Model theory of algebraically valued fields

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1. Algebra and quantifiers

Let us start by recalling the language in which we will, mostly, be working:

Definition 1.1 (The three sorted language)**:** *Let $\mathcal{L}_{\mathrm{k},\Gamma}$ be the language consisting of three sorts* K, k *and* Γ_∞*. On* K *and* k *we have the ring language and on* Γ *the ordered group language with an additional constant* ∞*. Finally, we also have a map* $\mathrm{val} : \mathrm{K} \to \Gamma_\infty$ *and a map* $\mathrm{r} : \mathrm{K}^2 \to \mathrm{k}$*.*

Any valued field K can be made into a $\mathcal{L}_{\mathrm{k},\Gamma}$-structure by interpreting K as K, k as the residue field $\mathcal{O}_K/\mathfrak{M}_K$ and Γ_∞ as the value group $K^\star/\mathcal{O}_K^\star \sqcup \{\infty\}$. The map val is interpreted as the valuation and $\mathrm{r}(a,b) = \mathrm{res}(a/b)$ if $a/b \in \mathcal{O}_K$ and 0 otherwise. We denote by ACVF the $\mathcal{L}_{\mathrm{k},\Gamma}$-theory of algebraically closed non trivially valued fields. Recall that if K is an algebraically closed valued field, k_K is algebraically closed and Γ_K is divisble.

The first result we want to prove is:

Theorem 1.2 (Robinson, 1956)**:** *The theory* ACVF *eliminates quantifiers.*

A crucial element of the proof is the study of certain valued field extensions. Let us start with the purely residual ones. By convention the polynomial 0 has degree $+\infty$ and the minimal polynomial of transcendental elements is 0. Let $M \models \mathrm{ACVF}$ and $A \leqslant M$. Assume that $\mathrm{K}(A)$ and $\mathrm{k}(A)$ are fields.

Lemma 1.3 (Purely residual 1-types)**:** *Pick any $\alpha \in \mathrm{k}(M)$. Let $P \in \mathcal{O}(A/X)$ be an exact lifting of its minimal polynomial over $\mathrm{res}(\mathrm{K}(A))$.*

1. *For every $Q = \sum_i q_i X^i, R \in \mathrm{K}(A)[X]$, with degree smaller than P, and every $a \in \mathrm{res}^{-1}(\alpha)$:*
   - $\mathrm{val}(Q(a)) = \min_i \mathrm{val}(q_i) \neq \infty$*;*
   - $\mathrm{r}(Q(a), R(b)) = \mathrm{r}(q_{i_0}, r_{j_0})\mathrm{res}(Q_0)(\alpha)\mathrm{res}(R_0)(\alpha)^{-1}$*, where $Q = q_{i_0}Q_0$, $R = r_{j_0}R_0$ and $\mathrm{val}(q_{i_0})$ and $\mathrm{val}(r_{j_0})$ are minimal.*
2. *There exists $a \in \mathrm{K}(M)$ with $\mathrm{res}(a) = \alpha$ and $P(a) = 0$.*
3. *Such an a is uniquely determined up to $\mathcal{L}$-isomorphism by P and the minimal polynomial $R \in \mathrm{k}(A)[X]$ of α over $\mathrm{k}(A)$: for every $N \models \mathrm{ACVF}$, $\mathcal{L}$-embedding $f : A \to N$, every root $a \in \mathrm{K}(M)$ of P and every root $b \in \mathrm{K}(N)$ of $f(P)$, if $\mathrm{res}(a)$ is a root of R and $\mathrm{res}(b)$ is a root of $f(R)$, then f can be extended by sending a to b.*

Proof. Let i_0 be such that $\mathrm{val}(q_{i_0})$ is minimal. Write $Q = q_{i_0}Q_0$. Then $\mathrm{res}(Q_0) \neq 0$. By minimality of P, $\mathrm{res}(Q_0(a)) = \mathrm{res}(Q_0)(\alpha) \neq 0$ and hence $\mathrm{val}(Q_0(a)) = 0$. It follows that $\mathrm{val}(Q(a)) = \mathrm{val}(q_{i_0}) = \min_i \mathrm{val}(q_i)$. Similarly $\mathrm{res}(R_0(a)) = \mathrm{res}(R_0)(\alpha) \neq 0$. It follows that $Q(a)/R(a) \in \mathcal{O}$ if and only if $q_{i_0}/r_{j_0} \in \mathcal{O}$ and, in that case, $\mathrm{res}(Q(a)/R(a)) = \mathrm{res}(q_{i_0}/r_{j_0})\mathrm{res}(Q_0(a))\mathrm{res}(R_0(a))$.
Let $P = c\prod_j(X - e_j)$, where $c \in \mathcal{O}_A^\star$. Since $\mathcal{O}$ is integrally closed, we have $e_j \in \mathcal{O}$ for all j. For any $a \in \mathrm{res}^{-1}(\alpha)$, $\mathrm{res}(P)(\alpha) = \mathrm{res}(P(a)) = \mathrm{res}(c)\prod_j \mathrm{res}(a) - \mathrm{res}(e_j) = 0$. It follows that there exists an j such that $\mathrm{res}(a) - \mathrm{res}(e_j) = 0$.
Finally, let C be the structure generated by Aa and $g : C \to N$ extend f by sending a to b. Note that by 1, P is the minimal polynomial of a over $\mathrm{K}(A)$ and $f(P)$ is the minimal polynomial of b over $f(\mathrm{K}(A))$. So $g|_{\mathrm{K}}$ is a ring homomorphism. Note also that for any $Q = \sum_i q_i X^i \in \mathrm{K}(A)[X]$ with degree smaller than P, by 1, we have $\mathrm{val}(Q(a)) = \min_i \mathrm{val}(q_i) = \min_i \mathrm{val}(f(q_i)) = \mathrm{val}(f(Q)(b))$. Similarly, g is compatible with r. Note also that, by 1, $\Gamma_\infty(C) \subseteq \Gamma_\infty(A)$ — so g is compatible with all the structure on this sort — and $\mathrm{k}(C) \subseteq \mathrm{k}(A)(\alpha)$. Since the minimal polynomial of $g(\alpha)$ is the image by f of the minimal polynomial of α, $g|_{\mathrm{k}}$ is a ring homomorphism. □

We continue with purely ramified extensions:

Lemma 1.4 (Purely ramified 1-types): *Pick any $\gamma \in \Gamma(M)$. Let n be its order in $\Gamma(M)/\mathrm{val}(\mathrm{K}(A))$.*

1. *For every $Q = \sum_i q_i X^i, R \in \mathrm{K}(A)[X]$ with degree smaller than n and every $a \in \mathrm{val}^{-1}(\gamma)$:*
   - *$\mathrm{val}(Q(a)) = \min_i(\mathrm{val}(q_i) + i\gamma)$ and the minimum is attained only once;*
   - *$\mathrm{r}(Q(a), R(a)) = \delta_{i_0=j_0}\mathrm{r}(q_{i_0}, r_{j_0})$, where $\mathrm{val}(q_{i_0})$ and $\mathrm{val}(r_{j_0})$ are minimal.*
2. *For any $c \in \mathrm{K}(A)$ such that $n\gamma = \mathrm{val}(c)$, there exists $a \in \mathrm{K}(M)$ with $\mathrm{val}(a) = \gamma$ and $a^n = c$.*
3. *Such an a is uniquely determined, up to $\mathcal{L}$-isomorphism, by the order m of γ in $\Gamma(M)/\Gamma(A)$, the order n of γ in $\Gamma(M)/\Gamma(A)$, $m\gamma \in \Gamma(A)$, a choice of $c \in \mathrm{K}(A)$ such that $\mathrm{val}(c) = n\gamma$, and, when $m = \infty$, by the set $D := \{\delta \in \Gamma(A) : \delta < \gamma\}$: for every $N \vDash \mathrm{ACVF}$, $\mathcal{L}$-embedding $f : A \to N$, every n-th root $a \in \mathrm{K}(M)$ of c and every n-th root $b \in \mathrm{K}(N)$ of $f(c)$, if $m\mathrm{val}(a) = m\gamma$, $m\mathrm{val}(b) = f(m\gamma)$ and, when $m = \infty$, for all $\delta \in \Gamma(A)$, $\mathrm{val}(a) > \delta$ if and only if $\delta \in D$ if and only if $\mathrm{val}(b) > f(\delta)$, then f can be extended by sending a to b.*

