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Solutions to homework 5

Problem 1 :
Let $\mathcal{L}_< := \{X; <: X^2\}$. We say that a total order $X, <$ is discrete if for all $x \in X$, $\{y \in X : y < x\}$ has a maximal element and $\{y \in X : x < y\}$ has a minimal element. Note that this definition of discrete order implies that there are no end points.

1. Give a theory T whose models are exactly the non empty discrete total orders.

   Solution: We define:

$$\begin{aligned} T &:= \{\forall x\, \neg x < x\} \\ &\cup\{\forall x \forall y \forall z\, (x < y \wedge y < z) \to x < z\} \\ &\cup\{\forall x \forall y\, x < y \vee y < x \vee x = y\} \\ &\cup\{\forall x \exists y\, y < x \wedge (\forall z\, z < x \to (z < y \vee z = y))\} \\ &\cup\{\forall x \exists y\, x < y \wedge (\forall z\, x < z \to (y < z \vee z = y))\} \\ \cup\{\exists x\, x = x\}. \end{aligned}$$

   The last sentence is here just to ensure that the structure is non-empty.

2. Let $(X, <)$ be an order, show that the following are equivalent:
   a) $X \vDash T$;
   b) there exists a (non empty) total order $(Y, <)$ such that X is $\mathcal{L}_<$-isomorphic to $\mathbb{Z} \times Y$ with the right to left lexicographic order (i.e. $(n, x) < (m, y)$ if $x < y$ or $x = y$ and $n < m$).

   Solution: Let us assume that $X \vDash T$. Let $s : X \to X$ be the successor function defined below. Note that s is a bijection so s^{-1} makes sense. For x, $y \in X$, we define $x \sim y$ if $y = s^n(x)$ for some $n \in \mathbb{Z}$. This is an equivalence relation and let $Y = X/\sim$. We can order Y by $\overline{x} < \overline{y}$ if $x < y$ and $\overline{x} \neq \overline{y}$ (this is well defined because each equivalence class is convex).

   For each class $y \in Y$ let us fix a point $x_y \in y$. We define $f : X \to \mathbb{Z} \times Y$ by $f(x) = (n, \overline{x})$ where $n \in \mathbb{Z}$ is such that $x = s^n(x_{\overline{x}})$. The inverse of f is the map $(n, z) \mapsto s^n(x_z)$. So f is bijective. Moreover, if $\overline{x} \neq \overline{y}$, then $x < y$ if and only if $\overline{x} < \overline{y}$ if and only if $f(x) < f(y)$. If $\overline{x} = \overline{y} =: z$, then $x = s^n(x_z) < s^m(x_z) = y$ if and only if $n < m$ if and only if $f(x) < f(y)$. In both cases, we have $x < y$ if and only if $f(x) < f(y)$, so f is an $\mathcal{L}_<$-isomorphism.

   For the converse, it suffices to show that every $\mathbb{Z} \times Y \vDash T$. This is indeed a linear order and for every $(n, y) \in \mathbb{Z} \times Y$, the immediate successor is $(n+1, y)$ and the immediate predecessor is

3. Show that T does not eliminate quantifiers.

   Solution: Consider $2\mathbb{Z} \leqslant \mathbb{Z}$. Both are models of T but the extension is not elementary. Indeed, the formula $\mathcal{L}_<(2\mathbb{Z})$-sentence $\forall x\, x < 0 \vee x = 0 \vee x = 2 \vee 2 < x$ holds in $2\mathbb{Z}$ but not in $\mathbb{Z}$. It follows that T is not model complete and hence, does not eliminate quantifiers.

4. Let $\mathcal{L}_s := \mathcal{L}_> \cup \{s : X \to X\}$. Give a theory T_s whose models are exactly the models of T where s is interpreted as the successor function, i.e. $s(x) = \min\{y \in X : y > x\}$.

   Solution: We define $T_s := T \cup \{\forall x\, x < s(x) \wedge (\forall z\, x < z \to (s(x) < z \vee s(x) = z))\}$.

5. Show that T_s eliminates quantifiers and is complete.

   Solution: Let M, $N \models T_s$, $A \subseteq M$, $f : A \to N$ be a partial embedding. Assume that N is $(|A|^+ + \aleph_1)$-saturated and pick $b \in M$, we want to extend f to Ab.

