Model theory of valued fields*

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Contents

1. Valued fields

1.1. Definitions

All rings are commutative and unitary.

07/01

Definition 1.1.1. Let R be a ring, a *valuation* on R is a (surjective) map $v : R \to \Gamma$, where $(\Gamma, +, 0, <,)$ is an ordered commutative monoid[1], such that for every $x, y \in K$:

(a) $v(0) \neq v(1) = 0$;
(b) $v(x + y) \geqslant \min\{v(x), v(y)\}$;
(c) $v(xy) = v(x) + v(y)$.
(d) $v(R) \smallsetminus \{v(0)\} \subseteq \Gamma$ is cancellable; *i.e.* for every $z \in R$, if $v(x) + v(z) = v(y) + v(z)$ and $v(z) \neq v(0)$, then $v(x) = v(y)$.

Lemma 1.1.2. *Let* (R, v) *be a valued ring. For every* $x, y \in R$*:*

(1) $v(-x) = v(x)$*;*
(2) if $v(x) < v(y)$ *then* $v(x + y) = v(x)$[2]*;*
(3) $v(x) \leqslant v(0)$*;*
(4) $\{x \in R : v(x) = v(0)\} \subseteq R$ *is a (proper) prime ideal.*

Proof. (1) If $0 \,\square\, v(-1)$, with $\square \in \{\leqslant, \geqslant\}$, then $v(-1) \,\square\, v(-1) + v(-1) = v((-1)^2) = v(1) = 0$. So $v(-1) = 0$[3]. It follows that $v(-x) = v(-1) + v(x) = v(x)$.

(2) Assume $v(x) < v(y)$. Since $v(x+y) \geqslant \min\{v(x), v(y)\} = v(x)$ it suffices to rule out that $v(x+y) > v(x)$. If not, we would have $v(x) = v(x+y-y) \geqslant \min\{v(x+y), v(y)\} > v(x)$, a contradiction.

(3) If $v(x) > v(0)$, then $v(x) = v(x + 0) = v(0)$, a contradiction.

(4) Let $x, y \in R$ with $v(y) = v(0)$, then $v(xy) = v(x) + v(0) = v(x \cdot 0) = v(0)$. Also, if $v(x) = v(0)$, we have $v(x + y) \geqslant \min\{v(x), v(y)\} = 0$ and hence $v(x + y) = 0$. Finally if $x, y \in R$ are such that $v(x) + v(y) = v(xy) = v(0)$, then $v(x) = 0$ or $v(y) = 0$. ☐

Remark 1.1.3. 1. From now on, we will write $\infty := v(0)$, which is both maximal and annihilating.

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[1] That is, $<$ is a total order and for every $x, y, z \in \Gamma$:
(a) $(x + y) + z = x + (y + z)$;
(b) $x + y = y + x$;
(c) $x + 0 = x$;
(d) if $x \leqslant y$ then $x + z \leqslant y + z$.

[2] Equivalently, every triangle is isoceles.

[3] Exercise: Show that ordered monoids are torsion free.

2. If R is a field, then (d) always holds. Indeed, $v(R^\times) \leqslant \Gamma$ is a subgroup, but $\infty = v(0)$ is not invertible, so $\infty \notin v(R^\times)$. In fact, $v(R) = v(R^\times) \cup \{\infty\}$.
3. Let $(\Gamma, +, 0, <, \infty) = (\mathbb{R}_{\geqslant 0}, \cdot, 1, >, 0)$. Then a valuation $|.| : K \to \mathbb{R}_{\geqslant 0}$ is exactly a multiplicative norm on K verifying the strong triangular inequality:

$$|x + y| \leqslant \max\{|x|, |y|\} \leqslant |x| + |y|$$

for every $x, y \in K$.

Fix (K, v) a valued field.

Definition 1.1.4. We define:
- the *value group* $vK^\times := v(K^\times)$ and its associated monoid $vK := v(K) = vK^\times \cup \{\infty\}$;
- the *valuation ring* $\mathcal{O} = \mathcal{O}_v := \{x : v(x) \geqslant 0\} \subseteq K$, a local subring;
- its unique maximal ideal $\mathfrak{m} = \mathfrak{m}_v := \{x : v(x) > 0\} \subset \mathcal{O}$;
- the *residue field* $Kv := \mathcal{O}/\mathfrak{m}$ and $\mathrm{res} = \mathrm{res}_v : \mathcal{O} \to k$ the canonical projection.

Proof. We have $v(0) = \infty > 0$ et $v(1) = 0$, so $0, 1 \in \mathcal{O}$. Also, for every $x, y \in \mathcal{O}$, $v(x + y) \geqslant \min\{v(x), v(y)\} \geqslant 0$ and $v(xy) = v(x) + v(y) \geqslant 0 + 0 = 0$. So $\mathcal{O} \subseteq K$ is a subring. Similarly, if $x, y \in \mathfrak{m}$, $v(x+y) \geqslant \min\{v(x), v(y)\} > 0$ and if $x \in \mathcal{O}$ and $y \in \mathfrak{m}$, then $v(xy) = v(x) + v(y) > 0$ and hence $\mathfrak{m} \subseteq \mathcal{O}$ is a ideal. Note also that $x \in \mathcal{O}$ is invertible if and only if $x^{-1} \in \mathcal{O}$, i.e. $-v(x) = v(x^{-1}) \geqslant 0$, or, equivalently, $v(x) = 0$. So $\mathcal{O}^\times = \mathcal{O} \smallsetminus \mathfrak{m}$ and $\mathfrak{m}$ is indeed the unique maximal ideal in $\mathcal{O}$. □

Proposition 1.1.5. *Let R be a ring and $K \geqslant R$ be a field. The following are equivalent:*
- *(i) there exists a valuation v on K such that $R = \mathcal{O}_v$;*
- *(ii) for some prime ideal $\mathfrak{p} \subset R$, $(R, \mathfrak{p})$ is maximal for domination*[4] *in K —in particular, R is local and $\mathfrak{p}$ is its maximal ideal;*
- *(iii) for all $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$;*
- *(iv) principal ideals of R are totally ordered by inclusion and $K = R_{(0)}$;*
- *(v) sub-R-modules of K are totally ordered by inclusion;*
- *(vi) the monoid $(K/R^\times, \cdot, \overline{1})$ is totally ordered, where $\overline{a} \leqslant \overline{b}$ if $b \in R \cdot a$, and $\pi : K \to K/R^\times$ is a valuation.*

We say that R is a valuation ring if these equivalent conditions hold.

Proof. Note that (vi) trivially implies (i).

(i)⇒(ii) Let $\mathfrak{p} := \mathfrak{m}_v = \{x \in K : v(x) > 0\}$. and let us assume that $(R, \mathfrak{p})$ is dominated by some $(S, \mathfrak{q})$ with $S \leqslant K$. We want to show that $S \leqslant R$. Fix some $s \in S$. If $v(s) < 0$, then $s^{-1} \in \mathfrak{p}$. So $1 = ss^{-1} \in S \cdot \mathfrak{p} \subseteq \mathfrak{q}$, a contradiction. It follows that $v(s) \geqslant 0$ and $s \in R = \mathcal{O}_v$. Since $R \leqslant R_\mathfrak{p} \leqslant K$ and $R_\mathfrak{p} \cdot \mathfrak{p} \cap R = \mathfrak{p}$, $(R_\mathfrak{p}, R_\mathfrak{p} \cdot \mathfrak{p})$ dominates $(R, \mathfrak{p})$. By maximality, $R = R_\mathfrak{p}$ is local and $\mathfrak{p}$ is its maximal ideal.

[4] If $R_1, R_2 \leqslant K$ are subrings and $\mathfrak{p}_i \subset R_i$ are prime ideals, $(R_2, \mathfrak{p}_2)$ dominates $(R_1, \mathfrak{p}_1)$ whenever $R_1 \leqslant R_2$ and $\mathfrak{p}_2 \cap R_1 = \mathfrak{p}_1$.

(ii)⇒(iii) Let us first assume that $\mathfrak{p}[x] \subset R[x]$ and let $\mathfrak{m} \supseteq \mathfrak{p}[x]$ be some maximal ideal of $R[x]$. Then $\mathfrak{m} \cap R$ is a ideal of R containing $\mathfrak{p}$ but that does not contain 1. So it is equal to $\mathfrak{p}$ and $(R[x], \mathfrak{m})$ dominates $(R, \mathfrak{p})$. By maximality, we have $R[x] = R$ and thus $x \in R$. Applying this to x^{-1}, we see that, if $\mathfrak{p}[x^{-1}] \subset R[x^{-1}]$, then $x^{-1} \in R$. So we may assume that $1 \in \mathfrak{p}[x]$ and $1 \in \mathfrak{p}[x^{-1}]$ to derive a contradiction. Let m and n be minimal such that $\sum_{i \leqslant m} a_i x^i = 1 = \sum_{j \leqslant m} b_j x^{-j}$, for some $a_i, b_j \in \mathfrak{p}$. We may assume that $m \leqslant n$. Since $1 - a_0 \in R^{\times}$, we have $c_i = a_i(1-a_0)^{-1} \in \mathfrak{p}$ and $\sum_{i=1}^{m} c_i x^i = 1$ it follows that $1 = \sum_{j<n} b_j x^{-j} + \sum_{i=1}^{m} b_n c_i x^{-(n-i)}$, contradicting the minimality of n.

(iii)⇒(iv) It follows from (iii) that $K = R_{(0)}$. Fix $a, b \in R$. If $a^{-1}b \in R$, then $(b) \subseteq (a)$. If not, by (iii), we must have $b^{-1}a \in R$ and thus $(a) \subseteq (b)$.