Proof. We always have $\mathrm{val}(Q(a)) = \mathrm{val}(\sum_i(q_i a^i)) \geqslant \min_i \mathrm{val}(q_i a^i) = \min_i \mathrm{val}(q_i) + i\gamma$. If the inegality were strict, there would exist $i < j < n$ such that $\mathrm{val}(q_i a^i) = \mathrm{val}(q_j a^j)$, i.e. $(j - i)\mathrm{val}(a) = \mathrm{val}(q_i) - \mathrm{val}(q_j) \in \mathrm{val}(A)$, contradicting the minimality of n. We have also proved that all $\mathrm{val}(q_i a^i) = \mathrm{val}(q_i) + i\gamma$, in particular the minimum, must be distinct. It follows that $\mathrm{res}(Q(a)/q_{i_0}a^{i_0}) = 1$. Note also that $\mathrm{val}(Q(a)) - \mathrm{val}(R(a)) = 0$ if and only if $\mathrm{val}(q_{i_0}) - \mathrm{val}(r_{j_0}) + (i_0 - j_0)\gamma = 0$. Since $|i_0 - j_0| < n$, this is, in turn, equivalent to $i_0 = j_0$ and $\mathrm{val}(q_{i_0}) = \mathrm{val}(r_{j_0})$. In that case, $\mathrm{res}(Q(a)/R(a)) = \mathrm{res}(q_{i_0}a^{i_0}, r_{j_0}a^{j_0}) = \mathrm{r}(q_{i_0}, r_{j_0})$.
Now assume $n < +\infty$. For any a with $a^n = c$, we have $n\mathrm{val}(a) = \mathrm{val}(c) = n\gamma$ and hence $\mathrm{val}(a) = \gamma$.
Finally, let C be the structure generated by Aa and $g : C \to N$ extend f by sending a to b. By 1, the minimal polynomial of a over $\mathrm{K}(A)$ is $X^n - c$ and the minimal polynomial of b over $f(\mathrm{K}(A))$ is $X^n - f(c)$. So g is compatible with the ring structure on K. The computations in 1 indicate that g is compatible with val and r. Finally, $\mathrm{k}(C) \subseteq \mathrm{k}(A)$ so g is compatible with the structure on k. Also, $\Gamma(C) \subseteq \Gamma(A) + \mathbb{Z}\mathrm{val}(a)$. Since the order of $\mathrm{val}(a)$ over $\Gamma(A)$

and the order of $\mathrm{val}(b)$ over $f(\Gamma(A))$ are equal, g is compatible with the group structure on Γ_∞. If $m < \infty$, for all $\delta \in \Gamma(A)$, $\delta < \mathrm{val}(a)$ if and only if $m\delta < m\mathrm{val}(a)$ if and only $f(m\delta) < f(m\mathrm{val}(a))$ if and only if $f(\delta) < \mathrm{val}(b)$, so g is compatible with the order on Γ_∞. If $m = \infty$, we also have $\delta < \mathrm{val}(a)$ if and only if $f(\delta) < \mathrm{val}(b)$, by hypothesis, so g is also compatible with the order on Γ_∞. □

And finally, we deal with immediate extensions. However, we first need to introduce the notion of pseudo-convergence:

Definition 1.5: *Let $(a_i)_{i\in I}$ be a sequence of elements in some valued field K — with no maximal element.*

1. *We say that the sequence $(a_i)_i$ is pseudo-Cauchy if, for $i < j < k \in I$ sufficiently large, $\mathrm{val}(a_k - a_j) > \mathrm{val}(a_k - a_i)$.*
2. *We say that l is a pseudo-limit of the sequence a_i — and we write $a_i \rightsquigarrow l$, or $l \in \mathrm{plim}_i\, a_i$ — if, for $i < j \in I$ sufficiently large, $\mathrm{val}(l - a_j) > \mathrm{val}(l - a_i)$.*

Also, to make the computations shorter — and hopefully more comprehensible — we introduce the concept of leading terms. It is a map that packages together the valuation and the residue map while still having the nice properties that make valuative computations work.

Definition 1.6 (Leading terms)**:** *Let K be a valued field. We define the group of leading terms $RV_K^\star := K^\star/(1 + \mathfrak{M}_K)$. The projection is denoted $\mathrm{rv} : K^\star \to RV_K^\star$. We also define $RV_K := RV_K^\star \sqcup \{0\}$ and we set $\mathrm{rv}(0) = 0$.*

Since $1 + \mathfrak{M}_K \leqslant \mathcal{O}_K^\star$, the valuation factorizes through rv and we have the following short exact sequence of groups:

$$k_K^\star \to RV_K^\star \to \Gamma_K.$$

Furthermore, the addition induces a k_K-vector space structure on every fiber of the map $\mathrm{val} : RV_K^\star \to \Gamma_K$; formally, we consider that $0 \in RV_K$ is an element of every fiber and any time $\mathrm{val}(a + b) > \min(\mathrm{val}(a), \mathrm{val}(b))$, we set $\mathrm{rv}(a) + \mathrm{rv}(b) = 0$.
The main trick to compute leading terms is the following generalization of the fact that in a valuative setting all triangles are isosceles.

Lemma 1.7: *Let K be a valued field and $a, b, c \in K$ be such that $\mathrm{val}(a - b) > \mathrm{val}(a - c)$, then $\mathrm{rv}(a - c) = \mathrm{rv}(b - c)$.*

Proof. We have $(b - c)/(a - c) = 1 + (b - a)/(a - c) \in 1 + \mathfrak{M}_K$. □

Lemma 1.8: *Let $M \vDash \mathrm{ACVF}$, $A \leqslant \mathrm{K}(M)$ and $(a_i)_i \in A$ be pseudo-Cauchy. Let $P \in A[X]$ and $c \in A$. Then:*

- *if there is a root of P in $\mathrm{plim}_j\, a_i$, then $P(a_i) \rightsquigarrow 0$;*
- *otherwise, for any $a \in \mathrm{plim}_i\, a_i$ and any sufficiently large i, $\mathrm{rv}(P(a)) = \mathrm{rv}(P(a_i))$.*

Proof. Let $P = c\prod_j (X - e_j)$and $J_0 = \{j : e_j \in \mathrm{plim}_i\, a_i\}$. Let i_0 be sufficiently large such that for all $j \notin J_0$, $\mathrm{val}(e_j - a_{i_0}) \leqslant \mathrm{val}(a_{i_1} - a_{i_0})$ for some $i_1 < i_0$. Note that, since a_i is Cauchy, for every $j \notin J_0$ and $a \in \mathrm{K}(M)$ such that $\mathrm{val}(a - a_{i_0}) > \mathrm{val}(a_{i_1} - a_{i_0})$ — in particular if

$a \in \mathrm{plim}_i\, a_i$ or if $a = a_i$ for any $i > i_0$ — $\mathrm{rv}(a_i - e_j) = \mathrm{rv}(a_{i_0} - e_j)$. It follows that if $J_0 = \emptyset$, $\mathrm{rv}(P(a)) = \mathrm{rv}(P(a_{i_0}))$.
If $J_0 \neq \emptyset$, for all $i_0 < i < l$, $\mathrm{val}(P(a_l)) = \mathrm{val}(c) \prod_{j \in J_0} \mathrm{val}(a_l - e_j) \prod_{j \notin J_0} \mathrm{val}(a_{i_0} - e_j) > \mathrm{val}(c) \prod_{j \in J_0} \mathrm{val}(a_i - e_j) \prod_{j \notin J_0} \mathrm{val}(a_{i_0} - e_j) = \mathrm{val}(P(a_i))$. So $\mathrm{val}(P(a_i)) \rightsquigarrow 0$. □

Corollary 1.9 (Immediate 1-types)**:** *Let $M \models \mathrm{ACVF}$, $A \leqslant M$ and $(a_i)_i \in A$ be pseudo-Cauchy. Let $P \in \mathrm{K}(A)[X]$ have minimal degree among those polynomials such that $P(a_i) \rightsquigarrow 0$; or $P = 0$ if such a polynomial does not exist.*

1. *For every $Q \in \mathrm{K}(A)[X]$ with degree smaller than P, every $a \in \mathrm{plim}_i\, a_i$ and every sufficiently large i,*
$$\mathrm{rv}(Q(a)) = \mathrm{rv}(Q(a_i)).$$
2. *If $P \neq 0$, there exist $a \in \mathrm{K}(M)$ with $a_i \rightsquigarrow a$ and $P(a) = 0$.*
3. *For all P, such an a is uniquely determined, up to isomorphism, by the sequence of the a_i and P: for every $N \models \mathrm{ACVF}$, $\mathcal{L}$-embedding $f : A \to N$,root $a \in \mathrm{K}(M)$ and root $b \in \mathrm{K}(N)$ of $f(P)$, if $a_i \rightsquigarrow a$ and $f(a_i) \rightsquigarrow b$, then f can be extended by sending a to b.*

Proof. By minimality of P, $Q(a_i)$ does not pseudo-converge to 0. It follows, by Lemma **(1.8)**, that $\mathrm{val}(Q(a)) = \mathrm{val}(Q(a_i))$ for sufficiently large i. Similarly, by Lemma **(1.8)**, since $P(a_i) \rightsquigarrow 0$, one of the roots of P must be a pseudo-limit of the a_i.
Let C be the structure generated by Aa and and $g : C \to N$ extend f by sending a to b. The computations in 1, show that a and b have the same minimal (up to f) over $\mathrm{K}(A)$ and that $\mathrm{rv}(Q(a)) = \mathrm{rv}(Q(a_i))\mathrm{rv}(f(Q(a_i))) = \mathrm{rv}(f(Q)(f(a_i))) = \mathrm{rv}(f(Q)(b))$. Since $\mathrm{val}(c) = \mathrm{val}(\mathrm{rv}(c))$ and $\mathrm{r}(c,d) = \delta_{\mathrm{val}(c) \geqslant \mathrm{val}(d)} \mathrm{rv}(c)\mathrm{rv}(d)^{-1}$, we see that g is compatible with val and r and that $\Gamma_\infty(C) \subseteq \Gamma_\infty(A)$ and $\mathrm{k}(C) \subseteq \mathrm{k}(A)$. □

Before we prove the elimination of quantifiers in ACVF, let me recall my favorite quantifier elimination test:

Proposition 1.10 (Schoenfield, 1971)**:** *Let T be some $\mathcal{L}$-theory. The following are equivalent:*

1. *for every $M, N \models T$, where N is $|M|^+$-saturated and every $A \leqslant M$, every $\mathcal{L}$-embedding $f : A \to N$ can be extended to an $\mathcal{L}$-embedding $M \to N$.*
2. *T eliminates quantifiers.*

Proof (Theorem **(1.2)**). Let $M, N \models \mathrm{ACVF}$, $A \leqslant M$ and $f : A \to N$ an $\mathcal{L}$-embedding. Assume N is $|M|^+$-saturated. Our goal is to extend f step by step by using the cases we studied above. First note that f has a (unique) extension to $A \cup \mathrm{Frac}(\mathrm{K}(A)) \cup \mathrm{Frac}(\mathrm{k}(A))$, so we may always assume that $\mathrm{K}(A)$ and $\mathrm{k}(A)$ are fields.

