : \mathbf{K} < 2 >\to \mathbf{K}, \cdot : \mathbf{K} < 2 >\to \mathbf{K}\}$.
4. Characteristic p fields (for some fixed prime p) in $\mathcal{L}_{\text{rg}}$.
5. Characteristic 0 fields in $\mathcal{L}_{\text{rg}}$.
6. Algebraically closed fields in $\mathcal{L}_{\text{rg}}$ (the proof requires algebra beyond Math 113).