(iv)⇒(v) Let $\mathfrak{a}, \mathfrak{b} \leqslant K$ be sub-R-modules and let us assume that there is some $a \in \mathfrak{a} \setminus \mathfrak{b}$. We want to show that $\mathfrak{b} \subseteq \mathfrak{a}$. Let $b \in \mathfrak{b}$ and write $a = a_0/a_1$ and $b = b_0/b_1$ with $a_0, a_1, b_0, b_1 \in R$. By (iii) we either have $a_0 b_1 \in (b_0 a_1)$, in which case $a \in R \cdot b \subseteq \mathfrak{b}$, a contradiction, or $b_0 a_1 \in (a_0 b_1)$, in which case, $b \in R \cdot a \subseteq \mathfrak{a}$. 11/01

(v)⇒(vi) The ordered set $(K/R^{\times}, \leqslant)$ is isomorphic to the set of principal sub-R-modules of K which is totally ordered by (v). Let us show that it is a monoid order. If $b \in R \cdot a$, then for every $c \in K$, $bc \in R \cdot ac$. So $\overline{a} \leqslant \overline{b}$ does indeed imply $\overline{a} \cdot \overline{c} \leqslant \overline{b} \cdot \overline{c}$.

Let us now check that π is a valuation. We have $\pi(0) = \overline{0} \neq \overline{1} = \pi(1)$. For every $x, y \in K$ we have $x + y \in (x) \cap (y)$, so $\pi(x+y) \geqslant \min\{\pi(x), \pi(y)\}$, and, by definition, $\pi(xy) = \pi(x) \cdot \pi(y)$. □

The valuation in condition (i) is essentially unique:

Lemma 1.1.6. *Let v be a valuation on K, $f : K \to L$ a field morphism and w a valuation on L. The following are equivalent:*

(i) $\mathcal{O}_v \subseteq f^{-1}(\mathcal{O}_w)$;

(ii) $\mathcal{O}_v^{\times} \subseteq f^{-1}(\mathcal{O}_w^{\times})$

(iii) there is a unique morphism $g : vK \to wL$ such that:

$$\begin{array}{ccc} K & \xrightarrow{v} & vK \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ L & \xrightarrow[w]{} & wL \end{array}$$

commutes.

Proof.

(i)⇒(ii) Let $x \in \mathcal{O}_v^{\times}$. Then, by (i), $x^{-1} \in \mathcal{O}_v \subseteq f^{-1}(\mathcal{O}_w)$ and hence $f(x)^{-1} = f(x^{-1}) \in \mathcal{O}_w$, in other words, $f(x) \in \mathcal{O}_w^{\times}$.

(ii)⇒(i) Let $x \in \mathcal{O}_v$. If $x \in \mathcal{O}_v^{\times}$, then, by (ii), $x \in f^{-1}(\mathcal{O}_w^{\times}) \subseteq f^{-1}(\mathcal{O}_w)$. If $x \notin \mathcal{O}_v^{\times}$, then $1 + x \in \mathcal{O}_v^{\times}$ and thus $1 + f(x) = f(1+x) \in \mathcal{O}_w^{\times} \subseteq \mathcal{O}_w$ and hence $x = \mathcal{O}_w$.

(ii)⇒(iii) For the diagram to commute, we must have, for every $x \in K$, $g(v(x)) = w(f(x))$. There remains to show that this defines an ordered group morphism. Let $x \in K$ be such that $v(x) = 0$, then $x \in \mathcal{O}_v^{\times}$ and hence, by (ii), $f(x) \in \mathcal{O}_w^{\times}$. So $w(f(x)) = 0$. It follows that g is well defined. Indeed let $x, y \in K$ be such that $v(x) = v(y)$. If $v(x) = v(y) = \infty$, then

$x = y = 0$ and $w(f(x)) = \infty = w(f(y))$. If $v(x) = v(y) \neq \infty$, then $v(xy^{-1}) = 0$ and hence $w(f(xy^{-1})) = 0$, *i.e.* $w(f(x)) = w(f(y))$.

Checking that g is an ordered monoid morphism is a matter of straighforward verification. For every $x, y \in K$, $g(v(x)+v(y)) = g(v(xy)) = w(f(xy)) = w(f(x)) + w(f(y))$ and $v(x) \geqslant v(y)$ if and only if $x \in \mathcal{O}_v \cdot y$, which, by (i), implies that $f(x) \in \mathcal{O}_w \cdot f(y)$, *i.e.* $w(f(x)) \geqslant w(f(y))$.

(iii)⇒(i) Let $x \in \mathcal{O}_v$. Then, by (iii), $w(f(x)) = g(v(x)) \geqslant 0$ and hence $f(x) \in \mathcal{O}_w$. □

If $f = \mathrm{id}_K$, we say that v and w are dependent.

11/01

Corollary 1.1.7. *Let v be a valuation on K, $f : K \to L$ a field morphism and w a valuation on L. The following are equivalent:*

(i) $\mathcal{O}_v = f^{-1}(\mathcal{O}_w)$;

(ii) $\mathcal{O}_v^\times = f^{-1}(\mathcal{O}_w^\times)$;

(iii) $(\mathcal{O}_w, \mathfrak{m}_w)$ dominates $(f(\mathcal{O}_v), f(\mathfrak{m}_w))$;

(iv) the morphism $g : vK \to wL$ of lemma 1.1.6.(iii) is injective.

We say that f is a valued field embedding.

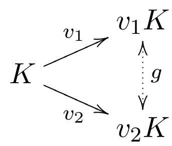
Proof. Note that (i) and (iii) imply lemma 1.1.6.(i) and (ii) implies lemma 1.1.6.(ii), all three statements imply lemma 1.1.6.(iii) and the existence of g. Now the morphism g is injective if and only if, for every $x \in K$

- $v(x) \geqslant 0$ if and only if $w(f(x)) \geqslant 0$, *i.e.* $\mathcal{O}_v = f^{-1}(\mathcal{O}_w)$;
- $v(x) = 0$ if and only if $w(f(x)) = 0$, *i.e.* $\mathcal{O}_v^\times = f^{-1}(\mathcal{O}_w^\times)$;
- $v(x) > 0$ if and only if $w(f(x)) > 0$, *i.e.* $f(\mathfrak{m}_v) \subseteq \mathfrak{m}_w \cap f(\mathcal{O}_v) \subseteq \mathfrak{m}_w \cap f(K) \subseteq f(\mathfrak{m}_v)$. Since, by lemma 1.1.6.(i), we have $f(\mathcal{O}_v) \subseteq \mathcal{O}_w$, this is equivalent to the domination of $(f(\mathcal{O}_v), f(\mathfrak{m}_w))$ by $(\mathcal{O}_w, \mathfrak{m}_w)$. □

Corollary 1.1.8. *Let $v_i : K \to \Gamma_i$, for $i = 1, 2$, be valuations. The following are equivalent:*

(i) $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$;

(ii) there is a unique isomorphism $g : v_1K \to v_2K$ such that:



commutes.

We say that v_1 and v_2 are equivalent valuations.

Proof. Exercise. □

11/01

Corollary 1.1.9. *Let R be a ring, $\mathfrak{p} \subset R$ be prime and $K \geqslant R$ a field. Then there exists a valuation ring $\mathcal{O} \subseteq K = \mathcal{O}_{(0)}$ with maximal ideal $\mathfrak{m}$ such that $(\mathcal{O}, \mathfrak{m})$ dominates $(R, \mathfrak{p})$.*

Proof. The set of pairs $(S, \mathfrak{q})$, where $S \leqslant K$ and $\mathfrak{q} \subset S$ is prime, ordered by domination is inductive. Indeed, let $(S_i, \mathfrak{q}_i)_i$ be a chain and let $S = \bigcup_i S_i \leqslant K$, a subring, and $\mathfrak{q} = \bigcup_i \mathfrak{q}_i = \bigcup_i S \cdot \mathfrak{q}_i \subset S$ an ideal. If, for some $a, b \in S$, $ab \in \mathfrak{q}$, let i be sufficiently large such that $a, b \in S_i$ and $ab \in \mathfrak{q}_i$. Then one of a or b is in $\mathfrak{q}_i \subseteq \mathfrak{q}$ and hence $\mathfrak{q} \subset S$ is prime. Moverover, for any i, by construction $\mathfrak{q}_i \subseteq \mathfrak{q} \cap S_i$ and if $a \in \mathfrak{q} \cap S_i$, then, for some $j \geqslant i$, $a \in \mathfrak{q}_j \cap S_i = \mathfrak{q}_i$, so $(S, \mathfrak{q})$ dominates $(S_i, \mathfrak{q}_i)$.

By Zorn's lemma, $(R, \mathfrak{p})$ is contained in a maximal element $(\mathcal{O}, \mathfrak{m})$. By proposition 1.1.5, $\mathcal{O}$ is a valuation ring and $\mathfrak{m}$ is its maximal ideal. □

Corollary 1.1.10. *Let (K, v) be a valued field and $K \leqslant L$ be a field extension. There exists a valuation w on L extending v.*

Proof. Applying corollary 1.1.9 to $(\mathcal{O}_v, \mathfrak{m}_v)$ in L, we find a valuation ring $\mathcal{O} \subseteq L = \mathcal{O}_{(0)}$, with maximal ideal $\mathfrak{m}$, that dominates $(\mathcal{O}_v, \mathfrak{m}_v)$. We now conclude with corollary 1.1.7. □

Lemma 1.1.11. *The ring $\mathcal{O}$ is integrally closed — that is, for every $P = X^d + \sum_{i<d} a_i x^i \in \mathcal{O}[x]$ and $c \in K = \mathcal{O}_{(0)}$, if $P(c) = 0$, then $c \in \mathcal{O}$.*

Proof. Assume $v(c) < 0$. Then, for every $i < d$, $v(c^d) = d \cdot v(c) < v(a_i) + iv(c)$ and hence $v(P(c)) = v(c^d + \sum_{i<d} a_i c^i) = d \cdot v(c) \neq \infty$. □

01/02

Theorem 1.1.12 (Weak approximation theorem). *Let K be a field and $(v_i)_{i<n}$ be valuations on K that are pairwise not dependent. Then, for every $a_i \in \mathcal{O}_{v_i}$, then exists $a \in K$ with $v_i(a - a_i) > 0$.*

04/02

Proof. Let $\mathcal{O}_i := \mathcal{O}_{v_i}$, $R := \bigcap_i \mathcal{O}_i$ and $\mathfrak{p}_i := \mathfrak{m}_{v_i} \cap R$.

