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Solutions to homework 10

Problem 1 :
Let $\mathcal{L}$ be a countable language. A complete $\mathcal{L}$-theory T is almost strongly minimal if there exists $M \vDash T$ $\aleph_0$-saturated, $A \subseteq M$ finite and $\varphi(x)$ an $\mathcal{L}(A)$-formula such that φ is strongly minimal and $M = \mathrm{acl}(\varphi(M) \cup A)$.

1. Show that there exists an $\mathcal{L}(A)$-formula $\psi(x,y)$ such that for all $a \in \varphi(M)^y$, $|\psi(M,a)| < \infty$ and $M = \bigcup_{a\in\varphi(M)^y} \psi(M,a)$.

   Solution: Consider the set

   $$\Sigma(y) = \{\neg(\exists t \bigwedge_i \varphi(t_i) \wedge \theta(y,t) : M \vDash \forall y \exists^{\leqslant n} y\ \theta(y,t) \text{ and } \varphi \text{ is an } \mathcal{L}(A)\text{-formula}\}.$$

   If Σ is finitely satisfiable, we can find $b \in M^y$ realizing Σ. By hypothesis, there exits $a \in \varphi(M)^n$ and $\theta(y,t)$ such that $M \vDash \theta(b,a)$ and $M \vDash \exists^{\leqslant n} y \theta(y,a)$. Let $\psi(y,t) = \theta(y,t) \wedge \forall t \exists^{\leqslant n} y \psi(y,t)$. Since $M \vDash \Sigma(b)$, we have $\neg(\exists t \bigwedge_i \varphi(t_i) \wedge \theta(t,y)$, but we also have $M \vDash \bigwedge_i \varphi(a_i) \wedge \theta(y,a)$, a contradiction.

   So Σ is not finitely satisfiable and there exists finitely many $\mathcal{L}(A)$-formulas $\psi_i(y,t)$ such that for all $a \in \varphi(M)^n$, $|\psi_i(y,a)| < \infty$ and for all $b \in M$, there exists i and $a \in \varphi(M)^n$ such that $M \vDash \psi_i(M,a)$. Let $\psi(y,t) = \bigcup_i \psi_i(y,t)$. Then we do have that for all $a \in \varphi(M)^y$, $|\psi(M,a)| < \infty$ and $M = \bigcup_{a\in\varphi(M)^y} \psi(M,a)$.

2. Show that T is ω-stable.

   Solution: Let $N \succcurlyeq M$ and $A \subseteq B \subseteq N$. We define $X = \langle \bigwedge_i \varphi(t_i) \rangle \subseteq \mathcal{S}_t(B)$. Since φ is strongly minimal, $|X| \leqslant |B| + \aleph_0$ — indeed, the type of t_i over $Bt_{<i}$ is either algebraic (there are at most $|Bt_i| + |\mathcal{L}| = |B| + \aleph_0$) of those, or non algebraic (and there is only one of those). Moreover, given $p \in X$, there are only finitely many types $q(y,t)$ containing $\varphi(y,t)$ such that $q|_t = p$ and any type over B is of this form for some p. It follows that $|\mathcal{S}_y(B)| \leqslant |B| + \aleph_0$.

   Now if $B \subseteq N \vDash T$, by completeness of T, there exists $N' \succcurlyeq M$ such that $B \subseteq N \preccurlyeq N'$, so by the previous, $|\mathcal{S}_y(B)| \leqslant |\mathcal{S}_y(B \cup A)| \leqslant |B \cup A| + \aleph_0 = |B| + \aleph_0$.

3. Let $N \succcurlyeq M$ and χ be a strongly minimal $\mathcal{L}(N)$-formula. Let $B \subseteq M$ such that φ and χ are B-definable, $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be sequences in $\varphi(N) \cup \chi(N)$ such that, for all i, $a_i \in \varphi(N)$ if and only if $b_i \in \varphi(N)$, $a_i \notin \mathrm{acl}(Ba_{<i})$ and $b_i \notin \mathrm{acl}(Bb_{<i})$. Show that the map sending a_i to b_i, for all i, and fixing B is a partial elementary embedding.

   Solution: We may assume N to be sufficiently saturated and homogeneous. Let us start by assuming that $(I,<)$ is well-ordered. We prove by induction on $i \in I$ that the map f_i sending $a_{j<i}$ to $b_{j<i}$ and fixing B is elementary. If $i = 0$ or i is limit, this is obvious. Let now assume that it holds for i. Then we find an map g extending f_i and defined at a_i. Let $c = g(a_i)$. It suffices to show that $\mathrm{tp}(c/Bb_{\geqslant i}) = \mathrm{tp}(b_{i+1}/Bb_{\geqslant i})$. But this follows immediately from the fact there is a unique non-algebraic type over $Bb_{\geqslant i}$ containing φ (respectively ψ).