Claim 1.11 (lifting k)**:** *Pick any $\alpha \in \mathrm{k}(M)$, then f can be extended to some $C \leqslant M$, with $\alpha \in \mathrm{res}(\mathrm{K}(C))$.*

Proof. Let $R \in \mathrm{k}(A)[X]$ be the minimal polynomial of α over $\mathrm{k}(A)$ and P be an exact lifting of the minimal polynomial of α over $\mathrm{res}(\mathrm{K}(A))$. Let $\beta \in \mathrm{k}(N)$ be a root of $f(R)$ — if $R = 0$, by saturation of N, we may assume β transcendental over $f(\mathrm{k}(A))$. By Lemma **(1.3)**.2, we find a root a of P in $\mathrm{K}(M)$ with $\mathrm{res}(a) = \alpha$ and a root b of $f(P)$ in $\mathrm{K}(N)$ with $\mathrm{res}(b) = f(\alpha)$. We now apply Lemma **(1.3)**.3. ◊

Claim 1.12 (lifting Γ)**:** *Pick any $\gamma \in \mathrm{k}(M)$, then f can be extended to some $C \leqslant M$, with $\gamma \in \mathrm{val}(\mathrm{K}(C))$.*

Proof. Let m be the order of γ in $\Gamma(M)/\Gamma(A)$ and n its order in $\Gamma(M)/\mathrm{val}(\mathrm{K}(A))$. Let $\delta \in \mathrm{K}(N)$ be such that $\delta^m = f(\gamma^m)$ — if $m = +\infty$, by saturation of N, we can find δ such that $\delta < f(\varepsilon)$ if and only if $\gamma < \varepsilon$, for every $\varepsilon \in \Gamma_\infty(A)$. By Lemma **(1.4)**.2, we find an n-th root a of c in $\mathrm{K}(M)$ with $\mathrm{val}(a) = \gamma$ and an n-th root b of $f(c)$ in $\mathrm{K}(M)$ with $\mathrm{val}(b) = f(\gamma)$. We conclude with Lemma **(1.4)**.3. ◊

Applying the those claims repetitively, we may assume that $\mathrm{k}(M) \subseteq \mathrm{res}(\mathrm{K}(A))$ and $\Gamma(M) \subseteq \mathrm{val}(\mathrm{K}(A))$. In particular, $\mathrm{K}(A) \leqslant \mathrm{K}(M)$ is an immediate extension and $M \smallsetminus A \leqslant \mathrm{K}$.

Claim 1.13 (immediate extensions)**:** *Let $(a_i)_i \in \mathrm{K}(A)$ be some pseudo-Cauchy sequence with a pseudo limit in $\mathrm{K}(M)$. Then f can be extended to some $C \leqslant M$ containing a pseudo-limit of the a_i.*

Proof. Let $P \in \mathrm{K}(A)[X]$ be minimal such that $P(a_i) \rightsquigarrow 0$. If $P \neq 0$, by Corollary **(1.9)**.2 we find $a \in \mathrm{K}(M)$ a root of P with $a_i \rightsquigarrow a$ and $b \in \mathrm{K}(N)$ a root of $f(P)$ with $f(a_i) \rightsquigarrow b$. If $P = 0$ take any $a \in \mathrm{plim}_i\, a_i$. By saturation of N, we also find $b \in \mathrm{K}(N)$ with $f(a_i) \rightsquigarrow b$. Now apply Corollary **(1.9)**.3. ◊

Applying this new claim repetitively, we may assume that every pseudo-Cauchy sequence of elements in $\mathrm{K}(A)$ with a limit in $\mathrm{K}(M)$ has a pseudo-limit in $\mathrm{K}(A)$.

Claim 1.14: $\mathrm{K}(M) \subseteq \mathrm{K}(A)$.

Proof. Pick some $a \in \mathrm{K}(M)$. For any $e \in \mathrm{K}(A)$, since $\mathrm{K}(A) \leqslant \mathrm{K}(A)(a)$ is immediate, there is $c \in \mathrm{K}(A)$ such that $\mathrm{rv}(a - e) = \mathrm{rv}(c)$, i.e. $\mathrm{val}(a - (e + c)) > \mathrm{val}(a - e)$. Note that if $e \neq a$, we may assume $c \neq a$. It follows that, by (transfinite) induction, we can build a maximal pseudo-Cauchy sequence $a_i \in \mathrm{K}(A)$ which pseudo-converges to $a \in \mathrm{K}(M)$. Let $c \in \mathrm{K}(A)$ be a pseudo-limit of the a_i. If $c \neq a$, the sequence (a_i) is not maximal. So $a = c \in \mathrm{K}(A)$. ◊

We have therefore extended f to the whole of M. By Proposition **(1.10)**, ACVF eliminates quantifiers. □

2. Definable sets and swiss cheeses

We will now derive of number of properties of definable set in ACVF from quantifier elimination. We start by giving a classification of its completions. For all p and q prime or 0, let $\mathrm{ACVF}_{p,q}$ be the theory of algebraically closed non trivially valued fields with characteristic p and residue characteristic q. Note that if $p > 0$, then we must have $q = p$.

Proposition 2.1: *For every p and q, $\mathrm{ACVF}_{p,q}$ is complete.*

Proof. By quantifier elimination, it suffices to find a common substructure to any two models of $\mathrm{ACVF}_{p,q}$. If $q = p > 0$, the trivially valued field $\mathbb{F}_p$ embeds (uniquely) in any model of $\mathrm{ACVF}_{p,p}$. If $q = p = 0$, the trivially valued field $\mathbb{Q}$ embeds (uniquely) in any model of $\mathrm{ACVF}_{0,0}$.

Finally, the field $\mathbb{Q}$ with the p-adic valuation embeds (uniquely) in every model of $\mathrm{ACVF}_{0,p}$. □

Let us now give a complete description of 1-types over models. We consider points to be closed balls of radius $+\infty$, and the whole field to be an open ball of radius $-\infty$.

Definition 2.2: *Let $M \vDash \mathrm{ACVF}$, $A \subseteq \mathrm{K}(M)$ and $B = (b_i)_i$ a chain of nested balls in A. An element $a \in M$ is said to be generic in B over A if:*

- *$a \in b_i$, for every i;*
- *for every ball b of A with $b \subset b_i$ for all i, $a \notin b$.*

We write $\eta_B|_A$ for the — a priori partial — type of generics of B over A.

Lemma 2.3: *Let $M \vDash \mathrm{ACVF}$, $A = A^{\mathrm{a}} \subseteq \mathrm{K}(M)$ and $a \in \mathrm{K}(M)$. Let $B = \{b \text{ ball in } A : a \in b\}$. Then:*

$$\eta_B|_A \vdash \mathrm{tp}(a/A).$$

In particular $\eta_B|_A$ is complete.

Proof. Note that, by construction, $a \vDash \eta_B|_A$. Let us first assume that B has a minimal element b_0 for inclusion and that b_0 is a closed ball. If b_0 is point (in A), then $\eta_B \vdash x = b_0$ which isolates a complete type. Otherwise, let $c \in A$ be such that $c \in b$, $d \in A$ with $\mathrm{val}(d) = \mathrm{rad}(b)$, $a' \vDash \eta_B|_A$ and $\alpha' = \mathrm{res}((a' - c)/d)$. If $\alpha' \in \mathrm{res}(A)^{\mathrm{a}} = \mathrm{res}(A)$, we find $e \in \mathrm{K}(A)$ with $\mathrm{res}(e) = \alpha'$, i.e. $\mathrm{val}(a' - (c + de)) > \mathrm{val}(d)$, so a' is in the open ball of radius $\mathrm{val}(d)$ around $c + de \in \mathrm{K}(M)$, a contradiction. It follows, from Lemma **(1.3)**.3, that $(a' - c)/d \equiv_A (a - c)/d$ and hence $a' \equiv_A a$.
Let us now assume that B does not have a minimal element and that $\bigcap_{b \in B} \cap A = \varnothing$. For all $b \in B$, let $a_b \in A$ belong to b but not to any $b' \in B$ with $b' \subset b$. Then $(a_b)_{b \in B}$ is a pseudo-Cauchy sequence. If the minimal polynomial of this sequence is not 0, by Corollary **(1.9)**.2, there exists $c \in A^{\mathrm{a}} = A$ with $a_b \rightsquigarrow c$, but then $c \in b$ for every $b \in B$, a contradiction. It follows that $P = 0$. Note also that any $a' \vDash \eta_B|_A$ is a pseudo-limit of the a_b. By Corollary **(1.9)**.3, we have $a' \equiv_A a$.
Let us now deal with the remaining case. We may assume that there is a $c \in \bigcap_{b \in B} \cap A$ and that the open balls are cofinal in B. Let $a' \vDash \eta_B|_A$. For all $d \in A$, we have $\mathrm{val}(a' - c) \geqslant \mathrm{val}(d)$ if and only the closed ball of radius $\mathrm{val}(d)$ around c is in B, equivalently, if $\mathrm{val}(d) \in \{\mathrm{rad}(b) : b \in B\}$. In particular, if $\mathrm{val}(a' - c) \in \mathbb{Q} \otimes \mathrm{val}(A) = \mathrm{val}(A)$, then the closed ball around c with radius $\mathrm{val}(a' - c)$ is in B and hence so is the open ball around c with radius $\mathrm{val}(a' - c)$. But a' is not in the open ball around c with radius $\mathrm{val}(a' - c)$, a contradiction. It now follows from Lemma **(1.4)**.3, that $(a' - c) \equiv_A (a - c)$ and hence $a' \equiv Aa$. □