Claim 1.1.12.1. $\mathcal{O}_i = R_{\mathfrak{p}_i}$.

Proof. We obviously have $R_{\mathfrak{p}_i} \subseteq \mathcal{O}_i$. There remains to show that $\mathcal{O}_i \subseteq R_{\mathfrak{p}_i}$. Fix $a \in \mathcal{O}_i$. Let $I := \{j : a \in \mathcal{O}_j\}$. For every $j \in I$, let $f_j \in \mathbb{Z}[x]$ be a monic polynomial with $f_j(a) \in \mathfrak{m}_j$ if it exists and $f_j = 1$ otherwise. Let also $f = 1 + x \prod_{j \in I} f_j$. If $j \in I$ and $f_j \neq 1$, we have $v_j(f(a)) = v_j(1) = 0$. If $f_j = 1$, by hypothesis, $f_j(a) \notin \mathfrak{m}_j$ and hence we also have $v_j(f(a)) = 0$. If $j \notin I$, then, since f is monic and $v_j(a) < 0$, $v_j(f(a)) = \deg(f) \cdot v_j(a)$. So $v_j(af(a)^{-1}) = (1 - \deg(f)) \cdot v_j(a) \geqslant 0$.

Let $c = f(a)^{-1}$. In both cases, we have $v_j(c) \geqslant 0$ and $v_j(ac) \geqslant 0$ and hence $c, ac \in R$. Also, $v_i(c) = 0$ and thus $c \notin \mathfrak{p}_i$. It follows that $a = ac/c \in R_{\mathfrak{p}_i}$. ◊

Claim 1.1.12.2. *The $\mathfrak{p}_i$ are the maximal ideals of R and they are distinct.*

Proof. Let $x \in R \setminus \bigcup_i \mathfrak{p}_i$. Then, $v_i(x) \leqslant 0$ for every i and thus $x^{-1} \in \bigcap_i \mathcal{O}_i = R$. So $R = (R \setminus \bigcup_i \mathfrak{p}_i)^{-1} R$ and any proper ideal $\mathfrak{a} \subseteq R$ is included in $\bigcup_i \mathfrak{p}_i$. Let I be minimal such that $\mathfrak{a} \subseteq \bigcup_{i \in I} \mathfrak{p}_i$. If $|I| > 1$, for every $i \in I$, by minimality, $\mathfrak{a} \cap \mathfrak{p}_i \setminus \bigcup_{i \neq j \in I} \mathfrak{p}_j$ contains some c_i. Pick any $i_0 \in I$ and let $a = c_{i_0} + \prod_{i \neq i_0} c_i$. Since $\mathfrak{p}_{i_0}$ is prime and non of the c_i, for $i \neq i_0$, are in $\mathfrak{p}_{i_0}$, it follows that $\prod_{i \neq i_0} c_i \notin \mathfrak{p}_{i_0}$. Since $c_{i_0} \in \mathfrak{p}_{i_0}$, it follows that $a \notin \mathfrak{p}_{i_0}$. However, for every $i \neq i_0$, $\prod_{i \neq i_0} c_i \in \mathfrak{p}_i$ and $c_{i_0} \notin \mathfrak{p}_i$ so $a \notin \mathfrak{p}_i$. This contradicts that $a \in \mathfrak{a} \subseteq \bigcup_{i \in I} \mathfrak{p}_i$. So $\mathfrak{a} \subseteq \mathfrak{p}_i$, for some i.

Note also that if $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, then $\mathcal{O}_j = R_{\mathfrak{p}_j} \subseteq R_{\mathfrak{p}_i} = \mathcal{O}_i$ and, by hypothesis, $i = j$. So the $\mathfrak{p}_i$ are indeed distinct and the maximal ideals of R. ◊

By the Chinese reminder theorem, the natural map $R \to \prod_i R/\mathfrak{p}_i \simeq \prod_i R_{\mathfrak{p}_i}/\mathfrak{p}_i R_{\mathfrak{p}_i} = \prod_i \mathcal{O}_i/\mathfrak{m}_i$ is surjective. It follows that we can find $a \in \bigcap_i a_i + \mathfrak{m}_i$. ☐

1.2. Topology

Definition 1.2.1. Fix $x \in K$ and $\gamma \in vK$. We define: 14/01

- the *closed ball* $\overline{\mathrm{B}}(x,\gamma) := \{y \in K : v(y-x) \geqslant \gamma\}$ of radius γ around x;
- the *open ball* $\mathring{\mathrm{B}}(x,\gamma) := \{y \in K : v(y-x) > \gamma\}$ of radius γ around x.

By convention, K is considered to be the open ball of radius $-\infty$. Note that points are closed balls of radius ∞.

Lemma 1.2.2. *Let b be a ball in K. Then $I_b := \{x - y : x, y \in b\} \subseteq K$ is a sub-$\mathcal{O}$-module and $b = x + I_b$ for any $x \in b$.*

Proof. Let us first assume that $b = \overline{\mathrm{B}}(c, v(d))$ with $c, d \in K$, then $\{x - c : x \in b\} = d\mathcal{O}$ is a sub-$\mathcal{O}$-module of K. Moreover, if $x, y \in b$, then $x - y = (x - c) - (y - c) \in d\mathcal{O}$ and hence $I_b \subseteq \{x - c : x \in b\} = d\mathcal{O} \subseteq I_b$. Similarly, if $b = \mathring{\mathrm{B}}(c, v(d))$, then $I_b = d\mathfrak{m}$ a sub-$\mathcal{O}$-module of K. By definition $b = c + I_b$ is an additive coset of I_b and hence $b = c + I_b = x + I_b$ for any $x \in b$. ☐

Lemma 1.2.3. *Let b_1, b_2 be balls of K, then at least one of the following holds:*

(i) $b_1 \cap b_2 = \varnothing$;
(ii) $b_1 \subseteq b_2$;
(iii) $b_2 \subseteq b_1$.

Proof. Note that we either have $I_{b_1} \subseteq I_{b_2}$ or $I_{b_2} \subseteq I_{b_1}$. So we may assume that $b_1 \cap b_2 \neq \varnothing$ and $I_{b_1} \subseteq I_{b_2}$. Let $x \in b_1 \cap b_2$, we then have $b_1 = x + I_{b_1} \subseteq x + I_{b_2} = b_2$. ☐

Lemma 1.2.4. *Open balls generate a totally disconnected*[5] *Hausdorff field topology.*

In other words, the ideals $\gamma\mathfrak{m} := \mathring{\mathrm{B}}(0,\gamma) \subseteq \mathcal{O}$, for $\gamma \in vK^{\times}_{\geqslant 0}$, form a basis of neighbourhoods of 0 and we consider the (unique) additive group topology generated by this basis of neighbourhoods of 0.

Proof. Let $U \subset K$ be open and $a, b \in K$ be such that $a + b \in U$. Then there exists $\gamma \in vK^{\times}$ such that $U \supseteq (a + b) + \gamma\mathfrak{m} = (a + \gamma\mathfrak{m}) + (b + \gamma\mathfrak{m})$, *i.e.* (a, b) is in the interior of $+^{-1}(U) \subseteq K^2$. So $+$ is continuous. Similarly, if $-a \in U$, for some $\gamma \in vK^{\times}$ we have $U \supseteq -a + \gamma\mathfrak{m} = -(a + \gamma\mathfrak{m})$, So $-$ is continuous. Finally, if $ab \in U$, there exists $\gamma \in vK^{\times}$, that we may assume larger than 0, with $U \supseteq ab + \gamma\mathfrak{m} \supseteq ab + a \cdot \delta\mathfrak{m} + b \cdot \delta\mathfrak{m} + \delta\mathfrak{m} \cdot \delta\mathfrak{m} \supseteq (a + \delta\mathfrak{m}) \cdot (b + \delta\mathfrak{m})$, provided $\gamma \leqslant \min\{\delta + v(a), \delta + v(b), \delta\}$.

Moreover, for every $a \in K^{\times}, \gamma \in vK^{\times}$ and $x \in (\max\{v(a), \gamma - 2 \cdot v(a)\})\mathfrak{m}$, by lemma 1.1.2.(2), we have $v(a + x) = v(a)$, and thus $v((a + x)^{-1} - a^{-1}) = v(xa^{-1}(a + x)^{-1}) = v(x) - 2 \cdot v(a) > \gamma$; so the inverse map is indeed continuous a a.

[5] A topology is totally disconnected if the only connected subsets are points. If it is Hausdorff, it is totally disconnected if and only if any two distinct points are separated by a clopen set.

The topology is Hausdorff since, for every $a, b \in K$ distinct, $\mathring{\mathrm{B}}(a, v(a-b)) \cap \mathring{\mathrm{B}}(b, v(a-b)) = \varnothing$. But since, for any $\gamma \in vK^\times$ and $c \notin \mathring{\mathrm{B}}(a,\gamma)$, $\mathring{\mathrm{B}}(a,\gamma) \cap \mathring{\mathrm{B}}(c,\gamma) = \varnothing$, any open ball is closed in the topology and hence the topology is totally disconnected. □

In fact, every non trivial ball is both open and closed in this topology and the topology is also generated by the non trivial closed balls.

