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Homework 7

Problem 1 :
Let T be an $\mathcal{L}$-theory. Let $M \models T$ and X be an $\mathcal{L}(M)$-definable set. We say that X is coded in T if X is $\mathcal{L}(\ulcorner X \urcorner \cap M)$-definable.

1. Show that the following are equivalent:
   a) T eliminates imaginaries;
   b) For all $M \models T$, every $\mathcal{L}(M)$-definable function is coded.

   Hint: A definable function has a domain.

2. Let $S_0, \ldots S_k$ be $\mathcal{L}$-sorts. Assume that every $\mathcal{L}(M)$-definable function whose domain is contained in S_0 is coded and that every $\mathcal{L}(M)$-definable function whose domain is a subset of $\prod_{i=1}^{k} S_i$ is coded. Show that every $\mathcal{L}(M)$-definable function whose domain is in $\prod_{i=0}^{k} S_i$ is coded.

3. Show that the following are equivalent:
   a) T eliminates imaginaries;
   b) for every $\mathcal{L}$-sort S and $M \models T$, every $\mathcal{L}(M)$-definable function f whose domain is contained in S is coded.

Problem 2 :
Let T be a complete $\mathcal{L}$-theory with one sort X and no function symbols or constants. Assume that T eliminates quantifiers and imaginaries and that, in models of T, the algebraic and definable closure coincide. Let $\mathcal{L}_{f,<}$ be the language $\mathcal{L}$ with a new sort Y, a function symbol $f : X \to Y$ and a predicate $<: Y^2$. Let $T_{f,<}$ be the theory axiomatizing the following:

- f is surjective;
- For all $a \in Y$, $f^{-1}(a)$ is a model of T;
- For all $\mathcal{L}$-predicate $R(x_1, \ldots, x_n)$ and tuple $x_1, \ldots, x_n \in X$, if $R(x_1, \ldots, x_n)$ holds then for all i, j, $f(x_i) = f(x_j)$.
- $(Y, <)$ is a dense linear order without end-points.

1. Show that $T_{f,<}$ eliminates quantifiers.
2. Let $M \models T_{f,<}$ and $A \leqslant M$. Assume M is strongly $|A|^+$-homogeneous and $|A|^+$-saturated. Pick $c \in Y(M) \smallsetminus Y(A)$ and for all $a \in Y(A)$ pick σ_a be an $\mathcal{L}$-automorphism of $f^{-1}(a)$. Show that there exists $\sigma \in \mathrm{Aut}_{\mathcal{L}_{f,<}}(M)$ such that for all $a \in Y(a)$, $\sigma|_{f^{-1}(a)} = \sigma_a$ and $\sigma(c) \neq c$.
3. Let $M \models T_{f,<}$ and $A \leqslant M$. For all $a \in Y(M)$, let dcl^a denote the $\mathcal{L}$-definable closure in the $\mathcal{L}$-structure $f^{-1}(a)$. Show that $\mathrm{dcl}(A) = Y(A) \cup \bigcup_{a \in Y(A)} \mathrm{dcl}^a(A \cap f^{-1}(a))$.
4. Let $M \models T_{f,<}$ and $g : X \to Y$ be an $\mathcal{L}_{f,<}(M)$-definable map. Show that there exists $(a_i)_{0 \leqslant i < k} \in Y(M)$ such that, for all x if $g(x) \neq f(x)$, then $g(x) = a_i$ for some i.

5. Let $M \models T_{f,<}$ and $g : X \to X$ be an $\mathcal{L}_{f,<}(M)$-definable map. Assume that for all x, $f(g(x)) = f(x)$. Show that there exists finitely many $a_i \in Y(M)$, $g_i : f^{-1}(a_i) \to f^{-1}(a_i)$ $\mathcal{L}(f^{-1}(a_i))$-definable, $W_j \subseteq Y$ open intervals and $h_j : X \to X$ $\mathcal{L}$-definable maps such that:

   - $g|_{f^{-1}(a_i)} = g_i$;
   - for all $c \in W_j$, $g|_{f^{-1}(c)} = h_j$.

6. Let $M \models T_{f,<}$ and $g : X \to X$ be an $\mathcal{L}_{f,<}(M)$-definable map. Assume that for all x, $f(g(x)) \neq f(x)$. Show that there exists finitely many $a_i \in X(M)$, finitely many $c_j \in Y(M)$, finitely many open intervals $W_k \subseteq Y$, $\mathcal{L}(f^{-1}(c_j))$-formulas $\varphi_{i,j}$ and $\mathcal{L}$-formulas $\psi_{i,k}$ such that, for all i,

$$g(x) = a_i \text{ if and only if } x \in \bigcup_j \varphi_{i,j}(f^{-1}(c_j)) \cup \bigcup_k \bigcup_{y \in W_k} \psi_{i,k}(f^{-1}(y)).$$

7. Show that $T_{f,<}$ eliminates imaginaries.