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Solutions to homework 7

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Problem 1 :
Let T be an $\mathcal{L}$-theory. Let $M \vDash T$ and X be an $\mathcal{L}(M)$-definable set. Recall that X is coded in T if X is $\mathcal{L}(\ulcorner X \urcorner \cap M)$-definable.

1. Show that the following are equivalent:

   a) T eliminates imaginaries;

   b) For all $M \vDash T$, every $\mathcal{L}(M)$-definable function is coded.

   Hint: A definable function has a domain.

   Solution: Since (graphs of) definable functions are particular cases of definable sets, b) is a consequence of a). On the other hand, If b) holds and X is a definable set, then the code for identity function on X is also a code for X. Indeed X is $\mathcal{L}(A)$-definable if and only if the identity function on X is $\mathcal{L}(A)$-defined over A.

2. Let $S_0, \dots S_k$ be $\mathcal{L}$-sorts. Assume that every $\mathcal{L}(M)$-definable function whose domain is contained in S_0 is coded and that every $\mathcal{L}(M)$-definable function whose domain is a subset of $\prod_{i=1}^k S_i$ is coded. Show that every $\mathcal{L}(M)$-definable function whose domain is in $\prod_{i=0}^k S_i$ is coded.

   Solution: Replacing M by an elementary extension, we may assume that M is $|\mathcal{L}|^+$-saturated. Let f be a definable function whose domain is contained in $\prod_{i=0}^k S_i$. for all $a \in S_0(M)$, let $f_a(x) = f(a,x)$. Then f_a is a definable function whose domain is contained in $\prod_{i=1}^k S_i$. By our hypothesis, f_a is $\mathcal{L}(\ulcorner f_a \urcorner \cap M)$-definable so there exists an $\mathcal{L}$-formula $\varphi(x,y,z)$ and $c \in \ulcorner f_a \urcorner \cap M$ such that $\varphi(x,y,c)$ holds if and only if $y = f(a,x)$. Since $\ulcorner f_a \urcorner \subseteq \mathrm{dcl}^{\mathrm{eq}}(\ulcorner f \urcorner \cup \{a\})$, there exists an $\mathcal{L}^{\mathrm{eq}}(\ulcorner f \urcorner)$-definable map g such that $c = g(a)$. By hypothesis g is $\mathcal{L}(\ulcorner g \urcorner \cap M)$-definable. Since g is $\mathcal{L}^{\mathrm{eq}}(\ulcorner f \urcorner)$-definable, $\ulcorner g \urcorner \subseteq \ulcorner f \urcorner$ and hence g is $\mathcal{L}(\ulcorner f \urcorner \cap M)$-definable. In particular, f_a is $\mathcal{L}(\{a\} \cup (\ulcorner f \urcorner \cap M))$-definable.

   Let $\pi(t) \coloneqq \{\neg(y = f(t,x) \leftrightarrow \theta(x,y,t)) : \theta \text{ is an } \mathcal{L}(\ulcorner f \urcorner \cap M)\text{-formula}\}$. We have just proved that π is not satisfiable in M. Note that, since $\ulcorner f \urcorner$ is the definable closure of a singleton its cardinality is smaller or equal to $|\mathcal{L}^{\mathrm{eq}}| = |\mathcal{L}|$. By $|\mathcal{L}|^+$-saturation of M, π is not finitely satisfiable. It follows that there exist $\mathcal{L}(\ulcorner f \urcorner \cap M)$-formulas $\theta_i(x,y,t)$, for $0 \leqslant i < k$ such that for all $a \in S_0(M)$, the graph of f_a is defined by $\theta_i(x,y,a)$ for some i. Let $X_i = \{t \in S_0 : \text{the graph of } f_t \text{ is defined by } \theta_i(x,y,t)\}$. The set X_i is $\mathcal{L}(\ulcorner f \urcorner \cap M)$-definable and $y = f(t,x)$ is equivalent to $\bigvee_i (t \in X_i \wedge \theta_i(x,y,t))$ which is obviously an $\mathcal{L}(\ulcorner f \urcorner \cap M)$-formula.

3. Show that the following are equivalent:

   a) T eliminates imaginaries;

   b) for every $\mathcal{L}$-sort S and $M \vDash T$, every $\mathcal{L}(M)$-definable function f whose domain is contained in S is coded.

Solution: Once again, b) is a particular case of a). Let us now assume that b) holds. We prove by induction on k that every $\mathcal{L}(M)$-definable function f whose domain is in $\prod_{i=0}^{k} S_i$ is coded. The case $k = 0$ is our hypothesis and the induction step is the previous question. We conclude by the first question that T eliminates imaginaries.