Let us now prove the following result — it is a version of the exchange principle for two strongly minimal sets and the proof is essentially the same. If $a \in \varphi(N) \smallsetminus \operatorname{acl}(B)$ and $b \in \psi(N) \smallsetminus \operatorname{acl}(Ba)$, then $a \notin \operatorname{acl}(Bb)$. Let $(a_i)_{i \in \omega}$ be an infinite set of realisations of $\operatorname{tp}(a/B)$, with $a_0 = a$ and let b' realize the unique non algebraic extension of $\operatorname{tp}(b/B)$ over $Ba_{\leqslant\omega}$. Then applying the previous paragraph to (a, b) and (a_i, b'), we see that, for all $i \in \omega$, $\operatorname{tp}(ab/B) = \operatorname{tp}(a_i b'/B) = \operatorname{tp}(a_0 b'/B)$. It follows that $\operatorname{tp}(a_0/Bb')$ has infinitely many realizations and hence is not algebraic. It follows that $\operatorname{tp}(a/Bb)$ is not algebraic either[1].

Now we can prove that for all $i \in I$, and all $i < j_1 < \ldots < j_n \in I$, $a_i \notin \operatorname{acl}(Ba_{<i} a_{j_{\leqslant n}})$. We proceed by induction on n. If $n = 0$, this is our hypothesis on a_i. If we assume this to hold for n, but not $n + 1$, we have $a_i \in \operatorname{acl}(Ba_{<i} a_{j_{\leqslant n+1}}) \smallsetminus \operatorname{acl}(Ba_{<i} a_{j_{\leqslant n}})$. By the symmetry result above, we have $a_{j_{n+1}} \in \operatorname{acl}(Ba_{\leqslant i} a_{j_{\leqslant n}}) \subseteq \operatorname{acl}(Ba_{<j_{n+1}})$, a contradiction. It follows that $a_i \notin \operatorname{acl}(Ba_{\neq i})$. Similarly, $b_i \notin \operatorname{acl}(Bb_{\neq i})$. It follows that our hypothesis on the sequences $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ also holds if we change the order on I, so we may as well assume it is well-ordered.

4. Let $N \succcurlyeq M$ and χ be a strongly minimal $\mathcal{L}(N)$-formula. Show that there exists $A \subseteq B \subseteq N$ finite such that χ is an $\mathcal{L}(B)$-formula and for all $b \in \chi(N) \smallsetminus \operatorname{acl}(B)$, there exists $c \in \varphi(N)$ such that $\operatorname{acl}(B \cup c) = \operatorname{acl}(B \cup b)$.

   Solution: Let $B_0 \subseteq N$ finite contain A and such that χ is B-definable. Pick any $b \in \chi(N) \smallsetminus \operatorname{acl}(B_0)$, and let $c \in \varphi(N)^n$ be such that $b \in \operatorname{acl}(Ac)$. We may assume that c is minimal length such that $b \in \operatorname{acl}(B_0 c)$. Let $B = B_0 c_{<n}$ and $c = c_n$. Then by minimality, $b \in \operatorname{acl}(Bc) \smallsetminus \operatorname{acl}(B)$ and hence $c \in \operatorname{acl}(Bb)$.

   Note that we have only prove that there exists B and b such that $b \in \chi(N) \smallsetminus \operatorname{acl}(B)$ and there exists $c \in \varphi(N)$ such that $\operatorname{acl}(B \cup c) = \operatorname{acl}(B \cup b)$. But this can be written as a first order statement over B so it holds of any realisation of $\operatorname{tp}(b/B)$, the unique non algeraic type over B containing χ, i.e. it holds of any $b \in \chi(N) \smallsetminus \operatorname{acl}(B)$.

5. Let $M \preccurlyeq M_1 \preccurlyeq N_1$ where M_1 and N_1 are $\aleph_0$-saturated. Let θ be an $\mathcal{L}(M_1)$-formula such that $\theta(M_1) = \theta(N_1)$ is infinite. Show that there exists a strongly minimal $\mathcal{L}(M_1)$-formula χ such that $\chi(M_1) = \chi(N_1)$.