04/02

Definition 1.2.5. Fix $\mathfrak{F}$ a (proper) filter[6] on K and $x \in K$.

(1) The filter $\mathfrak{F}$ is *Cauchy* if for every $\gamma \in vK$, there is an open ball of radius γ in $\mathfrak{F}$ — equivalently if there is a ball of radius γ in $\mathfrak{F}$.
(2) The filter $\mathfrak{F}$ *converges to* x, and we write $\lim \mathfrak{F} = x$, if for every $\gamma \in vK^\times$, $\mathring{\mathrm{B}}(x,\gamma) \in \mathfrak{F}$ — equivalently, if $\mathfrak{F} \supseteq \mathfrak{N}_x$, the neighbourhood filter of x.
(3) The field K is *complete* if every Cauchy filter on K converges to some $x \in K$.

11/01

Definition 1.2.6 (Leading terms)**.** Fix $\gamma \in vK^\times_{\geqslant 0}$. We define the multiplicative monoid of *γ-leading terms* $\mathrm{RV}_\gamma = \mathrm{RV}_{\gamma,v} := K/(1+\gamma\mathfrak{m})$. Let $\mathrm{rv}_\gamma = \mathrm{rv}_{\gamma,v} : K \to \mathrm{RV}_\gamma$ denote the canonical projection.

Remark 1.2.7.
1. It is naturally a multiplicative monoid and we have the following short exact sequence :
$$1 \to R_\gamma^\times \to \mathrm{RV}_\gamma^\times \to vK^\times \to 0,$$
where $R_\gamma = \mathcal{O}/\gamma\mathfrak{m}$ and $\mathrm{RV}_\gamma^\times := \mathrm{RV}_\gamma \setminus \{0\}$. We also denote v the natural map $\mathrm{RV}_\gamma \to vK$
2. There is also the trace of an additive structure on $\mathbf{RV}_\gamma$. We will describe it later.
3. We usually simply denote RV_0 as RV and rv_0 as rv.

04/02

Definition 1.2.8. Let
$$\widehat{K} = \widehat{K}_v := \varprojlim_{\gamma \in vK^\times_{\geqslant 0}} \mathrm{RV}_\gamma$$
as multiplicative monoids, where the transition maps $\mathrm{rv}_{\gamma,\delta} : \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$, for $\infty > \gamma > \delta \geqslant 0$, are the natural maps. We also define:

- $v : \widehat{K} \to vK$ by $v(x) = v(x_\gamma)$ for any $\gamma \in vK^\times_{\geqslant 0}$;
- $+ : \widehat{K}^2 \to \widehat{K}$ by $(x+y)_\gamma = \mathrm{rv}_\gamma(X_\varepsilon)$, where $X_\varepsilon := \mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)$ does not contain 0, for sufficiently large $\varepsilon \in vK^\times_{\geqslant 0}$. If $0 \in X_\varepsilon$ for all ε, then define $x + y = (0)_\gamma =: 0$.

Proof. Since $v = v \circ \mathrm{rv}_{\gamma,\delta}$, v is indeed well-defined on $\widehat{K}$. As for +, let us fix $x, y \in \widehat{K}$. Note that X_ε is the open ball of radius $\delta = \varepsilon + \min\{v(x), v(y)\}$ around any of its elements. If $0 \notin X_\varepsilon$, then $v(X_\varepsilon)$ is a singleton $\{\gamma\}$ and $\mathrm{rv}_{\delta-\gamma}(X_\varepsilon)$ is also a singleton. Since the X_ε form a chain, γ is independent of ε, whereas δ increases as ε increases. So $(x+y)_{\gamma-\delta}$ is well defined. □

Proposition 1.2.9. *The valued field* $(\widehat{K}, +, \cdot, v)$ *is complete.*

[6] That is, $\mathfrak{F} \subseteq \mathfrak{P}(K)$ such that:
(a) $K \in \mathfrak{F}$, $\varnothing \notin \mathfrak{F}$;
(b) for every $U, V \in \mathfrak{F}$, $U \cap V \in \mathfrak{F}$;
(c) for every $U \subseteq V \subseteq K$, if $U \in \mathfrak{F}$ then $V \subseteq K$;

Proof. The fact the $(\widehat{K}, +, \cdot)$ is a field follows more or less directly form the definitions. Let us check distributivity, for example. Let $x, y, z \in \widehat{K}$. Then $\mathrm{rv}_\varepsilon^{-1}(z_\varepsilon)(\mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)) = \mathrm{rv}_\varepsilon^{-1}(z_\varepsilon) \cdot \mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(z_\varepsilon) \cdot \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)$. Since these sets characterise both $z(x+y)$ and $zx + zy$, they must be equal.

As for completeness, for every $x \in \widehat{K}$, and $\gamma \in v\widehat{K} = vK$, $\mathring{\mathrm{B}}(x, \gamma) = \mathrm{rv}_{\gamma - v(x)}^{-1}(x_{\gamma - v(x)}) \subseteq \widehat{K}$. It follows that a Cauchy filter on $\widehat{K}$ is generated by sets of the form $\mathrm{rv}_\gamma^{-1}(\zeta_\gamma)$, for every $\gamma \in vK^\times_{\geqslant 0}$ and thus converges to $\zeta := (\zeta_\gamma)_\gamma \in \widehat{K}$. □

Remark 1.2.10. We have:
- $\iota : K \to \widehat{K}$ has dense image[7];
- $\mathcal{O}_{\widehat{K}} \simeq \varprojlim \mathcal{O}/\gamma\mathfrak{m}_K$ is the closure of $\mathcal{O}_K \subseteq \widehat{K}$;
- $\mathfrak{m}_{\widehat{K}} = \mathcal{O}_{\widehat{K}} \cdot \mathfrak{m}_K$ is the closure of $\mathfrak{m}_K \subseteq \widehat{K}$;
- $v\widehat{K} = vK$;
- $\widehat{K}v \simeq Kv$;
- $K/(1 + \mathfrak{m}_K) \simeq \widehat{K}/(1 + \mathfrak{m}_{\widehat{K}})$.

Proof. Exercise □

14/01

Definition 1.2.11. Fix $\mathfrak{F}$ a filter on K and $x \in K$.
(1) The filter $\mathfrak{F}$ is *pseudo Cauchy* if it is generated by balls.
(2) The filter $\mathfrak{F}$ accumulates at x, if any open ball around x meets any element of $\mathfrak{F}$ — equivalently, $x \in \bigcap_{U \in \mathfrak{F}} \overline{U} =: \overline{\mathfrak{F}}$.
(3) The field K is *spherically complete* if every pseudo Cauchy filter on K accumulates at some in $x \in K$.

Remark 1.2.12.
- Usually, the accumulation points of a pseudo Cauchy filter are called its pseudo limits. Since balls are closed, in that case, we have $\overline{\mathfrak{F}} := \bigcap_{u \in \mathfrak{F}} U$.
- The (potential) uniqueness of spherical completions is a much harder question that we'll come back to later.

Definition 1.2.13. Let $\mathfrak{F}$ be a filter on some set X and $f : X \to Y$. We denote by $f_\star\mathfrak{F}$ the filter generated by $\{f(U) : U \in \mathfrak{F}\}$ — that is, the filter $\{V \subseteq Y : f(U) \subseteq V$, for some $U \in \mathfrak{F}\}$.

Proof. Let $U, V \in \mathfrak{F}$. Note that, since $U \neq \varnothing$, $f(U) \neq \varnothing$. and since $f(U \cap V) \subseteq f(U) \cap f(V)$, the set $\{f(U) : U \in \mathfrak{F}\}$ has the finite intersection property and thus generates a filter. □

1.3. Examples

1.3.1. Adic valuations

07/01

Example 1.3.1. Fix K a field.

1. The map $-\deg : K(x) \to \mathbb{Z} \cup \{\infty\}$[8] is a valuation.
2. Define v_0 on $K(x)$ by $v_0(\sum_{i=0}^{n} a_i x^i) = \min\{i : a_i \neq 0\} \in \mathbb{Z} \cup \{\infty\}$ and $v_0(P/Q) = v_0(P) - v_0(Q)$. Then $v_0 : K(x) \to \mathbb{Z} \cup \infty$ is a valuation. Note that, if $f \in K(x)^\times$, $v_0(f)$ is the unique $n \in \mathbb{Z}$ such that $f = x^n P/Q$ where P, Q are prime to x.
   - We have $\mathcal{O}_{v_0} = \{f \in K(x) : v_0(f) \geqslant 0\} = \{P/Q : P, Q \in K[x] \text{ and } Q \notin (x)\} = K[x]_{(x)}$ and $\mathfrak{m}_{v_0} = \{f \in K(x) : v_0(f) \geqslant v(x)\} = x\mathcal{O}_{v_0}$. It follows that $Kv_0 = K[x]_{(x)}/x \simeq (K[x]/x)_{(0)} \simeq K$. The isomorphism is induced by the map: $K[x] \to K$ sending P to $P(0)$.
   - For every $f \in K(x)$, we have $-\deg(f) = v_0(f(x^{-1}))$. Indeed, let $P, Q \in K[X] \smallsetminus \{0\}$. Then $P(x^{-1}) = x^{-\deg(P)} P_1(x)$ where $P_1(x) = x^{\deg(P)} P(x^{-1}) \in K[X] \smallsetminus (X)$ and hence $v_0(P(x^{-1})/Q(x^{-1})) = v_0(x^{\deg(Q)-\deg(P)}) + v_0(P_1(x)Q_1(x)) = -\deg(P/Q)$. We say that $f \mapsto f(x^{-1})$ is a valued field isomorphism $(K(X), -\deg) \to (K(X), v_0)$
     It follows that $\mathcal{O}_{-\deg} = \{f \in K(x) : \deg(f) \leqslant 0\} = \{f(x^{-1}) : f \in \mathcal{O}_{v_0}\}$, $\mathfrak{m}_{-\deg} = \{f \in K(x) : \deg(f) < 0\} = \{f(x^{-1}) : f \in \mathfrak{m}_{v_0}\}$, and $K[x](-\deg) \simeq Kv_0 \simeq K$ where the isomorphisme is induced by the map $\mathcal{O}_{-\deg} \to K$ given by $\sum_{i=0}^{n} a_i x^i / \sum_{i=0}^{n} b_i x^i \mapsto a_n/b_n$ where $b_n \neq 0$. In a (very precise) sense, this is the map $f \mapsto f(\infty)$.