   First, let $C = \{s^n(a) \in M : n \in \mathbb{Z} \text{ and } a \in A\}$. We define $g : C \to N$ by $g(s^n(a)) = s^n(f(a))$. Let us show that g is well defined and injective. Pick any n_1, $n_2 \in \mathbb{Z}$ and a_1, $a_2 \in A$ and let $n = \min\{n_1, n_2\}$. Then $s^{n_1}(a_1) = s^{n_2}(a_2)$ if and only if $s^{n_1-n}(a_1) = s^{n_2-n}(a_2)$. Now, $n_i - n \geqslant 0$ so this is an atomic formula. It follows that the previous statement is equivalent to $s^{n_1-n}(f(a_1)) = s^{n_2-n}(f(a_2))$ which is itself equivalent to $s^{n_1}(f(a_1)) = s^{n_2}(f(a_2))$.

   Let us now show that g is an $\mathcal{L}_s$-embedding. First $s(g(s^n(a))) = s^{n+1}(f(a)) = g(s^{n+1}(a)) = g(s(s^n(a)))$. And, with the same notations as above $s^{n_1}(a_1) < s^{n_2}(a_2)$ if and only if $s^{n_1-n}(a_1) < s^{n_2-n}(a_2)$ if and only if $s^{n_1-n}(f(a_1)) < s^{n_2-n}(f(a_2))$ if and only if $g(s^{n_1}(a_1)) = s^{n_1}(f(a_1)) < s^{n_2}(f(a_2)) = g(s^{n_2}(a_2))$.

   If $b \in C$, we are done. Otherwise, let $\Sigma(x) := \{g(c) < x : c \in C \text{ and } c < b\} \cup \{x < g(c) : c \in C \text{ and } b < c\}$. Let us show that $\Sigma(x)$ is finitely satisfiable in N. Let $\Sigma_0 \subseteq \Sigma$ be finite. Then there are C_1, $C_2 \subseteq C$ finite such that $C_1 < b < C_2$ and $\Sigma_0 = \{g(c) < x : c \in C_1\} \cup \{x < g(c) : c \in C_2\}$. If C_1 and C_2 are non empty let $c_1 = \max C_1$ and $c_2 = \min C_2$, Since $c_1 < b < c_2$ we must have $s(c_1) < c_2$. It follows that $s(g(c_1)) < g(c_2)$ and Σ_0 is realized in N by $s(g(c_1))$. If C_2 is empty, we also have $N \models \Sigma_0(s(g(c_1)))$. And if $C_1 = \varnothing$, then $N \models \Sigma_0(s^{-1}(g(c_2)))$. If both C_1 and c_2 are empty, then Σ_0 is empty and any $d \in N$ realizes it.

   By our saturation hypothesis, we find $d \in N$ realizing Σ. Note that there are no n such that $s^n(b) \in C$ or $s^n(d) \notin g(C)$ (otherwise we would have either $b \in C$ or $d \in g(C)$). We extend g by defining $h(s^n(b)) = s^n(d)$ for all $n \in \mathbb{Z}$ (and $h|_C = g$). Then h is well defined and injective. Moreover $s^n(b) < c$ if and only if $b < c$, if and only if $d < g(c)$ if and only if $s^n(d) < g(c)$ (and symmetrically if $c < s^n(b)$). So h is a partial $\mathcal{L}_s$-embedding defined at b and extending f.

   By the criterion for elimination of quantifiers, we have that T eliminates quantifiers. Note that there are no constants in $\mathcal{L}_s$ so it follows from quantifier elimination that T is complete.

   If you are bothered by this whole empty substructure thing, note that we have indeed proved in the course of the proof that the the quantifier free type of any singleton is always the same, so the set of partial embeddings between models of T_s with finite domain is never empty.

6. Show that T is complete.

   Solution: Let M, $N \models T$. We can enrich M into $M_s \models T_s$ be interpreting s as the successor function. Similarly we can enrich N into $N_s \models T_s$. By the previous question, $M_s \equiv_{\mathcal{L}_s} N_s$. It follows that for all $\mathcal{L}_<$-sentence, $M \models \varphi$ if and only if $M_s \models \varphi$ if and only if $N_s \models \varphi$ if and only if $N \models \varphi$.

Problem 2 :
Let M be an $\mathcal{L}$-structure and λ be a cardinal. We say that M is a λ-model if $|M| = \lambda$ and for every $\mathcal{L}$-formula φ, either $\varphi(M)$ is finite or $|\varphi(M)| = \lambda$.

1. Let M be an $\mathcal{L}$ structure and λ be a cardinal greater or equal to $|M|$ and $|\mathcal{L}|$. Show that there exists $N \succcurlyeq M$ which is a λ-model.