We see from the proof that there is a correspondence between the description of the 1-types over models in algebraic terms and in terms of generics of balls:

1. Generics of closed balls correspond, up to translation and scaling, to residual extensions;
2. Generics of open balls correspond, up to translation, to ramified extensions where the cut is of the form γ^+;
3. Generics of non empty strict intersections of balls correspond, up to translation, to the other ramified extensions;

4. Generics of empty strict intersections of balls correspond to immediate extensions.

We have seen that every 1-type over a model can be described exclusively in terms of balls. It follows — by some abstract nonsense — that this is also the case of of every definable subset of K:

Proposition 2.4: *Let M be some $\mathcal{L}$-structure and $\Delta(x;t)$ a set of $\mathcal{L}$-formulas. Assume that for every $p \in \mathcal{S}_x(M)$, $p|_\Delta \vdash p$. Then every $\mathcal{L}(M)$-formula $\varphi(x)$ is equivalent to a Boolean combination of formulas from $\Delta(x;M)$.*

Definition 2.5: *Let K be a valued field.*

- *A Swiss cheese is a set of the form $b \smallsetminus \bigcup_i b_i$ where b is a ball and the b_i are finitely many (strict) subballs of b.*
- *A Swiss cheese $b \smallsetminus \bigcup_i b_i$ is nested inside some other Swiss cheese $d \smallsetminus \bigcup_j d_j$ if there exists a j such that $b = d_j$.*

Note that the union of two nested Swiss cheeses is still a Swiss cheese. Similarly, the union of two Swiss cheeses with a non empty intersection is still a Swiss cheese.

Lemma 2.6: *Let K be a valued field. Finite unions of Swiss cheeses are stable under Boolean combinations.*

Proof. It suffices to show that the intersection of two Swiss cheeses is a Swiss cheese and that the complement of a swiss cheese is a swiss cheese. Let $B = b \smallsetminus \bigcup_i b_i$ and $D = d \smallsetminus \bigcup_i d_i$ be two Swiss cheeses. We have $B \cap D = (b \cap d) \smallsetminus (\bigcup_i (b_i \cap d) \cup \bigcup_j b \cap d_j)$ where some of the intersections might be empty. Similarly $\mathrm{K} \smallsetminus B = (\mathrm{K} \smallsetminus b) \cup \bigcup_i b_i$. □

Theorem 2.7 (Holly, 1995)**:** *Any definable subset of* K *in* ACVF *has a unique decomposition as a finite disjoint union of non-nested Swiss cheeses.*

We say that ACVF is C-minimal.

Proof. Let $\varphi(x;s,t,u) := (u = 0 \wedge \mathrm{val}(x-s) > \mathrm{val}(t)) \vee (u = 1 \wedge \mathrm{val}(x-s) \geqslant \mathrm{val}(t))$. Note that for $M \vDash \mathrm{ACVF}$, $\varphi(x;M)$ is exactly the set of all $\mathrm{K}(M)$ balls. Lemma **(2.3)** implies that for all $p \in \mathcal{S}_x(M)$, $p|_\varphi \vdash p$. By Proposition **(2.4)**, every definable subset of K is a Boolean combination of balls. By Lemma **(2.6)**, it is a finite union of Swiss cheese. As noted above, we can assume that it is a finite disjoint union of non-nested Swiss cheeses.
Uniqueness of the decomposition follows immediately from the fact whenever a Swiss cheese is included in a finite disjoint union of non-nested Swiss cheeses, then it is included in one of those Swiss cheeses. □

We will now describe the structure induced on the residue field and the value group.

Definition 2.8: *Let T be an $\mathcal{L}$-theory and D be a $\varnothing$-definable set. We say that D is stably embedded if for every $M \vDash T$ and every $\mathcal{L}(M)$-definable $X \subseteq D^n$, X is $\mathcal{L}(D(M))$-definable.*

It then follows that for any $A \subseteq M$, the $\mathcal{L}(A)$-induced structure on D is a definable enrichment of the $\mathcal{L}$-induced structure.

Definition 2.9: *Let T be an $\mathcal{L}$-theory and assume we have some $\mathcal{L}'$-structure D interpretable in T. We say that D is a (stably embedded) pure $\mathcal{L}$-structure if:*

- *D is stably embedded;*
- *The $\mathcal{L}$-induced structure on D is exactly the $\mathcal{L}'$-structure.*

Proposition 2.10: *Let $M \vDash \mathrm{ACVF}$ and $A \leqslant M$. If $X \subseteq \mathrm{k}^n$ is A-definable, then it is $\mathcal{L}_{\mathrm{rg}}(\mathrm{k}(A))$-definable. In particular, the residue field k is a stably embedded pure ring.*

Proof. By quantifier elimination, and since for any $M \vDash \mathrm{ACVF}$, $\mathcal{L}_{\mathrm{rg}}(\mathrm{k}(A))$-definable sets are closed under Boolean combinations, it suffices to consider atomic formulas. So we may assume X is defined by $R(\overline{x}, \mathrm{r}(\overline{P}(a), \overline{Q}(a)), \alpha) = 0$ where $R \in \mathbb{Z}[\overline{x}, \overline{y}, \overline{z}]$, $\overline{P}, \overline{Q}\mathbb{Z}[\overline{t}]$ are tuples, $a \in \mathrm{K}(A)$ is a tuple and $\overline{\alpha} \in \mathrm{k}(A)$ is a tuple. We see immediately that this is equivalent to an $\mathcal{L}_{\mathrm{rg}}(\mathrm{k}(A))$-formula. □

Proposition 2.11: *Let $M \vDash \mathrm{ACVF}$ and $A \leqslant M$. If $X \subseteq \Gamma_\infty^n$ is A-definable, then it is $\mathcal{L}_{\mathrm{og}}(\Gamma_\infty(A))$-definable. In particular, the value group Γ_∞ is a stably embedded pure ordered (semi-)group.*

Proof. As above, it suffices to consider atomic formulas. So we may assume $X \subseteq \Gamma_\infty^n$ is defined by $L(\overline{x}, \mathrm{val}(\overline{P}(a)), \gamma) \square 0$ where L is a $\mathbb{Z}$-linear function, $\overline{P} \in \mathbb{Z}[\overline{t}]$ is a tuple, $a \in \mathrm{K}(M)$ is a tuple, $\gamma \in \Gamma_\infty(M)$ is a tuple and $\square \in \{=, <\}$. We see immediately that this is equivalent to an $\mathcal{L}(\mathrm{k}(M))$-formula and that if there are no parameters, it is equivalent to on $\mathcal{L}_{\mathrm{og}}$-formula. □

Definition 2.12: *Let T be a theory, two $\varnothing$-definable sets D_1 and D_2 are orthogonal if any definable set $X \subseteq D_1^{n_1} \times D_2^{n_2}$ is a finite union of definable boxes of the form $Y_1 \times Y_2$ where $Y_i \subseteq D_i^{n_i}$.*

Proposition 2.13: *The value group Γ_∞ and the residue field k are orthogonal.*

Proof. Since finite unions of boxes are closed under Boolean combinations, it suffices to consider atomic formulas. But these are easily seen to either be of the form $\Gamma_i^n \times Y_2$ for some $Y_2 \subseteq \mathrm{k}^m$ or $Y_1 \times \mathrm{k}^m$ for some $Y_1 \subseteq \Gamma_\infty^n$. □

We conclude by describing the algebraic closure in algebraically closed valued fields:

Proposition 2.14: *Let $A \subseteq M \vDash \mathrm{ACVF}$. We have* $\mathrm{acl}(A) = \mathrm{K}(A)^{\mathrm{a}} \cup \mathrm{k}(A)^{\mathrm{a}} \cup \mathbb{Q} \otimes \Gamma(A) \cup \{\infty\}$.

Proof. The fact that $\mathrm{k}(\mathrm{acl}(A)) \subseteq \mathrm{k}(A)^{\mathrm{a}}$ and $\Gamma(\mathrm{acl}(A)) \subseteq \mathbb{Q} \otimes \Gamma(A)$ follows immediately from Propositions **(2.10)** and **(2.11)** (and the characterization of the algebraic closure in ACF and DOAG).

Claim 2.15: *Let $f : \mathrm{k}^n \times \Gamma_\infty^m \to \mathrm{K}$ be definable. Then $f(\mathrm{k}^n \times \Gamma_\infty^m)$ is finite.*

Proof. If $f(\mathrm{k}^n \times \Gamma_\infty^m)$ is infinite, by Theorem **(2.7)**, it contains a ball. In particular, for any M over which f is definable, it is in bijection with $\mathrm{K}(M)$. But there exists $N \succcurlyeq M$ with $|\mathrm{k}(N)|, |\Gamma_\infty(N)| < |\mathrm{K}(N)|$. For example, take any maximal completion of M. ◊

Claim 2.16: $\mathrm{K}(\mathrm{acl}(A)) \subseteq \mathrm{K}(A)^{\mathrm{a}}$

Proof. It follows from Claim **(2.15)**, that $\mathrm{K}(\mathrm{acl}(A)) \subseteq \mathrm{K}(\mathrm{acl}(\mathrm{K}(A)))$. Now, by Lemma **(2.3)**, any type over $\mathrm{K}(A)^{\mathrm{a}}$ concentrating on K is the generic of some chain of nested ball. All of these types are not algebraic except for the generic of closed balls of infinite radius, a.k.a. points of $\mathrm{K}(A)^{\mathrm{a}}$. ◊

This concludes the proof $\square$

Remark 2.17: It follows from Proposition **(2.14)**, that there are no definable section of the valuation or the residue map in ACVF. Note that C-minimality also prevents the existence of definable angular components. Indeed, the fibers of an angular component are of the form $\bigcup_{\gamma\in\Gamma} b_\gamma$ where b_γ is an open ball with $\mathrm{val}(b_\gamma) = \gamma$. This set is not a finite union of Swiss cheeses.