Definition 1.3.2. Let R be an integral domain and $p \in R$ be prime. For every $x \in R$, define the *p-adic valuation* $v_p(x) = \max\{n \in \mathbb{Z}_{\geqslant 0} : x \in (p)^n\} \in \mathbb{Z} \cup \{\infty\}$.

In particular, it induces a valuation $v_p : K_p := (R/\bigcap_n (p)^n)_{(0)} \to \mathbb{Z} \cup \{\infty\}$.

Proof. We have $1 \in (p)^n$ if and only if $n = 0$ — by convention $p^0 = 1$. So $v_p(1) = 0$. And we have $0 \in \bigcap_{n \geqslant 0}$, so $v_p(0) = \infty \neq 0$. Let $x, y \in R$. If $x, y \in (p)^n$ — i.e. $v_p(x), v_p(y) \geqslant n$ — for some $n \in \mathbb{Z}_{\geqslant 0}$, then $x + y \in (p)^n$ and thus $v_p(x+y) \geqslant n$. Taking $n = \min\{v_p(x), v_p(y)\}$, we see that $v_p(x+y) \geqslant \min\{v_p(x), v_p(y)\}$, as required. Finally, if $x \in (p)^m$ and $y \in (p)^n$, for some $m, n \in \mathbb{Z}_{\geqslant 0}$, then $xy \in (p)^{n+m}$. If $xy \in (p)^{n+m+1}$, let $x_0, y_0, z_0 \in R$ be such that $x = p^m x_0$, $y = p^n y_0$ and $p^{n+m} x_0 y_0 = xy = p^{n+m+1} z_0$. It follows that $x_0 y_0 \in (p)$ and thus $x_0 \in (p)$, in which case $x \in (p)^{n+1}$, or $y_0 \in (p)$, in which case $y \in (p)^{m+1}$. It follows that $v_p(xy) = v_p(x) + v_p(y)$. So v_p is a valuation on R.

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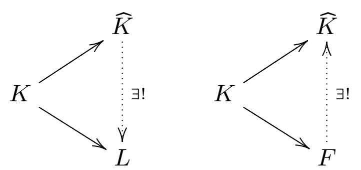
[7] In fact, $\widehat{K}$ is the unique, up to unique K-isomorphism, complete dense valued field extension of K. It also has the following universal properties: for every $f : K \to L$ with L complete and $g : K \to F$ with dense image, we have:



[8] By convention, $\deg(0) = -\deg(0) = \infty$.

Note that if $v(x) = \infty$, for any $y \in R$, we have $v_p(y + x) = v_p(y)$, so v_p factorises through the quotient by $\mathfrak{p} = v_p^{-1}(\infty) = \bigcap_n (p)^n$. Moreover, for any $x, y \in R$, if $v(x) < \infty$ or $v(y) < \infty$, then $v(xy) = v(x) + v(y) < \infty$. Thus $\mathfrak{p} \subseteq R$ is prime and $R/\mathfrak{p}$ is an integral domain. For every $\overline{x} \in R/\mathfrak{p}$ and non zero $\overline{y} \in R/\mathfrak{p}$, we define $v(\overline{x}/\overline{y}) = v(x) - v(y) \in \mathbb{Z} \cup \{\infty\}$.

We have $v_p(\overline{0}) = \infty \neq 0 = v_p(\overline{1})$ and for every $\overline{x}, \overline{y}, \overline{r}, \overline{s} \in R/\mathfrak{p}$, with $\overline{y}, \overline{s}$ non zero, we have $v_p(\overline{x}/\overline{y} + \overline{r}/\overline{s}) = v_p((\overline{xs} + \overline{ry})/(\overline{ys})) = v_p(xs + ry) - v_p(y) - v_p(s) \geqslant \min\{v_p(x) + v_p(s), v_p(r) + v_p(y)\} - v_p(y) - v_p(s) = \min\{v_p(x) - v_p(y), v_p(r) - v_p(s)\} = \min\{v_p(\overline{x}/\overline{y}), v_p(\overline{r}/\overline{s})\}$ and $v_p(\overline{x}/\overline{y} \cdot \overline{r}/\overline{s}) = v_p((\overline{xr})/(\overline{ys})) = v_p(xr) - v_p(ys) = v_p(x) - v_p(y) + v_p(r) - v_p(s) = v_p(\overline{x}/\overline{y}) + v_p(\overline{r}/\overline{s})$. □

Example 1.3.3.
1. $v_x : K[x] \to \mathbb{Z} \cup \{\infty\}$ induces $v_0 : K(X) \to \mathbb{Z} \cup \{\infty\}$.
2. For every, prime $p \in \mathbb{Z}$ and $x \in \mathbb{Q}^\times$, $v_p(x)$ is the unique $n \in \mathbb{Z}$ such that $x = p^n y/z$ with $y, z \in \mathbb{Z}_{\neq 0}$ prime to p.

04/02

Lemma 1.3.4. *Fix R an integral domain and $p \in R$ be prime. Then $\mathcal{O}_{\widehat{K}_p} \simeq \varprojlim_n R/(p)^n$.*

Proof. Exercise □

Example 1.3.5.
1. $\widehat{K}_{v_0} \simeq K((x)) := \{\sum_{i \geqslant i_0} a_i x^i : a_i \in K\}$ — the Laurent series field. We have:
    - $v(\sum_{i \geqslant i_0} a_i x^i) = \min\{i : a_i \neq 0\}$, $\sum_{i \geqslant i_0} a_i x^i + \sum_{i \geqslant i_0} c_i x^i = \sum_{i \geqslant i_0} (a_i + c_i) x^i$ and $(\sum_{i \geqslant i_0} a_i x^i) \cdot (\sum_{j \geqslant j_0} c_j x^j) = \sum_{k \geqslant i_0 + j_0} (\sum_{i+j=k} a_i c_j) x^k$.
    - $\mathcal{O}_{\widehat{K}_{v_0}} \simeq K[[x]] := \{\sum_{i \geqslant 0} a_i x^i : a_i \in K\}$ — the power series ring.
2. $\mathbb{Q}_p := \widehat{\mathbb{Q}}_{v_p}$ and $\mathbb{Z}_p := \mathcal{O}_{\mathbb{Q}_p} = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

1.3.2. Hahn fields

07/01

Fix k a field and Γ an ordered abelian group.

Definition 1.3.6 (Hahn Fields). We define $k((\Gamma)) := \{f : \Gamma \to k : \mathrm{supp}(f) := \{\gamma \in \Gamma : f(\gamma) \neq 0\}$ is well ordered$\}$. We also define, for every $f, g \in k((\Gamma))$ and $\gamma \in \Gamma$:
- $(f + g)(\gamma) := f(\gamma) + g(\gamma)$;
- $(f \cdot g)(\gamma) := \sum_{\varepsilon + \delta = \gamma} f(\varepsilon) g(\delta)$;
- $v(f) := \min\{\gamma \in \Gamma : f(\gamma) \neq 0\} \in \Gamma \sqcup \{\infty\}$.

Proof. The sum in the definition of $\cdot$ is finite. If it is infinite, we can find an increasing sequence in the set $\{\varepsilon : f(\varepsilon) \neq 0$ and $g(\gamma - \varepsilon) \neq 0\}$. But then the sequence of $\gamma - \varepsilon$ is then decreasing, contradicting the fact that $\mathrm{supp}(g)$ is well-ordered. Also $\mathrm{supp}(f + g) \subseteq \mathrm{supp}(f) \cup \mathrm{supp}(g)$ is well ordered.

There remains to show that any $X \subseteq \mathrm{supp}(f \cdot g)$ has a minimum. Let $Y := \{\gamma \in \mathrm{supp}(f) :$ for all $\gamma' \in \mathrm{supp}(f)$ and $\delta' \in \mathrm{supp}(g)$, $\gamma' + \delta' \in X$ and $\gamma' + \delta' \leqslant (\gamma + \mathrm{supp}(g)) \cap X \neq \varnothing$ implies $\gamma' \geqslant \gamma\}$. Note that the minimal γ such that $\gamma + \mathrm{supp}(g) \cap Z \neq \varnothing$ is in Y and that for every $\gamma \in Y$, there exists $\delta \in \mathrm{supp}(g)$ with $\gamma + \delta \in X$. Let δ_0 be minimal such that $Y + \delta \cap X \neq \varnothing$ and $\gamma_0 \in Y$ be such that $\gamma_0 + \delta_0 \in X$. If $X_{<\gamma_0 + \delta_0} \neq \varnothing$, let γ_1 be minimal such that $\gamma + \mathrm{supp}(g) \cap X_{<\gamma_0 + \delta_0} \neq \varnothing$ and

$\delta_1 \in \operatorname{supp}(g)$ be minimal such that $\gamma_1+\delta \in X_{<\gamma_0+\delta_0}$. Note that, since $\gamma_0 \in Y$ and $\gamma_1+\delta_1 < \gamma_0+\delta_0$, we have $\gamma_0 \leqslant \gamma_1$. Also, if $\gamma' \in \operatorname{supp}(f)$ and $\delta' \in \operatorname{supp}(g)$ are such that $\gamma'+\delta' \leqslant \gamma_1+\delta_1 < \gamma_0+\delta_0$, then by minimality of γ_1, $\gamma_1 \leqslant \gamma'$, *i.e.* $\gamma_1 \in Y$. By minimality of δ_0, we have $\delta_0 \leqslant \delta_1$ and hence $\gamma_0+\delta_0 \leqslant \gamma_1+\delta_1 < \gamma_0+\delta_0$, a contradiction. So $X_{<\gamma_0+\delta_0} = \varnothing$ and $\gamma_0+\delta_0$ is minimal in X. □

We usually write elements of $k((\Gamma))$ as formal power series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$.