Problem 2 :
Let T be a complete $\mathcal{L}$-theory with one sort X and no function symbols or constants. Assume that T eliminates quantifiers and imaginaries and that, in models of T, the algebraic and definable closure coincide. Let $\mathcal{L}_{f,<}$ be the language $\mathcal{L}$ with a new sort Y, a function symbol $f : X \to Y$ and a predicate $<: Y^2$. Let $T_{f,<}$ be the theory axiomatizing the following:

- f is surjective;
- For all $a \in Y$, $f^{-1}(a)$ is a model of T;
- For all $\mathcal{L}$-predicate $R(x_1, \ldots, x_n)$ and tuple $x_1, \ldots, x_n \in X$, if $R(x_1, \ldots, x_n)$ holds then for all i, j, $f(x_i) = f(x_j)$.
- $(Y, <)$ is a dense linear order without end-points.

1. Show that $T_{f,<}$ eliminates quantifiers.

   Solution: Let M, $N \vDash T_{f,<}$, g a partial embedding from M into N whose domain is A and $a \in M$. Assuming that N is $|A|^+$-saturated, we have to extend g to a. We may assume that A is closed under f.

   Let us first assume that $a \in Y(M) \smallsetminus A$. Then $f^{-1}(a) \cap A = \emptyset$. Let $D = \{c \in Y(A) : c < a\}$. Pick any $b \in Y(N) \smallsetminus g(A)$ such that for all $c \in A$, $g(c) < b$ if and only if $c \in D$ — such a b exists by saturation and the fact that $Y(N)$ is a dense linear order without endpoints. Then g can be extended by sending a to b.

   If $a \in X(M)$, let $c := f(a) \in A$. We have that $f^{-1}(c)$, $f^{-1}(g(c)) \vDash T$ and since T is complete, we have $f^{-1}(c) \equiv f^{-1}(g(c))$. Let g_c be the restriction of g to $f^{-1}(c)$. The map g_c is a partial embedding from $f^{-1}(c)$ into $f^{-1}(g(c))$. By quantifier elimination, g_c is a partial elementary embedding — when the domain of g_c is empty, we are using the fact that T is complete. Note that, since N is $|A|^+$-saturated, so is $f^{-1}(g(c))$. It follows that g_c can be extended to a. This extension is also an extension of g since predicates of $\mathcal{L}$ are always false when applied to points in distinct fibers.

2. Let $M \vDash T_{f,<}$ and $A \leqslant M$. Assume M is strongly $|A|^+$-homogeneous and $|A|^+$-saturated. Pick $c \in Y(M) \smallsetminus Y(A)$ and for all $a \in Y(A)$ pick σ_a be an $\mathcal{L}$-automorphism of $f^{-1}(a)$. Show that there exists $\sigma \in \mathrm{Aut}_{\mathcal{L}_{f,<}}(M)$ such that for all $a \in Y(a)$, $\sigma|_{f^{-1}(a)} = \sigma_a$ and $\sigma(c) \neq c$.

   Solution: Note first that, by quantifier elimination, for any a, $b \in Y(M)$, $\mathrm{tp}(a) = \mathrm{tp}(b)$ so there exists $\sigma \in \mathrm{Aut}_{\mathcal{L}_{f,<}}(M)$ such that $\sigma(a) = b$. In particular, there is an $\mathcal{L}$-isomorphism $\theta_{a,b} : f^{-1}(a) \to f^{-1}(b)$.

   Now, pick an order isomorphism τ of $Y(M)$ fixing $Y(A)$ and moving c. This can always be done because $Y(M)$ is a $|A|^+$-saturated and strongly $|A|^+$-homogeneous dense linear order without endpoints. Let us define σ as follows. If $y \in Y$, $\sigma(y) = \tau(y)$. If $f(x) \in Y(A)$, let $\sigma(x) = \sigma_{f(x)}(x)$. If $f(x) \notin Y(A)$, let $\sigma(x) =$

$\theta_{f(x),\tau(f(x))}(x)$. Note that σ preserves f, $\sigma|_Y = \tau$ is an order isomorphism and, fiber by fiber, σ is an $\mathcal{L}$-isomorphism, so it is an $\mathcal{L}_{f,<}$-automorphism of M.

If $a \in Y(A)$, by definition of σ, we have that $\sigma|_{f^{-1}(a)} = \sigma_a$. Also, $\sigma(c) = \tau(c) \neq c$.