3. Imaginaries

Let us start by recalling various definitions and constructions pertaining to imaginaries.

Definition 3.1: *A theory T is said to eliminate imaginaries if for every $\varnothing$-definable sets $X \subseteq Y \times Z$, there exists an $\varnothing$-definable map $f : Y \to E$ such that, for all $y_1, y_2 \in Y$:*

$$X_{y_1} = X_{y_2} \text{ if and only if } f(y_1) = f(y_2),$$

where $X_y := \{z \in Z : (y, z) \in X\}$.

We now recall Shelah's construction to add in imaginaries as actual points:

Definition 3.2: *Let T be an $\mathcal{L}$-theory.*
- *We define the enrichment $\mathcal{L}^{\mathrm{eq}}$ by adding a sort $E_{\varphi(x;y)}$ for each $\mathcal{L}$-formula $\varphi(x, y)$ and map $f_{\varphi(x;y)} : S_y \to E_X$, where S_y is the product of sorts over which y ranges.*
- *We define $T^{\mathrm{eq}} := T \cup \{f_{\varphi(x;y)}$ surjective $\wedge \forall y_1 y_2,\ (\forall x,\ \varphi(x; y_1) \leftrightarrow \varphi(x, y_2)) \leftrightarrow f(y_1) = f(y_2) : \varphi(x; y)$ $\mathcal{L}$–formula$\}$.*

Lemma 3.3: *Let T be an $\mathcal{L}$-theory:*
1. *For every $M \vDash T$, there is a unique $M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$ such that $M^{\mathrm{eq}}|_{\mathcal{L}} = M$.*
2. *For every $\mathcal{L}^{\mathrm{eq}}$-formula $\varphi(x)$ where x is a tuple of $\mathcal{L}$-variables, there exists an $\mathcal{L}$-formula $\psi(x)$ with $T^{\mathrm{eq}} \vdash \forall x,\ \varphi(x) \leftrightarrow \psi(x)$.*
3. *T^{eq} eliminates imaginaries.*

Lemma 3.4: *Let $M \vDash T$ and X be M-definable. Let $\varphi(x; y)$ and $m \in M$ a tuple, be such that $X = \varphi(x; m)$. Then $\ulcorner X \urcorner := \mathrm{dcl}^{\mathrm{eq}}(f_{\varphi(x;y)}(m))$ does not depend on the choice of φ or m.*

The set $\ulcorner X \urcorner$ is called the code of X.

Lemma 3.5: *Let T be an $\mathcal{L}$-theory. The following are equivalent:*
1. *T eliminates imaginaries;*
2. *For every $M \vDash T$ and $e \in m^{\mathrm{eq}}$, $e \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{dcl}^{\mathrm{eq}}(e) \cap M)$;*
3. *For every $M \vDash T$ and M-definable X, X is $\ulcorner X \urcorner \cap M$-definable.*

Let us now consider imaginaries in algebraically closed fields.

Remark 3.6: Let $M \vDash \mathrm{ACVF}$ be $\aleph_0$-saturated and b be a ball of M with $b \cap \mathrm{dcl}(\varnothing) = \varnothing$ and $\mathrm{rad}(b) \notin \mathrm{dcl}(\varnothing)$. Then $\ulcorner b \urcorner \cap M = \mathrm{dcl}(\mathrm{rad}(b))$ and b is not $\mathrm{rad}(b)$-definable. So ACVF (in the three sorted language) does not eliminate imaginaries.

We can, however, describe a — in a sense, minimal — set of imaginaries which is sufficient to eliminate all imaginaries.

Definition 3.7 (Geometric sorts)**:** *We define:*
- $\mathrm{S}_n := \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O})$;
- $\mathrm{T}_n := \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_{n,n}(\mathcal{O})$, *where* $\mathrm{GL}_{n,n}(\mathcal{O})$ *is the subgroup of* $\mathrm{GL}_n(\mathcal{O})$ *of matrices whose coordinate-wise reduction modulo* $\mathfrak{M}$ *has zeros on the last column, except for a* 1 *on the diagonal.*
- *We denote by* $\mathcal{L}^{\mathcal{G}}$ *the language with sort* K, S_n *and* T_n *for all* $n \in \mathbb{Z}_{>0}$. *We have the ring language on* K *and maps* $s_n : \mathrm{GL}_n(\mathrm{K}) \to \mathrm{S}_n$ *and* $t_n : \mathrm{GL}_n(K) \to \mathrm{T}_n$.

Every valued field can be naturally made into an $\mathcal{L}^{\mathcal{G}}$-structure. We denote by $\mathcal{G} := \{\mathrm{K}\} \cup \{\mathrm{S}_n, \mathrm{T}_n : n \geqslant 1\}$ the set of geometric sorts.

Remark 3.8:
- S_n is the moduli space for rank n free $\mathcal{O}$-submodules of K^n.
- $\mathrm{T}_n = \bigcup_{s \in \mathrm{S}_n} (s/\mathfrak{M}s) \smallsetminus \mathfrak{M}s$.
- $\mathrm{S}_1 = \mathrm{K}^\star / \smallsetminus \mathcal{O}^\star = \Gamma$.
- $\mathrm{T}_1 = \mathrm{K}^\star/(1 + \mathfrak{M}) = \mathrm{RV}$.
- The set of closed balls with finite radius can be $\varnothing$-definably embedded in S_2;
- The set of open balls with finite radius can be $\varnothing$-definably embedded in T_2;

The last result whose proof we'll sketch is:

Theorem 3.9 (Haskell-Hrushovski-Macpherson, 2006)**:** *The* $\mathcal{L}^{\mathcal{G}}$*-theory* $\mathrm{ACVF}^{\mathcal{G}}$ *of algebraically closed valued fields eliminates imaginaries.*

The proof we will follow is more recent. It is a improvement of Johnson's on a proof of Hrushovski. Let us first recall the notion of definable types since they will play a central role in classifying imaginaries in ACVF:

Definition 3.10: *Let* M *be an* $\mathcal{L}$*-structure,* $A \subseteq M$ *and* $p \in \mathcal{S}_x(M)$*. We say that* p *is* A*-definable if for every* $\mathcal{L}$*-formula* $\varphi(x; y)$*, there exists a* $\mathcal{L}(A)$*-formula* $\theta(y) =: \mathrm{d}_p x\, \varphi(x; y)$ *such that for all tuple* $m \in M^y$*,*

$$\varphi(x; m) \in p \text{ if and only if } M \vDash \theta(m).$$

When $p \in \mathcal{S}_x(M)$ is definable. The set $\ulcorner p \urcorner := \bigcup_{\varphi(x;y)} \ulcorner \mathrm{d}_p x\, \varphi(x; M) \urcorner$ is called the code of p. We say that a sort R is dominant in T if for all $M \vDash T$, $M \subseteq \mathrm{dcl}(R(M))$.

Lemma 3.11 (Hrushovski)**:** *Let* T *be an* $\mathcal{L}$*-theory with the sort* R *dominant. Assume:*
1. *for every* $M \vDash T$ *and non-empty* M*-definable* $X \subseteq R$*, there exists an* $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$*-definable* $p \in \mathcal{S}_R(M)$ *which concentrates on* X*;*
2. *for every definable type* $p \in \mathcal{S}(M)$*,* p *is* $\mathrm{acl}^{\mathrm{eq}}(\ulcorner p \urcorner) \cap M$*-definable;*
3. *for every finite* $X \subseteq M^n$*,* X *is* $\ulcorner X \urcorner \cap M$*-definable.*

Then T *eliminates imaginaries.*

Some of these hypotheses are easier than others to prove in $\mathrm{ACVF}^{\mathcal{G}}$. The density of definable types — Hypothesis Lemma **(3.11)**.1 — is an easy consequence of C-minimality.

Lemma 3.12: *Let b be a ball in some $M \vDash \mathrm{ACVF}^{\mathcal{G}}$. Then*

1. *the type $\eta_b|_M$ is definable;*
2. *$\ulcorner \eta_b|_M \urcorner = \ulcorner b \urcorner$.*

Proposition 3.13: *For any definable $\varnothing \neq X \subseteq \mathrm{K}$ in some $M \vDash \mathrm{ACVF}^{\mathcal{G}}$, there exists $p \in \mathcal{S}_{\mathrm{K}}(M)$ which is $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable and concentrates on X.*

Proof. Let b be any outer ball of a swiss cheese appearing in the canonical decomposition of X as a finite union of disjoint non nested swiss cheeses. Then b is $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-definable. So $\eta_b|_M$ is $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$ and it concentrates on X. □

The computation of the canonical basis of types, and the proof that it is inter-definable with its geometric part — Hypothesis Lemma **(3.11)**.2 — is much more involved. Hrushovski's proof of that fact involves stably dominated types and the canonical representation of definable types as limits along the value group of stably dominated types. Instead, we follow Johnson's more hands on approach. The following notion plays a crucial role:

Definition 3.14: *Let K be a valued field and V be a K-vector space. A valuation on V is a map $v : V \setminus \{0\} \to \Sigma_V \cup \{\infty\}$ where Σ_V is a totally ordered set with a free Γ_K-action such that:*

- *for all $\gamma \leqslant \delta \in \Gamma_K$ and $\sigma \leqslant \tau \in \Sigma_V$, $\gamma\sigma \leqslant \delta\tau$;*
- *for all $\lambda \in K$ and $x \in V$, $v(\lambda x) = \mathrm{val}(\lambda)v(x)$;*
- *for all $x, y \in V \setminus \{0\}$, $v(x + y) \geqslant \min\{v(x), v(y)\}$;*
- *$v(0) = \infty$.*

When Σ_V is Γ_K acting on itself by multiplication. We say that V is strictly valued.