04/02

Proposition 1.3.7. *The valued field* $(k((\Gamma)),+,\cdot,v)$ *is spherically complete.*

Proof. One can easily compute that $k((\Gamma))$ is a ring with $0(\gamma) = 0$, $1(0) = 1$, $1(\gamma) = 0$ and $(-f)(\gamma) = -f(\gamma)$. We do have $v(1) = 0$, $v(0) = \infty$. Since $\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$, we have $v(f+g) \geqslant \min\{v(f), v(g)\}$. Finally, if $\gamma < v(f)$, $\delta < v(g)$, then $\sum_{\gamma'+\delta'=\gamma+\delta} f(\gamma)g(\delta) = 0$ and thus $v(f\cdot g) = v(f)+v(g)$. So $(k((\Gamma)),+,\cdot,v)$ is a valued ring.

Let us now show that it is spherically complete. Note that, for every $f \in k((\Gamma))$ and $\gamma \in \Gamma$, $\mathring{\mathrm{B}}(f,\gamma) := \{g \in k((\Gamma)) : \text{for every } \delta \leqslant \gamma, g(\delta) = f(\delta)\}$. Let $\mathfrak{B}$ be a pseudo Cauchy filter. If $\mathfrak{B}$ is principal, then $\overline{\mathfrak{B}} \neq \varnothing$. So we may assume that it is not principal and therefore generated by open balls. For every $\gamma \in \Gamma$, let $f(\gamma) = h(\gamma)$, where $\mathring{\mathrm{B}}(h,\gamma) \in \mathfrak{B}$ and $f(\gamma) = 0$ if no such ball exists. Let $I \subseteq \operatorname{supp}(f)$ be non empty and pick some $\gamma \in I$. Then there is some $\mathring{\mathrm{B}}(h,\gamma) \in \mathfrak{B}$ and $\operatorname{supp}(f) \cap (-\infty,\gamma] = \operatorname{supp}(h) \cap (-\infty,\gamma]$. In particular, $I \cap (-\infty,\gamma] \subseteq \operatorname{supp}(h)$ has a minimal element. So $f \in k((\Gamma))$ and by construction $f \in \overline{\mathfrak{B}}$.

There remains to show that $k((\Gamma))$ is a field. Fix any $x \in k((\Gamma)) \smallsetminus 0$ and, for every $\gamma \in \Gamma$, let $b_\gamma = \{y \in k((\Gamma)) : v(xy-1) > \gamma + v(x)\}$. For every $y \in b_\gamma$ and $e \in k((\Gamma))$, $v(x(y+e)-1) > \gamma + v(x)$ if and only if $v(e) > \gamma$. So either b_γ is an open ball of radius γ or it is empty. Let $\mathfrak{B}$ be the filter generated by the non empty b_γ. By spherical completeness, we find $y \in \overline{\mathfrak{B}}$. If $v(xy-1) = \varepsilon + v(x) < \infty$, let $z(\varepsilon) = x(v(x))^{-1}(\mathbb{1}_{\varepsilon+v(x)=0} - \sum_{\gamma+\delta=\varepsilon+v(x),\delta<\varepsilon} x(\gamma)y(\delta))$ and $z(\gamma) = y(\gamma)$ otherwise. Then $v(xz-1) > \varepsilon + v(x)$, so $b_\varepsilon \neq \varnothing$. However, $v(y-z) = \varepsilon$, contradicting that $y \in b_\varepsilon$. It follows that $xy-1 = 0$, *i.e.* $y = x^{-1}$. □

07/01

Remark 1.3.8. We have:

- $vk((\Gamma)) = \Gamma \cup \{\infty\}$;
- $k((\Gamma))v \simeq k$. The isomorphism is induced by the map $x \mapsto x(0)$.

01/03

1.3.3. Witt vectors

We now wish to build mixed characterstic valued fields with prescribed (perfect) residue field. Fix p a prime.

Definition 1.3.9 (Witt polynomials). For every $n \in \mathbb{Z}_{\geqslant 0}$, let $w_{p^n}(x) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} \in \mathbb{Z}[x]$ and $w(x) = (w_{p^n}(x))_{n\geqslant 0}$.

Note that $w_{p^{n+1}}(x) = w_{p^n}(x^p) + p^{n+1}x_{n+1}$.

Lemma 1.3.10. *Let* $P(y) \in \mathbb{Z}[y]$ *where* y *is a tuple. There exists unique* $P_n \in \mathbb{Z}[z_0,\ldots,z_n]$, *where* $|z_i| = |y|$ *such that for every* $n \in \mathbb{Z}_{\geqslant 0}$, $w_{p^n}((P_i(z))_i) = P(w_{p^n}(z))$.

In other terms, $w((P_i(z))_i) = (P(w_{p^i}(z)))_i$.

Proof. Note that $w_{p^n}((P_i(z))_i) = p^n P_n(z) + w_{n-1}((P_i(z)^p)_i)$. It follows that $P_0 = P$ and that, by induction on n, there is a unique $P_n \in \mathbb{Q}[z]$ with the required properties. There remains to show that $P_n \in \mathbb{Z}[z]$. We also proceed by induction on n.

Claim 1.3.10.1. *Let A be some ring, $\mathfrak{a} \subseteq A$ contain p, $a, b \in A$ and $n \in \mathbb{Z}_{>0}$ such that $a \equiv b \mod \mathfrak{a}^n$, then $a^p \equiv b^p \mod \mathfrak{a}^{n+1}$.*

Proof. We have $(a+c)^p = a^p + \sum_{i=1}^{p-1} \binom{p}{i} a^{n-i} c^i + c^p$. For every $0 < i < p$, we have $\binom{p}{i} \in \mathfrak{a}$ and $n+1 \leqslant np$. So if $c \in \mathfrak{a}^n$, we have $(a+c)^p - a^p \equiv 0 \mod \mathfrak{a}^{n+1}$. ◊

In particular, since, for all $i < n$, $P_i(z^p) \equiv P_i(z)^p \mod p$, we have $P_i(z^p)^{p^{n-i-1}} \equiv P_i(z)^{p^{n-i}} \mod p^{n-i}$ and hence: 03/03

$$\begin{aligned}
p^n P_n(z) &= w_{p^n}((P_i(z))_i) - w_{p^{n-1}}((P_i(z)^p)_i) \\
&= P(w_{p^n}(z)) - w_{p^{n-1}}((P_i(z)^p)_i) \\
&\equiv P(w_{p^{n-1}}(z^p)) - \sum_{i<n-1} p^i P_i(z)^{p^{n-i}} && \mod p^n \\
&\equiv w_{p^{n-1}}((P_i(z^p))_i) - \sum_{i<n-1} p^i P_i(z^p)^{p^{n-1-i}} && \mod p^n \\
&= 0
\end{aligned}$$

It follows that $P_n(z) \in \mathbb{Z}[x]$. □

Let $S_n, P_n \in \mathbb{Z}[x,y]$ be the unique polynomials such that $w_{p^n}(S(x,y)) = w(x) + w(y)$ and $w(P(x,y)) = w(x) \cdot w(y)$.

Definition 1.3.11 (Witt vectors). For $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, , we define the functors $\mathrm{W}_{p^n} : \mathfrak{Ring} \to \mathfrak{Ring}$ of *length n Witt vectors*, by $\mathrm{W}_{p^n}(A) := (A^n, (S_i)_{i<n}, (P_i)_{i<n})$ and $\mathrm{W}_{p^n}(f) : \mathrm{W}_{p^n}(A) \to \mathrm{W}_{p^n}(B) := a \mapsto (f(a_i))_{i<n}$, for every ring morphism $f : A \to B$.

Furthermore, we have natural morphisms $g_{p^n} : \mathrm{W}_{p^n}(A) \to A^n := a \mapsto (w_{p^i}(a))_{i<n}$ and $\mathrm{res}_{p^n,p^m} : \mathrm{W}_{p^m}(A) \to \mathrm{W}_{p^n}(A) := a \mapsto (a_i)_{i<n}$, for every $n \leqslant m \in \mathbb{Z}_{>0} \cup \{\infty\}$.

The g_{p^n} are usually called the ghost component maps. We will often write W for W_{p^∞}.