3. Let $M \vDash T_{f,<}$ and $A \leqslant M$. For all $a \in Y(M)$, let dcl^a denote the $\mathcal{L}$-definable closure in the $\mathcal{L}$-structure $f^{-1}(a)$. Show that $\mathrm{dcl}(A) = Y(A) \cup \bigcup_{a \in Y(A)} \mathrm{dcl}^a(A \cap f^{-1}(a))$.

   Solution: Going to an elementary extension, we may assume that M is strongly $|\mathcal{L}(A)|^+$-homogeneous and $|\mathcal{L}(A)|^+$-saturated. Then an element $c \in M$ is in $\mathrm{dcl}(A)$ if and only if c is fixed by all $\mathcal{L}_{f,<}$-automorphisms of M that fix A. If $c \in Y(M) \smallsetminus Y(A)$ or $c \in X(M)$ but $f(c) \notin Y(A)$, then we have build in the previous question an $\mathcal{L}_{f,<}$-automorphism of M that does not fix c (take all the σ_a to be the identity). If $c \in X(M)$, $a = f(c) \in Y(A)$ and $c \notin \mathrm{dcl}^a(f^{-1}(a))$, then, since $f^{-1}(A)$ is strongly $|\mathcal{L}(A)|^+$-homogeneous and $|\mathcal{L}(A)|^+$-saturated, we can find an $\mathcal{L}(f^{-1}(a) \cap A)$-automorphism σ_a of $f^{-1}(a)$ which does not fix c. In the previous question, we showed that we can find an $\mathcal{L}_{f,<}$-automorphism of M equal to σ_a on $f^{-1}(a)$. This automorphism does not fix c. It follows that $\mathrm{dcl}(A) \subseteq Y(A) \cup \bigcup_{a \in Y(A)} \mathrm{dcl}^a(A \cap f^{-1}(a))$.

   The converse inclusion is easier. Let $c \in Y(A) \cup \bigcup_{a \in Y(A)} \mathrm{dcl}^a(A \cap f^{-1}(a))$. If $c \in Y(A) \subseteq A$, then we are done. Otherwise, we have $c \in X(M)$. Let $a = f(c)$. There exists an $\mathcal{L}(f^{-1}(a) \cap A)$-formula $\varphi(x)$ such that $\varphi(f^{-1}(a)) = \{c\}$. Then c is defined in M by $f(c) = a \wedge \varphi_a(c)$ (where φ_a is the relativization of φ to $f^{-1}(a)$).

4. Let $M \vDash T_{f,<}$ and $g : X \to Y$ be an $\mathcal{L}_{f,<}(M)$-definable map. Show that there exists $(a_i)_{0 \leqslant i < k} \in Y(M)$ such that if $g(x) \neq f(x)$, then $g(x) = a_i$ for some i.

   Solution: Let $A \leqslant M$ be such that g is $\mathcal{L}(A)$-definable. We may assume that M is $|A|^+$-saturated. By the previous question, $g(x) \in Y(A) \cup \{f(x)\}$. So the set of $\mathcal{L}(A)$-formulas $\pi(x) := \{g(x) \neq a : a \in A\} \cup \{g(x) \neq f(x)\}$ is not satisfiable in M and, by saturation, π is not finitely satisfiable. It follows that there exists finitely many $a_i \in Y(A)$ such that $M \vDash g(x) = f(x) \vee \bigvee_i g(x) = a_i$.

5. Let $M \vDash T_{f,<}$ and $g : X \to X$ be an $\mathcal{L}_{f,<}(M)$-definable map. Assume that for all x, $f(g(x)) = f(x)$. Show that there exists finitely many $a_i \in Y(M)$, $g_i : f^{-1}(a_i) \to f^{-1}(a_i)$ $\mathcal{L}(f^{-1}(a_i))$-definable, $W_j \subseteq Y$ open intervals and h_j $\mathcal{L}$-definable maps such that:

   - $g|_{f^{-1}(a_i)} = g_i$;
   - for all $c \in W_j$, $g|_{f^{-1}(c)} = h_j$.

   Solution: As before, let $A \leqslant M$ be such that g is $\mathcal{L}(A)$-definable and let us assume that M is $|A|^+$-saturated. Pick $y \in Y(M) \smallsetminus Y(A)$, then $g|_{f^{-1}(y)}$ is an $\mathcal{L}_{f,<}(A \cup \{y\})$-definable subset of $f^{-1}(Y)$. By quantifier elimination in $T_{f,<}$ (and induction on quantifier free $\mathcal{L}_{f,<}$-formulas), $g|_{f^{-1}(y)}$ is an $\mathcal{L}$-definable in $f^{-1}(y)$.