As always, we set $\infty > \Sigma_V$ and $\infty x = x\infty = \infty$. Note that a valuation on V is determined (up to isomorphism of Σ_V) by the relation $x|y$ defined by $v(x) \leqslant v(y)$.

Definition 3.15: *Let K be a valued field and (V, v) be a valued K-vector space. For all $\sigma \in v(V)$, define $V_{\geqslant\sigma} := \{x \in V : v(x) \geqslant \sigma\}$ and $V_{>\sigma} := \{x \in V : v(x) > \sigma\}$. Both are $\mathcal{O}$-modules and $V_{\geqslant\sigma}/V_{>\sigma}$ is a k-vector space.*

We fix $M^{\mathcal{G}} \vDash \mathrm{ACVF}^{\mathcal{G}}$. When all the data required to define a valued K-vector space (V, v) is definable, we say that (V, v) is a definable valued K-vector space. The first step to encode the canonical bases of definable types is to see that definable types are completely determined by a collection of definable valued vector spaces:

Lemma 3.16: *For every $p \in \mathcal{S}_{\mathrm{K}^n}(M^{\mathcal{G}})$ and $d \in \mathbb{Z}_{\geqslant 0}$, there exists a $\ulcorner p \urcorner$-definable valuation v_p^d on $\mathrm{K}^{(d+1)n}$ such that p is completely determined by the sequence of the v_p^d.*

Proof. let $p \in \mathcal{S}_{\mathrm{K}^n}(M^{\mathcal{G}})$ definable. For every $d \in \mathbb{Z}_{\geqslant 0}$, we define $\mathrm{K}_{\leqslant d}[X] := \{\sum_{i_j \leqslant d} a_i \prod_j X_j^{i_j}\}$. We identify $\mathrm{K}_{\leqslant d}[X]$ with $\mathrm{K}^{n(d+1)}$. For all $P, Q \in \mathrm{K}_{\leqslant d}[X]$, define $P|_p Q$ as $p(x) \vdash \mathrm{val}(P(x)) \leqslant \mathrm{val}(Q(x))$. It is quite obvious from quantifier elimination that the v_p^d characterize p. □

We continue by encoding definable valuations on K^n in finite collections of definable $\mathcal{O}$-submodules of K^n.

Lemma 3.17: *Let v be a definable valuation on K^n. There exists, at most n, $\ulcorner v \urcorner$-definable $\mathcal{O}$-submodules of K^n which completely determine v.*

Proof. The crucial point is the following:

Claim 3.18: *Each Γ-orbit in $v(\mathrm{K}^n)$ contains a $\ulcorner v \urcorner$-definable element.*

Now let $\gamma_i \in v(\mathrm{K}^n)$ be $\ulcorner v \urcorner$-definable elements in each orbits. Then v is entirely determined by the $R_i := \{x \in \mathrm{K}^n : v(x) \geqslant \gamma_i\}$, which are indeed $\mathcal{O}$-submodules of K^n. □

In turn, $\mathcal{O}$-submodules of K^n give rise to definable *strict* valuations:

Lemma 3.19: *Let $R \leqslant \mathrm{K}^n$ be a definable $\mathcal{O}$-submodule. There exists an $\ulcorner R \urcorner$-definable K-vector space $V \leqslant \mathrm{K}^n$ and an $\ulcorner R \urcorner$-definable strict valuation v on V, and a $\ulcorner R \urcorner$-definable k-vector space $W \leqslant V_{\geqslant 0}/V_{>0}$ such that R is completely determined by the triple (V, v, W).*

Proof. Let V be the K-span of R. For all $x \in V$, we define $v(x) := \sup\{\mathrm{val}(\lambda) : x \in \lambda R\}$. We have $V_{>0} \leqslant R \leqslant V_{\geqslant 0}$ and hence R is entirely determined by its image in $V_{\geqslant 0}/V_{>0}$. □

Let us now encode definable strict valuations:

Lemma 3.20: *Let v be a definable strict valuation on some definable K-vector subspace $V \leqslant \mathrm{K}^n$. There exists $a \in \mathrm{K}^r$ and $s \in \mathrm{S}_l$ both $\ulcorner v \urcorner$-definable such that v is as-definable and such that $V_{\geqslant 0}/V_{>0}$ is as-definably isomorphic to $s/\mathfrak{M}s$.*

Proof. Let $W := \{x \in V : v(x) = \infty\}$. This is a K-vector subspaceof V. We then find a $\ulcorner W \urcorner$-definable basis a of W and a $\ulcorner V \urcorner \ulcorner W \urcorner$-definable basis a' of V/W. So we may assume that $W = 0$ and $V = K^l$. But since there are maximally complete models of (any completion of) ACVF, K^l, v is definable isomorphic to $\mathrm{K}^l, \mathrm{val}$. In particular, $V_{\geqslant 0}$ is definably isomorphic to $\mathcal{O}^l$, i.e. it is a lattice $s \in \mathrm{S}_l$ and $V_{\geqslant 0}/V_{>0} = s/\mathfrak{M}s$. □

There remains to encode k-vector spaces $W \leqslant s/\mathfrak{M}s$ for any $s \in \mathrm{S}_n$:

Lemma 3.21: *Let W be a k-vector subspace of $s/\mathfrak{M}s$ for some $s \in \mathrm{S}_n$. Then there exists a $\ulcorner W \urcorner$-definable tuple $a \in \bigcup_n \mathrm{T}_n$ such that W is a-definable.*

Therefore, we have proved:

Proposition 3.22: *Any definable $p \in \mathcal{S}_{\mathrm{K}^n}(M^{\mathcal{G}})$ is $\ulcorner p \urcorner \cap M^{\mathcal{G}}$-definable.*

To conclude, we have to code finite sets of geometric tuples. This is also surprisingly hard. The proof in the original Haskell-Hrushovsi-Macpherson paper is over five pages long. Once again, we follow Johnson's alternative argument. His proof requires some more advanced model theoretic technonology:

Definition 3.23: *Let M be an $\mathcal{L}$-structure and $A \subseteq M$.*

- *Let $f : X \to Y$ and $p \in \mathcal{S}_X(M)$ be A-definable. We define $f_\star p$ as the A-definable type whose definition scheme is given by $\mathrm{d}_{f_\star p} y \varphi(y, t) := \mathrm{d}_p x \varphi(f(x), t)$.*
- *Let $p \in \mathcal{S}_x(M)$and $q \in \mathcal{S}_y(M)$. Assume p and q are A-definable. We define $p \otimes q \in \mathcal{S}(M)$ as the A-definable type whose definition scheme is given by $\mathrm{d}_{p \otimes q} xy\ \varphi(x, y, t) := \mathrm{d}_p x(\mathrm{d}_q y \varphi(x, y, t))$.*

- *We say that $p \in \mathcal{S}(M)$ is generically stable if it is definable and for every definable type $q \in \mathcal{S}(M)$, $p \otimes q = q \otimes p$.*

Note that for all $A \subseteq C \subseteq M$, $b \models f_\star p|_C$ if and only if $b = f(a)$ for some $a \models p|_C$ and $(a,b) \models p \otimes q|_C$ if and only if $a \models p|_C$ and $b \models q|_{Ca}$. When $f_\star p$ is a realized type — equivalently there exists some $c \in \operatorname{dcl}(A)$ such that for all $a \models p|_A$, $f(a) = c$ — we write $f_\star p = c$.

Proposition 3.24: *For all $M \models \mathrm{ACVF}^{\mathcal{G}}$ and $a \in \mathcal{G}(M)$, there exists an a-definable generically stable type $p_a \in \mathcal{S}_{\mathrm{K}^n \times \mathrm{k}^n}(M)$ and a $\varnothing$-definable map f such that $f_\star p = a$.*

Proposition 3.25: *For all $M \models \mathrm{ACVF}^{\mathcal{G}}$ and X a finite set of tuples in $\mathcal{G}(M)$, X is $\ulcorner X \urcorner \cap \mathcal{G}(M)$-definable.*

Proof. We may assume that the elements $(a_i)_{i \leqslant d}$ of X are identically sorted tuples. By Proposition **(3.24)**, we find stably dominated $p_i \in \mathcal{S}_{\mathrm{K}^m \times \mathrm{k}^n}(M)$ which are a_i-definable and f such that $f_\star p_i = a_i$. Symmetric polynomials provide us with a $\varnothing$-definable map from $\mathfrak{S} : (\mathrm{K}^m \times \mathrm{k}^n)^d \to \mathrm{K}^l \times \mathrm{k}^r$ whose fibers correspond to enumerations of the same set of size at most d. Let $q = \mathfrak{S}_\star(\bigotimes_i p_i)$. Then q is $\ulcorner X \urcorner$-definable and there exists g $\varnothing$-definable such that $g_\star q = X$. By Proposition **(3.22)**, g, and hence X, is $\ulcorner X \urcorner \cap \mathcal{G}(M)$-definable. □

A. Eliminating quantifiers with more algebra

Theorem A.1: *Let (K, val) be a valued field, $L \geqslant K$ a finite extension and $(\mathcal{O}_i)_i$ enumerating all valuation rings on L. Let k_i denote the residue field of $\mathcal{O}_i$ and Γ_i its value group. Then $[L : K] = p^d \sum_i [k_i : k_K][\Gamma_i : \Gamma_K]$, where $d \in \mathbb{Z}_{\geqslant 0}$ and p is the residue characteristic of K, if it is positive, and 1 otherwise.*

Theorem A.2 (Conjugation theorem)**:** *Let (K, val) be a valued field and $L \geqslant K$ be an algebraic extensions. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two valuations rings on L extending $\mathcal{O}$. Then, there exists $\sigma \in \mathrm{Aut}(L/K)$ such that $\mathcal{O}_2 = \sigma(\mathcal{O}_1)$.*

Let us now (re-)prove quantifier elimination for ACVF.