Proof. Let $0 := (0)_{i<n} \in \mathrm{W}_{p^n}(A)$, $1 := (\mathbb{1}_{i=0})_{i<n}$ and $M_i \in \mathbb{Z}[x]$ be such that $w_{p^n}(M(x)) = -w_{p^n}(x)$. We can now check that all the required equality for $\mathrm{W}_{p^n}(A)$ to be a ring hold us-

ing lemma 1.3.10[9]. Now, if $f : A \to B$ is a ring morphism. then for any $a, b \in \mathrm{W}_{p^n}(A)$, $\mathrm{W}_{p^n}(f)(a+b) = \mathrm{W}_{p^n}(f)((S_i(a,b))_i) = (f(S_i(a,b)))_i = (S_i(f(a), f(b)))_i = \mathrm{W}_{p^n}(f)(a) + \mathrm{W}_{p^n}(f)(b)$, and similarly for multiplication. So $\mathrm{W}_{p^n}(f)$ is a ring morphism. The fact that the g_{p^n} and the res_{p^n,p^m} are morphism is an immediate consequence of their definitions. □

Remark 1.3.12. It seems as if, to compute in the Witt vectors, it suffices to compute ghost component equalities for polynomials over $\mathbb{Z}$. And indeed, the g_n, being bijective over $\mathbb{Q}[x]$, are injective on $\mathbb{Z}[x]$, where x is an arbitrary tuple. Since, for any ring A generated by a tuple a, there is a natural surjection $\mathbb{Z}[a] \to A$, ghost component equalities translate to actual equalities in $\mathrm{W}_{p^n}(\mathbb{Z}[x])$ which are transported functorially to any $\mathrm{W}_{p^n}(A)$.

For example, for every $a \in A$, let $[a] = (a \cdot \mathbb{1}_{i=0})_{i<n} \in \mathrm{W}_{p^n}(A)$. In $\mathbb{Z}[x,y]$, we have $w_{p^n}(P([x],[y])) = w_{p^n}([x]) \cdot w_{p^n}([y]) = x^{p^n} \cdot y^{p^n} = w_{p^n}([x \cdot y])$. It follows that $[x]\cdot[y] = [x \cdot y]$ in $\mathrm{W}_{p^n}(\mathbb{Z}[x,y])$ and, since, for any $f : \mathbb{Z}[x,y] \to A$, $\mathrm{W}_{p^n}(f)([x]) = [f(x)]$, the equality also holds over any ring.

Definition 1.3.13. A *p-ring* is a ring A with a choice of ideal $\mathfrak{a} \leqslant A$ such that its residue ring $A/\mathfrak{a}$ is characteristic p, $\phi_p : A/\mathfrak{a} \to A/\mathfrak{a} := x \mapsto x^p$ is bijective and A is Hausdorff complete in its $\mathfrak{a}$-adic topology — *i.e.* $A \simeq \varprojlim_n A/\mathfrak{a}^n$.

We say that $(A, \mathfrak{a})$ is unramified if $\mathfrak{a} = (p)$.

Example 1.3.14. $\mathbb{Z}_p$ is an unramified p-ring.

For every ring R, let $\mathfrak{m}_n(R) \subseteq \mathrm{W}_{p^n}(R)$ be the kernel of $\mathrm{res}_{0,n} : \mathrm{W}_{p^n}(R) \to \mathrm{W}_1(R) \simeq R$.

Lemma 1.3.15. *If R is a characteristic p ring with $\phi_p : R \to R$ bijective, then $(\mathrm{W}_{p^n}(R), \mathfrak{m}_n(R))$ is an unramified p-ring with residue ring R.*

Proof. The only non-obvious statement is that $\mathfrak{m}_n(R)^i = (p^i)$.

Claim 1.3.15.1. *For all ring R and $x \in \mathrm{W}(R)$, $(p \cdot x)_0 \equiv 0 \mod p$ and for every n, $(p \cdot x)_{n+1} \equiv x_n^p \mod p$.*

Proof. Let $y = p \cdot x$, $z_i = \mathbb{1}_{i>0} x_{i-1}^p$. We have $w_1(y) = p \cdot w_1(x) \equiv 0 = z_0 = w_1(z) \mod p$ and for all $n > 0$, $w_{p^n}(z) = \sum_{i<n} p_{i+1} x_i^{p^{1+n-i-1}} = p \cdot w_{p^{n-1}}(x^p) \equiv p \cdot w_{p^n}(x) = w_{p^n}(y) \mod p^{n+1}$. It follows, by induction on n, that, if $A = \mathbb{Z}[x]$, we have $y_n = z_n \mod p$. We conclude by functoriality. ◊

[9] In $\mathbb{Z}[x,y,z]$:

$$
\begin{aligned}
w_{p^n}(S(S(x,y),z)) &= w_{p^n}(S(x,y)) + w_{p^n}(z) = w_{p^n}(x) + w_{p^n}(y) + w_{p^n}(z)\\
w_{p^n}(S(x,S(y,z))) &= w_{p^n}(x) + w_{p^n}(S(y,z)) = w_{p^n}(x) + w_{p^n}(y) + w_{p^n}(z)\\
w_{p^n}(S(x,0)) &= w_{p^n}(x) + w_{p^n}(0) = w_{p^n}(x) + 0 = w_{p^n}(x)\\
w_{p^n}(S(x,M(x))) &= w_{p^n}(x) + w_{p^n}(M(x)) = w_{p^n}(x) - w_{p^n}(x) = 0\\
w_{p^n}(P(P(x,y),z)) &= w_{p^n}(P(x,y)) \cdot w_{p^n}(z) = w_{p^n}(x) \cdot w_{p^n}(y) \cdot w_{p^n}(z)\\
w_{p^n}(P(x,P(y,z))) &= w_{p^n}(x) \cdot w_{p^n}(P(y,z)) = w_{p^n}(x) \cdot w_{p^n}(y) \cdot w_{p^n}(z)\\
w_{p^n}(P(x,y)) &= w_{p^n}(x) \cdot w_{p^n}(y)\\
w_{p^n}(P(y,x)) &= w_{p^n}(y) \cdot w_{p^n}(x) = w_{p^n}(x) \cdot w_{p^n}(y)\\
w_{p^n}(P(x,1)) &= w_{p^n}(x) \cdot w_{p^n}(1) = w_{p^n}(x) \cdot 1 = w_{p^n}(x)\\
w_{p^n}(P(S(x,y),z)) &= w_{p^n}(S(x,y)) \cdot w_{p^n}(z) = (w_{p^n}(x) + w_{p^n}(y)) \cdot w_{p^n}(z)\\
w_{p^n}(S(P(x,z),P(y,z))) &= w_{p^n}(P(x,z)) + w_{p^n}(P(y,z)) = w_{p^n}(x) \cdot w_{p^n}(z) + w_{p^n}(y) \cdot w_{p^n}(z)
\end{aligned}
$$

If R is characteristic p, it follows that, for all $a \in \mathrm{W}_{p^n}(a)$, $p \cdot a = (0, a_0^p, \ldots, a_i^p, \ldots)$. Since ϕ_p is surjective on R, it follows that any element in $\mathfrak{m}_n(R)^i$ is a multiple of p^i and since $p = (\mathbb{1}_{j=1})_j \in \mathfrak{m}_n(R)$, that $\mathfrak{m}_n(R)^i = (p^i)$. □

Lemma 1.3.16. *Let $(A, \mathfrak{a})$ be a p-ring.*

1. *There is a unique multiplicative section $[.] : A/\mathfrak{a} \to A$ of the projection $A \to A/\mathfrak{a}$.*
2. *For every $a \in A$, we have $a \in [A]$ if and only if $a \in c = \bigcap_n A^{p^n}$.*

Proof. Fix some $\alpha \in A/\mathfrak{a}$. For every n, let $U_n := \{x^{p^n} : x/\mathfrak{a} = \phi_p^{-n}(\alpha)\}$. Note that, for every $m \geqslant n$, $U_m \subseteq U_n$. In particular, $U_m/\mathfrak{a} = U_0/\mathfrak{a} = \alpha$ and that. By claim 1.3.10.1, the U_n forms a Cauchy filter for the $\mathfrak{a}$-adic topology and let $[\alpha] = \lim_n U_n$. This defines a section as sets. Note that, $[\alpha] \in U_n \subseteq A^{p^n}$ and hence $[\alpha] \in A^{p^\infty}$. Conversely, if $a \in A^{p^\infty}$ has residue α, let $a_n \in A$ be such that $a_n^{p^n} = a$. Then $\phi_p^n(a_n/\mathfrak{a}) = \alpha$ and hence $a \in U_n$. So $a = [\alpha]$.

Now, for every $\alpha, \beta \in A/\mathfrak{a}$, $[\alpha]\cdot[\beta] \in A^{p^\infty}$ and hence $[\alpha\cdot\beta] = [\alpha]\cdot[\beta]$. Finally, if $f : A/\mathfrak{a} \to A$ is another multiplicative section, then, for every $\alpha \in A$, $f(\alpha) = f(\phi_p^{-n}(\alpha)^{p^n}) = f(\phi_p^{-n}(\alpha))^{p^n}$ and hence $f(\alpha) \in A^{p^\infty}$. So $f(A) = [A]$ and $f = [.]$. □

Proposition 1.3.17. *Let $(A, \mathfrak{a})$ be a p-ring and $R = A/\mathfrak{a}$. For every $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, there is a unique ring morphism $f_n : \mathrm{W}_{p^n}(R) \to A/\mathfrak{a}^n$, where $\mathfrak{a}^\infty = (0)$, such that*

$$\begin{array}{ccc} \mathrm{W}_{p^n}(R) & \overset{f_n}{\dashrightarrow} & A/\mathfrak{a}^n \\ & \searrow \quad \swarrow & \\ & R & \end{array}$$

commutes. Moreover:

1. *f_n is surjective if and only if $(A, \mathfrak{a})$ is unramified;*
2. *f_n is bijective if and only if it is surjective and for all $i \in \mathbb{Z}_{>0}$, and $c \in A$, $p^i c \in \mathfrak{a}^n$ implies $c \in \mathfrak{a}$* — i.e. *when $n = \infty$, p is not a zero divisor.*