   It follows that the set $\pi(y) := \{y \neq a : a \in Y(A)\} \cup \{\exists x_1 \exists x_2\, f(x_1) = y \wedge \neg(x_2 = g(x_1) \leftrightarrow \varphi(x_1, x_2)) : \varphi\ \mathcal{L}\text{-formula}\}$ is not satisfiable in M and it is therefore not finitely satisfiable either. So there exists finitely many $a_i \in Y(A)$ and $\mathcal{L}$-formulas φ_j such that if $y \neq a_i$ for any i, then $g|_{f^{-1}(y)}$ is defined in $f^{-1}(y)$ by φ_j, for some j. The set $W_j := \{y \in Y : g|_{f^{-1}(y)} = \varphi_j(f^{-1}(y))\}$ is an $\mathcal{L}_{f,<}$-definable subset of Y. It follows from quantifier elimination in $T_{f,<}$ (and induction on quantifier free $\mathcal{L}_{f,<}$-formulas) that W_j is a finite union of points and open intervals. Making C

bigger, we may assume that W_j is a finite union of open intervals. Renumbering these intervals, we may assume that W_j is an open interval.

Finally, for every a_i, $g|_{f^{-1}(a_i)}$ is an $\mathcal{L}_{f,<}(A)$-definable map, so, as above, it is of the form $\varphi_i(f^{-1}(a_i))$ for some $\mathcal{L}(f^{-1}(a_i) \cap A)$-formula.

6. Let $M \models T_{f,<}$ and $g : X \to X$ be an $\mathcal{L}_{f,<}(M)$-definable map. Assume that for all x, $f(g(x)) \neq f(x)$. Show that there exists finitely many $a_i \in X(M)$, finitely many $c_j \in Y(A)$, finitely many open intervals $W_k \subseteq Y$, $\mathcal{L}(f^{-1}(c_j))$-formulas $\varphi_{i,j}$ and $\mathcal{L}$-formulas $\psi_{i,k}$ such that, for all i,

$$g(x) = a_i \text{ if and only if } x \in \bigcup_j \varphi_{i,j}(f^{-1}(c_j)) \cup \bigcup_k \bigcup_{y \in W_k} \psi_{i,k}(f^{-1}(y)).$$

Solution: Once again, let $A \leqslant M$ be such that g is $\mathcal{L}(A)$-definable and let us assume that M is $|A|^+$-saturated. By Question 2.2, $g(x) \in \mathrm{dcl}^{f(x)}((A \cap f^{-1}(x)) \cup \{x\}) \cup \bigcup_{a \in Y(a)} \mathrm{dcl}^a(A \cap f^{-1}(a))$. Since $f(g(x)) \neq f(x)$, we have that $g(x) \in \bigcup_{a \in Y(a)} \mathrm{dcl}^a(A \cap f^{-1}(a))$. It follows that $\pi(x) := \{g(x) \neq c : c \in \bigcup_{a \in Y(a)} \mathrm{dcl}^a(A \cap f^{-1}(a))\}$ is not satisfiable in M. So it is not finitely satisfiable either and the image of g must be some finite set $\{a_i : 0 \leqslant i < n\} \subseteq A$.

For all $c \in Y(M) \smallsetminus Y(A)$, $g^{-1}(a_i) \cap f^{-1}(c)$ is an $\mathcal{L}_{f,<}(A \cup \{c\})$-definable subset of $f^{-1}(c)$. Then, there exists a formula ψ such that $g^{-1}(a_i) \cap f^{-1}(c) = \psi(f^{-1}(c))$. Therefore, the set $\pi(y) := \{y \neq a : a \in Y(A)\} \cup \{\exists x\, f(x) = y \wedge \neg(g(x) = a_i \leftrightarrow \psi(x)) : \psi\ \mathcal{L}\text{-formula}\}$ is not satisfiable in M. So it is not finitely satisfiable and we find a finite set $C \subseteq Y(A)$ and m $\mathcal{L}$-formulas $\psi_{i,l}$ such that, if $y \in Y(M) \smallsetminus C$, $g^{-1}(a_i) \cap f^{-1}(y) = \psi_{i,l}(f^{-1}(y))$ for some l. Let $k : n \to m$. The set $W_k = \{y \in Y : \forall x\, f(x) = y \to \bigwedge_i (g(x) = a_i \leftrightarrow \psi_{i,k(i)}(x))\}$ is an $\mathcal{L}_{f,<}$-definable subset of Y. It is a finite union of points and open intervals. Making C bigger, we may assume that W_k is a finite union of open intervals. Renumbering these intervals, we may assume that W_k is an open interval.