Proof (Theorem **(1.2)**). Let $M, N \models \mathrm{ACVF}$, $A \leqslant M$ and $f : A \to N$ an $\mathcal{L}$-embedding. Assume N is $|M|^+$-saturated. Our goal is to extend f to M. As before, since f has a (unique) extension to $A \cup \mathrm{Frac}(\mathrm{K}(A)) \cup \mathrm{Frac}(\mathrm{k}(A))$, so we may always assume that $\mathrm{K}(A)$ and $\mathrm{k}(A)$ are fields.

Claim A.3: *For every $\gamma \in \Gamma(A) \cap \mathbb{Q} \otimes \mathrm{val}(\mathrm{K}(A))$, we may extend f to some C with $\gamma \in \mathrm{val}(\mathrm{K}(C))$.*

Proof. Let $n \in \mathbb{Z}_{>0}$ be the order of γ in $\Gamma(M)/\mathrm{val}(\mathrm{K}(A))$ and $c \in \mathrm{K}(A)$ be such that $n\gamma = \mathrm{val}(c)$. Find $a \in \mathrm{K}(M)$ such that $a^n = c$ and $b \in \mathrm{K}(N)$ such that $b^n = f(c)$. Note that by minimality of n, the minimal polynomial of a over $\mathrm{K}(A)$ is $P := X^n - c$ and the minimal polynomial of b over $f(\mathrm{K}(A))$ is $f(P) = X^n - f(c)$. We extend $f|_{\mathrm{K}}$ to $g : \mathrm{K}(A)[a] \to N$ by sending a to b. Then $g|_{\mathrm{K}}$ is an $\mathcal{L}_{\mathrm{rg}}$-isomorphism.
Since $[\mathrm{K}(A)[a] : A] = n \leqslant [\mathrm{val}(\mathrm{K}(A)[a]) : \mathrm{val}(\mathrm{K}(A))]$, by Theorem **(A.1)**, $\mathrm{val}(\mathrm{K}(A)[a])$ is generated by γ over $\mathrm{val}(\mathrm{K}(A))$, $\mathrm{res}(\mathrm{K}(A)[a]) = \mathrm{res}(\mathrm{K}(A))$ and $\mathcal{O}(\mathrm{K}(A)[a])$ is the unique valuation ring of $\mathrm{K}(A)[a]$ extending $\mathcal{O}(\mathrm{K}(A))$ on $\mathrm{K}(A)$. So $\mathcal{O} = g^{-1}(\mathcal{O})$ and hence there

exists two isomorphisms $\tau : \mathrm{val}(\mathrm{K}(A)[a]) \to \Gamma_\infty(N)$ and $\rho : \mathrm{res}(\mathrm{K}(A)[a]) = \mathrm{res}(\mathrm{K}(A)) \to \mathrm{k}(N)$ with $\mathrm{val} \circ g = \tau \circ \mathrm{val}$ and $\mathrm{res} \circ g = \rho \circ \mathrm{res}$. Computing, we see that $\tau|_{\mathrm{val}(\mathrm{K}(A))} = f|_{\mathrm{val}(\mathrm{K}(A))}$ and that $\tau = restr f \mathrm{res}(\mathrm{K}(A))$. Since the only automorphism of $\mathbb{Q} \otimes \mathrm{val}(\mathrm{K}(A))$ over $\mathrm{val}(\mathrm{K}(A))$ is the identity, τ and $f|_{\Gamma_\infty}$ coincide when both are defined. It is now straightforward to check that the natural extension of g to the structure generated by a over A extends f and is an $\mathcal{L}_{\mathrm{k},\Gamma}$-isomorphism. Note that $\mathrm{K}(A)[a]$ is a field so compatibility with r is equivalent to compatibility with res. ◊

Claim A.4: *For every $\alpha \in \mathrm{k}(A) \cap \mathrm{res}(\mathrm{K}(A))^{\mathrm{a}}$, we may extend f to some $C \leqslant M$ with $\alpha \in \mathrm{res}(\mathrm{K}(C))$.*

Proof. Let $P \in \mathrm{K}(A)[X]$ be an exact lifting of the minimal polynomial of α over $\mathrm{res}(\mathrm{K}(A))$, $a \in \mathrm{K}(M)$and $b \in \mathrm{K}(N)$ such that $P(a) = 0 = f(P)(b)$ and $\mathrm{val}(b) = f(\alpha) = f(\mathrm{val}(a))$. We extend $f|_{\mathrm{K}}$ to $g : \mathrm{K}(A)[a] \to N$ by sending a to b. Then $g|_{\mathrm{K}}$ is an $\mathcal{L}_{\mathrm{rg}}$-isomorphism.
Since $[\mathrm{K}(A)[a] : A] = n \leqslant [\mathrm{res}(\mathrm{K}(A)[a]) : \mathrm{res}(\mathrm{K}(A))]$, by Theorem **(A.1)**, $\mathrm{res}(\mathrm{K}(A)[a])$ is generated by α over $\mathrm{res}(\mathrm{K}(A))$, $\mathrm{val}(\mathrm{K}(A)[a]) = \mathrm{val}(\mathrm{K}(A))$ and there exists two isomorphisms $\tau : \mathrm{val}(\mathrm{K}(A)[a]) = \mathrm{val}(\mathrm{K}(A)) \to \Gamma_\infty(N)$ and $\rho : \mathrm{res}(\mathrm{K}(A)[a]) = \mathrm{res}(\mathrm{K}(A)/\alpha) \to \mathrm{k}(N)$ with $\mathrm{val} \circ g = \tau \circ \mathrm{val}$ and $\mathrm{res} \circ g = \rho \circ \mathrm{res}$. Computing, we see that τ and $f|_\Gamma$ coincide, and since ρ sends $\alpha = \mathrm{val}(a)$ to $\mathrm{val}(b) = f(\alpha)$, ρ and $f|_\Gamma$ also coincide. It is now straightforward to check that the natural extension of g to the structure generated by a over A extends f and is an $\mathcal{L}_{\mathrm{k},\Gamma}$-isomorphism. ◊

Claim A.5: *We can extend f to some $C \leqslant M$ with $\mathrm{K}(C)^{\mathrm{a}} \subseteq C$.*

Proof. By the two previous claims, we may assume that $\mathrm{val}(\mathrm{K}(A)^{\mathrm{a}}) \cap \Gamma_\infty(A) \subseteq \mathrm{val}(\mathrm{K}(A))$ and that $\mathrm{res}(\mathrm{K}(A)^{\mathrm{a}}) \cap \mathrm{k}(A) \subseteq \mathrm{res}(\mathrm{K}(A))$. We can extend $f|_{\mathrm{K}}$ to an $\mathcal{L}_{\mathrm{rg}}$-isomorphism $g : \mathrm{K}(A)^{\mathrm{a}} \to N$. By Theorem **(A.2)**, we may assume that $g^{-1}(\mathcal{O}) = \mathcal{O}$. Since $\mathrm{val}(\mathrm{K}(A)^{\mathrm{a}}) \cap \Gamma_\infty(A) \subseteq \mathrm{val}(\mathrm{K}(A))$ and $\mathrm{res}(\mathrm{K}(A)^{\mathrm{a}}) \cap \mathrm{k}(A) \subseteq \mathrm{res}(\mathrm{K}(A))$, the isomorphisms induced by g on k and Γ_∞ coincide with f. So the natural extension of g to the structure generated by A and $\mathrm{K}(A)^{\mathrm{a}}$ extends f and is an $\mathcal{L}_{\mathrm{k},\Gamma}$-isomorphism. ◊

So we can always assume that $\mathrm{K}(A)^{\mathrm{a}} \subseteq A$.

Claim A.6: *For every $\gamma \in \Gamma(M)$, we may extend f to some $C \leqslant M$ with $\gamma \in \mathrm{val}(\mathrm{K}(C))$.*

Proof. We may assume that $\gamma \notin \mathrm{val}(\mathrm{K}(A))$. Let $a \in \mathrm{K}(N)$ be such that $\mathrm{val}(a) = \gamma$. Let us now prove that for all $P \in \mathrm{K}(A)(X)$, $\mathrm{rv}(P(a)) = \alpha \mathrm{rv}(a)^l$ where $\alpha \in \mathrm{rv}(\mathrm{K}(A))$ and $l \in \mathbb{Z}$. Since rv is multiplicative and $\mathrm{K}(A)^{\mathrm{a}} = \mathrm{K}(A)$, we may assume that $P = X - c$. If $\mathrm{val}(c) < \mathrm{val}(a)$, we have $\mathrm{rv}(P(a)) = \mathrm{rv}(-c) \in \mathrm{rv}(\mathrm{K}(A))$. If $\mathrm{val}(c) < \mathrm{val}(a)$, we have $\mathrm{rv}(P(a)) = \mathrm{rv}(a)$. It follows that $\mathrm{val}(\mathrm{K}(A)(a)) = \mathrm{val}(\mathrm{K}(A)) + \mathbb{Z}\gamma$ and $\mathrm{res}(\mathrm{K}(A)(a)) = \mathrm{res}(\mathrm{K}(A))$ since $\gamma \notin \mathbb{Q} \otimes \mathrm{val}(K(A))$.
Let n be the order of γ in $\Gamma(M)/\Gamma(A)$. Let $\eta \in \Gamma(N)$ be such that $n\eta = f(m\gamma)$. If $n = \infty$, by saturation, we may also assume that, for all $\delta \in \Gamma(A)$, $f(\delta) < \eta$ if and only if $\delta < \gamma$. Let $b \in \mathrm{K}(N)$ be such that $\mathrm{val}(b) = \eta$. We extend f to some g sending a to b. By the above computation, $g|_{\mathrm{K}}$ is a ring isomorphism and g respects both val and r. Since the residue field does not grow, $g|_{\mathrm{k}}$ is a ring isomorphism. Since γ and eta have the same order, and when this order is infinite, they realize the same cut (over f), $g|_\Gamma$ is an ordered group isomorphism. ◊