   Solution: Let χ be a minimal $\mathcal{L}(M_1)$-formula such that $\chi(M_1) \subseteq \theta(M_1)$. We have $\chi(N_1) \subseteq \theta(N_1) = \theta(M_1)$. It follows that $\chi(M_1) = \chi(N_1) \cap \theta(M_1) = \chi(N_1)$. Also, since M_1 is $\aleph_0$-saturated, any minimal formula over M_1 is strongly minimal.

6. Notations as above, show that any $b \in \varphi(N_1) \smallsetminus \varphi(M_1)$ is in $\operatorname{acl}(Bc)$ for some finite $B \subseteq M_1$ and $c \in \chi(N_1)$. Conclude that $N_1 = M_1$.

   Solution: By Question 4, there exists $B \subseteq M_1$ finite such that for any $b \in \varphi(N_1) \smallsetminus \operatorname{acl}(B) \subseteq \varphi(N_1) \smallsetminus \varphi(M_1)$, there exists $c \in \chi(N_1)\chi(N_1)$ such that $\operatorname{acl}(Bb) = \operatorname{acl}(Bc) \subseteq M_1$, and hence $b \in \varphi(M_1)$, a contradiction. It follows that $\varphi(N_1) = \varphi(M_1)$ ahd therefore, $N_1 = \operatorname{acl}(A \cup \varphi(N_1)) \subseteq \operatorname{acl}(A \cup \varphi(M_1)) = M_1$.

7. Show that T is κ-categorical for all uncountable cardinal κ.

   Solution: We know that T is ω-stable, it suffices to show that T does not have Vaughtian pairs. Assume T has a Vaughtian pair then, it has a Vaughtian pair (N_1, M_1) which is $|M|^+$-saturated, and hence M can be elementarily embedded in M_1. By Questions 5 and 6, we must have $M_1 = N_1$, contradicting the fact that it is a Vaughtuian pair.

[1] We can also see that result as a consequence of symmetry of non forking: by hypothesis, $b \mathop{\downarrow}_B a$, so $a \mathop{\downarrow}_B b$, i.e. $\operatorname{MR}(a/Bb) = \operatorname{MR}(a/B) = 1$.

8. Show that there exists a tuple $b \in M$ such that $\mathrm{tp}(b)$ is isolated, χ a strongly minimal $\mathcal{L}(b)$-formula and $B \subseteq M$ finite such that $M = \mathrm{acl}(\chi(M) \cup B)$.

   Solution: Let $M_0 \vDash T$ its prime model. Since T is totally transcendental, we find an $\mathcal{L}$-formula $\chi(y,t)$ and $b \in M_0^t$ such that $\varphi(y,b)$ is strongly minimal. Since M_0 is atomic, $\mathrm{tp}(b)$ is isolated. By Question 4, there exists $B \subseteq M$ finite, such that for all $b \in \varphi(M)$, there exists $c \in \chi(M)$ such that $\mathrm{acl}(Bb) = \mathrm{acl}(Bc)$. We may assume $A \subseteq B$. Then $M = \mathrm{acl}(A \cup \varphi(M)) \subseteq \mathrm{acl}(B \cup \chi(M)) \subseteq M$.

9. Show that there exists a tuple $b \in M$ such that $\mathrm{tp}(b)$ is isolated and χ a strongly minimal $\mathcal{L}(b)$-formula such that $M = \mathrm{acl}(\chi(M) \cup b)$.

   Solution: Let $c \in M^z$ enumerate B. By the same compactness argument than in Question 1, we find an $\mathcal{L}$-formula $\psi(y,t,z)$ such that $M \vDash \forall t \exists^{\leqslant n} y\ \psi(y,t,c) \wedge \forall y\ \exists t \bigwedge_i \chi(t_i,b) \wedge \psi(y,t,c) =: \zeta(c,b)$. By density of isolated types in totally transcedental theories, we may assume that $\mathrm{tp}(c/b)$ is isolated. Since $\mathrm{tp}(b)$ is isolated, it follows that $\mathrm{tp}(bc)$ is isolated. Moreover, we do have $M \subseteq \bigcup_{a \in \chi(M,b)^n} \psi(M,a,c) \subseteq \mathrm{acl}(bc\chi(M,b))$.

10. Let $\mathcal{L}$ be the language with two sorts X and Y and functions $f_i : X \to Y$ for $1 \leqslant i \leqslant n$. Let T be the theory stating that $\forall y_1 \ldots \forall y_n \exists^{=1} x \bigwedge_i f_i(x) = y_i$ and Y is infinite. Show that T eliminates quantifiers, Y is strongly minimal, T is almost strongly minimal and T is κ-categorical for all infinite cardinals κ (not just the uncountable ones).