Finally, for all $c_j \in C$, $g^{-1}(a_i) \cap f^{-1}(c_j)$ is an $\mathcal{L}_{f,<}(A)$-definable subset of $f^{-1}(c_j)$ and it is of the form $\varphi_{i,j}(f^{-1}(c_j))$ for some $\mathcal{L}(f^{-1}(c_j) \cap A)$-formula.

7. Show that $T_{f,<}$ eliminates imaginaries.

Solution: By Question 1.3, it suffices to show that every function, whose domain is a subset of some sort, is coded. Note that if the image of the function is inside a product of sorts, it suffices to code each of the components to code the function. Let $g : X \to X$ be a definable function. Let $F := \{x \in X : f(g(x)) = f(x)\}$. The functions $g_1 := g|_F$ and $g_2 := g|_{F^c}$ are both $\ulcorner g \urcorner$-definable and, since $g = g_1 \cup g_2$, g is $\ulcorner g_1 \urcorner \cup \ulcorner g_2 \urcorner$-definable. So we may assume that for all x, $f(g(x)) = f(x)$ or that for all x, $g(x) \neq f(x)$.

Let us first assume that for all x, $f(g(x)) = f(x)$. Let a_i, g_i, W_j and h_j be as in Question 2.5. Reordering the W_j and making W_1 bigger, we may assume that it is the largest interval which appears first in the order, on which $g|_{f^{-1}(y)} = g_1$. Then W_1 is $\ulcorner g \urcorner$-definable and it suffices to encode $g|_{f^{-1}(W_1)}$ and $g|_{f^{-1}(W_1^c)}$. Let $W_1 = (a, b)$, then $a, b \in \ulcorner g \urcorner \cap Y(M)$ and $g|_{f^{-1}(W_1)}$ is $\mathcal{L}_{f,<}(\{a, b\})$-definable, i.e. it is coded. By induction, we can remove all the W_j and we may assume that the domain of g is included in finitely many fibers (which are among the $f^{-1}(a_i)$).

By removing some of the a_i, we may assume that $f^{-1}(a_i)$ always intersect the domain of g. Note that, since Y is ordered, $a_i \in \ulcorner g \urcorner$. By elimination of imaginaries in T, $g|_{f^{-1}(a_i)}$ is coded by some tuple $c_i \in f^{-1}(a_i) \cap \ulcorner g|_{f^{-1}(a_i)} \urcorner \subseteq \ulcorner g \urcorner$ and each $g|_{f^{-1}(a_i)}$ is $\mathcal{L}_{f,<}(\{a_i, c_i\})$-definable. It follows that g is coded.

If $f(g(x)) \neq f(x)$ for all x, let a_i, c_j, W_k, $\varphi_{i,j}$ and $\psi_{i,k}$ be as in Question 2.6. Note that since Y is ordered and algebraic and definable closure coincide in T, $a_i \in \ulcorner g \urcorner$ whenever $g^{-1}(a_i) \neq \varnothing$. Reordering and enlarging $W_1 := (b, c)$, we may assume that it is the largest interval that appears first on which, for all i, $g^{-1}(a_i)$ is given by $\psi_{i,1}$. Note that $g|_{f^{-1}(W_1) \cap g^{-1}(a_i)}$ is $\mathcal{L}_{f,<}(\{a_i, b, c\})$-definable. By induction, we can remove each of the W_1 until g is defined on finitely many fibers (among the $f^{-1}(c_j)$). Removing the c_j where $f^{-1}(c_j) \cap \mathrm{dom}(g) = \varnothing$, we have that $c_j \in \ulcorner g \urcorner$. By elimination of imaginaries in T, $g^{-1}(a_i) \cap f^{-1}(c_j)$ is coded by some $d_{i,j} \in \ulcorner g \urcorner$. It follows that g is coded by the tuple of the $d_{i,j}$.

If $g : Y \to X$ then g is coded if and only if $g \circ f$ is coded. But these functions where just taken care of.

If $g : X \to Y$, let $F := \{x \in X : g(x) = f(x)\}$. For all $y \in Y$, $F_y := \{x \in F : f(x) = y\}$ is coded by some $c_y \in \mathrm{dcl}^{\mathrm{eq}}(\ulcorner g \urcorner \cup \{y\})$. Using that functions $Y \to X$ are coded, and the proof of Question 2.2, we can show that the map $y \mapsto c_y$ (and therefore F) is coded. So it suffices to code $g|_{F^c}$. Let a_i be as in Question 2.3. Because Y is ordered, $a_i \in \ulcorner g \urcorner$ (provided we remove the useless ones). Moreover, as for F, the set $X_i = g^{-1}(a_i)$ is coded.

Finally, assume $g : Y \to Y$. Then g is coded since $f \circ g$ is.