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Homework 10

Problem 1 :
Let $\mathcal{L}$ be a countable language. A complete $\mathcal{L}$-theory T is almost strongly minimal if there exists $M \vDash T$ $\aleph_0$-saturated, $A \subseteq M$ finite and $\varphi(x)$ an $\mathcal{L}(A)$-formula such that φ is strongly minimal and $M = \mathrm{acl}(\varphi(M) \cup A)$.

1. Show that there exists an $\mathcal{L}(A)$-formula $\psi(x,y)$ such that for all $a \in \varphi(M)^y$, $|\psi(M,a)| < \infty$ and $M = \bigcup_{a\in\varphi(M)^y} \psi(M,a)$.
2. Show that T is ω-stable.
3. Let $N \succcurlyeq M$ and χ be a strongly minimal $\mathcal{L}(N)$-formula. Let $B \subseteq M$ such that φ and χ are B-definable, $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be sequences in $\varphi(N) \cup \chi(N)$ such that, for all i, $a_i \in \varphi(N)$ if and only if $b_i \in \varphi(N)$, $a_i \notin \mathrm{acl}(Ba_{<i})$ and $b_i \notin \mathrm{acl}(Bb_{<i})$. Show that the map sending a_i to b_i, for all i, and fixing B is a partial elementary embedding.
4. Let $N \succcurlyeq M$ and χ be a strongly minimal $\mathcal{L}(N)$-formula. Show that there exists $A \subseteq B \subseteq N$ finite such that χ is an $\mathcal{L}(B)$-formula and for all $b \in \chi(N) \smallsetminus \mathrm{acl}(B)$, there exists $c \in \varphi(N)$ such that $\mathrm{acl}(B \cup c) = \mathrm{acl}(B \cup b)$.
5. Let $M \preccurlyeq M_1 \preccurlyeq N_1$ where M_1 and N_1 are $\aleph_0$-saturated. Let θ be an $\mathcal{L}(M_1)$-formula such that $\theta(M_1) = \theta(N_1)$ is infinite. Show that there exists a strongly minimal $\mathcal{L}(M_1)$-formula χ such that $\chi(M_1) = \chi(N_1)$.
6. Notations as above, show that any $b \in \varphi(N_1) \smallsetminus \varphi(M_1)$ is in $\mathrm{acl}(Bc)$ for some finite $B \subseteq M_1$ and $c \in \chi(N_1)$. Conclude that $N_1 = M_1$.
7. Show that T is κ-categorical for all uncountable cardinal κ.
8. Show that there exists a tuple $b \in M$ such that $\mathrm{tp}(b)$ is isolated, χ a strongly minimal $\mathcal{L}(b)$-formula and $B \subseteq M$ finite such that $M = \mathrm{acl}(\chi(M) \cup B)$.
9. Show that there exists a tuple $b \in M$ such that $\mathrm{tp}(b)$ is isolated and χ a strongly minimal $\mathcal{L}(b)$-formula such that $M = \mathrm{acl}(\chi(M) \cup b)$.
10. Let $\mathcal{L}$ be the language with two sorts X and Y and functions $f_i : X \to Y$ for $1 \leqslant i \leqslant n$. Let T be the theory stating that $\forall y_1 \dots \forall y_n \exists^{=1} x \bigwedge_i f_i(x) = y_i$ and Y is infinite. Show that T eliminates quantifiers, Y is strongly minimal, T is almost strongly minimal and T is κ-categorical for all infinite cardinals κ (not just the uncountable ones).