   Solution: We construct a elementary chain $(M_i)_{i\in\omega}$ such that $M_0 = M$, for every $\mathcal{L}(M_i)$-formula φ, if φ has infinitely many realizations in M_i, $|\varphi(M_{i+1})| = \lambda$ and $|M_i| \leqslant \lambda$. If M_i is built, we obtain M_{i+1} by compactness and Löwenheim-Skolem. Let Φ_i be the set of all $\mathcal{L}(M_i)$-formulas with infinitely many realizations in M_i. For all $\varphi(x) \in \Phi_i$, let $(c_{i,\varphi})_{i\in\lambda}$ be new constants sorted like x. We consider $\Sigma := \mathcal{D}^{\mathrm{el}}(M_i) \cup \{\varphi(c_{i,\varphi}) : i \in \lambda, \varphi \in \Phi_i\} \cup \{c_{i,\varphi} \neq c_{i,\varphi} : i \neq j, \varphi \in\in \Phi_i\}$. Since the φ have infinitely many realizations, Σ is finitely satisfiable (in M_i). Note that, there are at most λ formulas φ, so the language of Σ is size λ and we can find a model M_{i+1} of Σ of size λ.

   Let $N = \bigcup_i M_i$. Then $|N| = \lambda$, $N \succcurlyeq M$. Moreover, any $\mathcal{L}(N)$-formula $\varphi(x)$ is an $\mathcal{L}(M_i)$-formula for some i. If φ has infinitely realizations in N, the since $M_i \preccurlyeq N$, it alos has infinitely many realizations in M_i. By construction, $\varphi(M_{i+1}) \subseteq \varphi(N)$ has cardinality λ.

2. Let T be an $\mathcal{L}$-theory and λ be a cardinal greater than $|\mathcal{L}|$. Assume that all the λ-models of T are isomorphic. Show that T is complete.

   Solution: Let M, $N \vDash T$. By downwards Löwenheim-Skolem, we find $M_0 \preccurlyeq M$ and $N_0 \preccurlyeq N$ such that $|M_0| = |N_0| = |\mathcal{L}|$. By the previous question, we can find λ-models $M_1 \succcurlyeq M_0$ and $N_1 \succcurlyeq N_0$. By our hypothesis $M_1 \cong N_1$ and hence $M \equiv M_0 \equiv M_1 \equiv N_1 \equiv N_0 \equiv N$.

3. Assume that T is λ-categorical for some $\lambda \geqslant |\mathcal{L}|$. Show that every model of T of cardinality λ is a λ-model.

   Solution: Let $M \vDash T$ have cardinality λ, then, by Question 1, we can find a λ-model $N \succcurlyeq M$. Since T is λ-categorical, $M \cong N$ is also a λ-model.

Problem 3 :
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. Let $\kappa = \aleph_0 + |A|$. Assume M is strongly κ^+-homogeneous and κ^+-saturated. Let $X \subseteq M^x$ be $\mathcal{L}(M)$-definable and assume that for all $\sigma \in \mathrm{Aut}(M/A)$, $\sigma(X) = X$. Show that X is $\mathcal{L}(A)$-definable.

Solution: We have $X = \varphi(M, m)$ for some $m \in M^y$. Let $p = \mathrm{tp}(m/A)$ and $\Sigma(y) := p(y) \cup \{\exists x \neg\varphi(x,y) \wedge \varphi(x,m)\}$. If Σ is satisfiable, by κ^+-saturation, we find $n \in M^y$ such that $\mathrm{tp}(n/A) = \mathrm{tp}(m/A)$ and $\varphi(M,n) \neq \varphi(M,m)$. By strong κ^+-homogeneity, we can find $\sigma \in \mathrm{Aut}(M/A)$ such that $\sigma(m) = n$, but then $\sigma(X) = \varphi(M,n) \neq \varphi(M,m) = X$ a contradiction. It follows that we can find $\psi(y,a) \in \mathrm{tp}(m/A)$ such that, for all $n \in M^y$, if $M \vDash \psi(n)$, then $X \subseteq \varphi(M,n)$. By a symmetric argument, we find $\chi(y) \in \mathrm{tp}(m/A)$ such that, for all $n \in M^y$, if $M \vDash \psi(n)$, then $\varphi(M,n) \subseteq X$. It follows that $\theta(x) := \exists y\, \psi(y) \wedge \chi(y) \wedge \varphi(x,y)$ is and $\mathcal{L}(A)$-formula defining X.