Claim A.7: *For every $\alpha \in \mathrm{k}(M)$, we may extend f to some $C \leqslant M$ with $\alpha \in \mathrm{val}(\mathrm{K}(C))$.*

Proof. We may assume that $\alpha \notin \mathrm{res}(\mathrm{K}(A))$. Let $a \in \mathrm{K}(N)$ be such that $\mathrm{res}(a) = \alpha$. Let us now prove that for all $P \in \mathrm{K}(A)(X)$, $\mathrm{rv}(P(a)) = \eta Q(\alpha)$ where $\eta \in \mathrm{rv}(\mathrm{K}(A))$ and $Q \in resf(\mathrm{K}(A))(X)$. We may assume that $P = X - c$ with $c \in \mathrm{K}(A)$. If $\mathrm{val}(c) < 0$, $\mathrm{rv}(a-c) = \mathrm{rv}(-c) \in \mathrm{rv}(\mathrm{K}(A))$. If $\mathrm{val}(c) \geqslant 0$, since $\mathrm{rv}(a-c) = \mathrm{res}(a-c) = \alpha - \mathrm{res}(c)$. It follows that $\mathrm{val}(\mathrm{K}(A)(a)) = \mathrm{val}(\mathrm{K}(A))$ and $\mathrm{res}(\mathrm{K}(A)(a)) = \mathrm{res}(\mathrm{K}(A))(\alpha)$.
Let R be the minimal polynomial of α over $\mathrm{k}(A)$. Let $\beta \in \mathrm{k}(N)$ be a root of $f(R)$. Let $b \in \mathrm{K}(N)$ be such that $\mathrm{res}(b) = \beta$. We extend f to some g sending a to b. By the above computation, $g|_{\mathrm{K}}$ is a ring isomorphism and g respects both val and r. Since the value group does not grow, $g|_{\Gamma}$ is an ordered group isomorphism. Since α and β have the same minimal polynomial, $g|_{\mathrm{k}}$ is a ring isomorphism. ◊

We can now assume that $\Gamma_\infty(M) \subseteq \mathrm{val}(\mathrm{K}(A))$ and $\mathrm{k}(M) \subseteq \mathrm{res}(\mathrm{K}(A))$. So $\mathrm{rv}(\mathrm{K}(M)) \subseteq \mathrm{rv}(\mathrm{K}(A))$

Claim A.8: *For every $a \in \mathrm{K}(M)$, we may extend f to some $C \leqslant M$ with $a \in \mathrm{K}(C)$.*

Proof. For every $c \in \mathrm{K}(A)$, we have $\mathrm{rv}(a-c) \in \mathrm{rv}(\mathrm{K}(M)) = \mathrm{rv}(\mathrm{K}(A))$. Let $e \in \mathrm{K}(A)$ be such that $\mathrm{rv}(a-c) = \mathrm{rv}(e)$, i.e. $\mathrm{val}(a-(c+e)) > \mathrm{val}(a-c)$. So, by transfinite induction, we can build a maximal pseudo-Cauchy sequence $a_i \in \mathrm{K}(A)$ with $a \in \mathrm{plim}_i\, a_i$. For all $P \in \mathrm{K}(A)(X)$, let us now prove that $\mathrm{rv}(P(a)) = \mathrm{rv}(P(a_i))$ for sufficiently large i. We may assume $P = X - c$ for some $c \in \mathrm{K}(A)$. By maximality, $c \notin \mathrm{plim}_i\, a_i$ and hence we have $\mathrm{rv}(a-c) = \mathrm{rv}(a_i - c)$ for sufficiently large i.
By saturation, we find $b \in \mathrm{K}(N)$ with $f(a_i) \rightsquigarrow b$. By the above computation, the extension g of f sending a to b is an $\mathcal{L}_{\mathrm{k},\Gamma}$-isomorphism. ◊

This concludes the proof. □

B. The leading term language

Using leading terms, we can combine Lemmas **(1.3)** and **(1.4)**:

Lemma B.1: *Let $M \models \mathrm{ACVF}$, $A \leqslant \mathrm{K}(M)$ and $\alpha \in \mathrm{rv}(M)$. Let $P = \sum_i p_i X^i \in A[X]$ have minimal degree among those polynomials such that $\mathrm{val}(P(c)) > \min_i(\mathrm{val}(p_i) + i\mathrm{val}(\alpha))$; or $P = 0$ if such a polynomial does not exists. Then, for every $Q = \sum_i q_i X^i \in \mathcal{O}_A[X]$ with degree smaller than P and every $a \in \mathrm{res}^{-1}(\alpha)$,*

$$\mathrm{rv}(Q(a)) = \sum_{i \in I_0} \mathrm{rv}(q_i)\alpha^i \neq 0,$$

where $I_0 := \{i : \mathrm{val}(q_i) + i\mathrm{val}(\alpha)$ is minimal$\}$. If $\deg(P) \leqslant \min\{n > 0 : n\mathrm{val}(\alpha) \in \mathrm{val}(A)\}$[1]— or $P = 0$ — then I_0 is a singleton.
Moreover, there exists $a \in \mathrm{rv}^{-1}(\alpha)$ such that $P(a) = 0$.

1. Since, $\mathrm{val}(P(c)) > \min_i(\mathrm{val}(p_i) + i\mathrm{val}(\alpha))$, there are some $i < j$ such that $\mathrm{val}(p_i) + i\mathrm{val}(\alpha) = \mathrm{val}(p_j) + i\mathrm{val}(\alpha)$ and hence $(j-i)\mathrm{val}(\alpha) = \mathrm{val}(p_i) - \mathrm{val}(q_j)$ where $j - i \leqslant \deg(P)$. It follows that we always have $\deg(P) \geqslant \min\{n > 0 : n\mathrm{val}(\alpha) \in \mathrm{val}(A)\}$.

Proof. By minimality of P, we have $\mathrm{val}(Q(a)) \leqslant \min_i(\mathrm{val}(q_i) + i\mathrm{val}(\alpha)) = \min_i \mathrm{val}(q_i a^i) \leqslant \mathrm{val}(\sum_{i\in I_0} q_i a^i) \leqslant \mathrm{val}(Q(a))$. It follows that all these terms are equal and that $\mathrm{rv}(Q(a)) = \mathrm{rv}(\sum_{i\in I_0} q_i a^i) = \sum_{i\in I_0} \mathrm{rv}(q_i)\alpha^i \neq 0$. If I_0 is not a singleton, there are$i < j$ with $\mathrm{val}(q_i) + i\mathrm{val}(\alpha) = \mathrm{val}(q_j) + i\mathrm{val}(\alpha)$ and hence $(j - i)\mathrm{val}(\alpha) = \mathrm{val}(q_i) - \mathrm{val}(q_j) \in \mathrm{val}(A)$ where $j - i \leqslant \deg(Q) < \deg(P)$. It would follow that $\deg(P) > \min\{n > 0 : n\mathrm{val}(\alpha) \in \mathrm{val}(A)\}$.
Let $P = c\prod_j(X - e_j)$, where $c \in A$ and assume that there are no e_j with $\mathrm{rv}(e_j) = \alpha$. Let $J_0 := \{j : \mathrm{val}(e_j) > \mathrm{val}(\alpha)\}$, $J_\alpha := \{j : \mathrm{val}(e_j) = \mathrm{val}(\alpha)\}$ and $J_\infty := \{j : \mathrm{val}(e_j) < \mathrm{val}(\alpha)\}$. For any $a \in \mathrm{rv}^{-1}(\alpha)$, $\mathrm{val}(P(a)) = \mathrm{val}(c) + \sum_{j\in J_0} \mathrm{val}(a) + \sum_{j\in J_\alpha} \mathrm{rv}(a - e_j) + \mathrm{val}_{j\in J_\infty}\mathrm{val}(e_j)$. Note that for any $j \in J_\alpha$, since $\mathrm{rv}(e_j) \neq \alpha$, $\mathrm{val}(\alpha) = \mathrm{val}(a) = \mathrm{val}(a - e_j) = \mathrm{val}(e_j)$. It follows that $\mathrm{val}(P(a)) = \mathrm{val}(c) + \sum_{j\in J_\infty} \mathrm{val}(e_j) + (d - |J_\infty|)\mathrm{val}(\alpha)$, where $d = \deg(P)$.
Note, moreover, that $p_i = c\sum_{|J|=d-i}\prod_{j\in J} e_j$ and thus $\mathrm{val}(p_i) - \mathrm{val}(c) \geqslant \min_{|J|=d-i}\sum_{j\in J}\mathrm{val}(e_j) \geqslant \min_{|J|=d-i}\sum_{j\in J_\infty\cap J}\mathrm{val}(e_j) + \sum_{j\in J\smallsetminus J_\infty}\mathrm{val}(\alpha)$. It follows that $\mathrm{val}(p_i) + i\mathrm{val}(\alpha) \geqslant \mathrm{val}(c) + \sum_{j\in J_\infty}\mathrm{val}(e_j) + (d - |J| - |J_\infty \smallsetminus J| + |J \smallsetminus J_\infty|)\mathrm{val}(\alpha) = \mathrm{val}(P(a))$. It would follow that $\mathrm{val}(P(a)) = \min_i(\mathrm{val}(p_i) + i\mathrm{val}(\alpha))$, a contradiction. □

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