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Proof. Let us first assume that $n < \infty$. Let $\pi_n : A \to A/\mathfrak{a}^n$ be the natural projection. The map $\pi_n \circ w_{p^n} : \mathrm{W}_{p^n}(A) \to A/\mathfrak{a}^n$ factorises through $\mathrm{W}_{p^n}(\pi_1) : \mathrm{W}_{p^n}(A) \to \mathrm{W}_{p^n}(R)$. Indeed, since $p \in \mathfrak{a}$, $w_n(\mathfrak{a}) \subseteq \mathfrak{a}^n$. Let $h_n : \mathrm{W}_{p^n}(R) \to A/\mathfrak{a}^n$ be such that $h_n \circ \mathrm{W}_{p^n}(\pi_1) = \pi_n \circ w_n$ and $f_n = h_n \circ \mathrm{W}_{p^n}(\phi_p^{-n})$. Note that, $\mathrm{W}_{p^n}(\pi_1) \circ \mathrm{W}_{p^n}([.]) = \mathrm{W}_{p^n}(\mathrm{id}) = \mathrm{id}$ and hence, $h_n = \pi_n \circ w_n \circ \mathrm{W}_{p^n}([.])$. So, for every $x \in \mathrm{W}_{p^n}(R)$, $f_n(x) = h_n(\mathrm{W}_{p^n}(\phi_p^{-n})(x)) = h_n((x_i^{p^{-n}})_i) = w_n(([(x_i^{p^{-n}})])_i) = \sum_i p^i [x_i^{p^{-n}}]^{p^{n-i}} = \sum_i [x_i^{p^{-i}}] p^i$. In particular, $f_n(x)/\mathfrak{a} = [x_0]/\mathfrak{a} = x_0$. Note also that the f_n form a projective system and allow us to define $f_\infty : \mathrm{W}_{p^\infty}(R) = \varprojlim_n \mathrm{W}_{p^n}(R) \to \varprojlim_n A/\mathfrak{a}_n \simeq A$, which, by construction commutes with reduction to $R = \mathrm{W}_1(R)$.

If f_n is surjective, then any $a \in \mathfrak{a}$ is of the form $f_n(x)$ with $x_0 = a/\mathfrak{a} = 0$. So $a = \sum_{i>0} [x_i^{p^{-i}}] p^i \in (p)$. Conversely, if $\mathfrak{a} = (p)$, then, by induction on i any element of $A/\mathfrak{a}^i$ is of the form $\sum_i [x_i^{p^{-i}}] p^i$ and hence f is surjective. Also, since $\mathrm{W}_{p^n}(R)$ is itself unramified, any element of $\mathrm{W}_{p^n}(R)$ is of the form $\sum_i [x_i^{p^{-i}}] p^i$. By lemma 1.3.16, if $f_n' : \mathrm{W}_{p^n}(R) \to A/\mathfrak{a}^n$ makes the above diagram commute, for any $x \in R$, $f_n'([x]) = [x]$ and hence, if $n < \infty$, $f_n'(\sum_i [x_i^{p^{-i}}] p^i) = \sum_i [x_i^{p^{-i}}] p^i = f_n(x)$. If $n = \infty$, then $\pi_n \circ f_\infty' = f_n$ by uniqueness and hence $f_\infty' = \varprojlim_n f_n = f_\infty$.

If f_n is bijective, then for every $c \in A$ with $p^i c = 0$, we have $c = f_n(x)$ and $f_n(0, \dots, 0, x_0^{p^i}, \dots) = p^i c = 0$ and hence $x_0^{p^i} = 0 = x_0$, so $c = f_n(x) \in (p) = \mathfrak{a}$. Conversely, for every $c \in \mathrm{W}_{p^n}(R)$, with $c_j = 0$ for all $j < i < n$, if $0 = f_n(c) = p^i(\sum_{j \geqslant i} [c_j^{p^{-j}}] p^{j-i})$, then $\sum_{j \geqslant i}[c_j^{p^{-j}}]p^{j-i} \in \mathfrak{a}$. So $c_i^{p^{-i}} = 0$ and hence $c_i = 0$. It follows that $c = 0$. □

Corollary 1.3.18. *Let R be a characteristic p ring with bijective φ_p. Then $(\mathrm{W}_{p^n}(R), \mathfrak{m}_n(R))$ is the unique, up to unique isomorphism, unramified p-ring where p is not a zero divisor up to power n — that is, for every $i < n$ and $c \in \mathrm{W}_{p^n}(R)$ if $p^i c = 0$ then, $c \in (p)$.*

Proposition 1.3.19. *Let k be a characteristic p perfect field. Then $\mathrm{W}(k)$ is a complete valuation ring with associated valuation $v : \mathrm{W}(k) \to \mathbb{Z} \cup \{\infty\}$ defined by $v(a) = \min\{i : a_i \neq 0\}$.*

Proof. Let us show that v is a valuation. We have $v(0) = \infty \neq 0 = v(1)$. For every $x, y \in \mathrm{W}(k)$, let $n = \min\{v(x), v(y)\}$. Then $\mathrm{res}_{p^{n-1}}(x + y) = \mathrm{res}_{p^{n-1}}(x) + \mathrm{res}_{p^{n-1}}(y) = 0$ and hence $v(x + y) \geqslant n$. Also, et $x = p^{v(x)}s$ and $y = p^{v(y)}t$. Since $s, t \notin (p)$, we have $s_0, t_0 \neq 0$ and hence $(st)_0 = s_0 t_0 \neq 0$. Since, by claim 1.3.15.1, $p^i \cdot u = (\mathbb{1}_{j \geqslant i} u_{j-i}^{p^i})_j$, it follows that $v(xy) = v(x) + v(y)$.

Completeness follows the fact that the valuation induced by v is exactly the p-adic valuation. It then follows that any element in $1 + (p)$ is invertible and hence, since k is a field, so is any element of valuation 0. So $\mathrm{W}(k)$ is the valuation ring associated to v. □

Corollary 1.3.20. *Let k be a characteristic p perfect field. Then $\mathrm{W}(k)_{(0)}$ is the unique (up to unique isomorphism) complete unramified charactersitic zero valued field with residue field k and value group $\mathbb{Z}$.*

2. Algebraically closed valued fields

Definition 2.0.1. Let $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$ be the three sorted language with: 11/01

- a sort $\mathbf{K}$ with the ring language $(+, -, 0, \cdot, 1)$;
- a sort $\mathbf{\Gamma}$ with the ordered group language $(+, -, 0, <)$ and a constant ∞;
- a sort $\mathbf{RV}$ with the ring language;
- a map $\mathrm{v} : \mathbf{RV} \to \mathbf{\Gamma}$;
- a map $\mathrm{rv} : \mathbf{K} \to \mathbf{RV}$.

Any valued field (K, v) can be made into a $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-structure by interpreting $\mathbf{K}$ as the field K, $\mathbf{\Gamma}$ as vK — with its ordered moinoid structure, $0 = v(1)$, $\infty = v(0)$ and $-$ is interpreted as the inverse on $vK^\times$ and $-\infty = \infty$ — and $\mathbf{RV}$ as $K/(1 + \mathfrak{m})$ — with its multiplicative structure, $0 = \mathrm{rv}(0)$, $+$ and $-$ defined as the additive structure on $\mathbf{k} \subseteq \mathbf{RV}$ and 0 elsewhere. The maps v and rv are interpreted as the canonical projections.

We will usually also write v for $\mathrm{v} \circ \mathrm{rv} : \mathbf{K} \to \mathbf{\Gamma}$, relying on the context to avoid any confusion. We will denote by $\mathbf{k}^\times$ (respectively $\mathbf{k}$), the definable subset $\mathrm{v}^{-1}(0) \subseteq \mathbf{RV}$ (respectively $\mathrm{v}^{-1}(\{0, \infty\}) \subseteq \mathbf{RV}$).

Definition 2.0.2. Let VF denote the $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-theory of valued fields and ACVF denote the $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-theory of algebraically closed non trivially valued fields.

Remark 2.0.3. 1. For all $n \in \mathbb{Z}_{>0}$, ACVF $\vDash (\forall x : \mathbf{\Gamma})(\exists y : \mathbf{\Gamma})\ ny = x$ — that is, the group $\mathbf{\Gamma}^{\times} := \mathbf{\Gamma} \smallsetminus \{\infty\}$ is divisible.
2. For every $P \in \mathbb{Z}[xy]$ where y is a tuple, ACVF $\vDash (\forall y : \mathbf{k})(\exists x : \mathbf{k})\ P(xy) = 0$ — that is, the residue field $\mathbf{k}$ is algebraically closed.
3. Any $M \vDash$ VF embeds into $N \vDash$ ACVF. If M is not trivially valued, we may assume $\mathbf{K}(N) = \mathbf{K}(M)^{\mathrm{a}}$.

In other words, VF $\vDash$ ACVF$_{\forall}$, the set of universal consequences of ACVF.

Proof. 1. Let $c \in K$ be such that $\mathrm{v}(c) = x$ and $a \in K$ such that $a^n = c$. We have $n\mathrm{v}(a) = \mathrm{v}(a^n) = \mathrm{v}(c)$.
2. Let $c \in \mathcal{O}$ be such that $\mathrm{res}(c) = y$ and $a \in K$ such that $P(c, a) = 0$. By lemma 1.1.11, $a \in \mathcal{O}$ and we have $P(\mathrm{res}(c), \mathrm{res}(a)) = \mathrm{res}(P(c, a)) = 0$.
3. If M is trivially valued, then $(\mathbf{K}(M), \mathrm{v})$ embeds in $(\mathbf{K}(M)(x), v_0)$ which is non trivially valued. So we may assume M is not trivially valued. This is then an immediate consequence of corollary 1.1.10. ☐

2.1. Elimination of quantifiers

Let us start by recalling the characterisation of 1-types for the residue field (ACF) and the value group (DOAG):

Fact 2.1.1. *Let $M, N \vDash$ ACF, $A \leqslant M$, $f : A \to M$ an $\mathfrak{L}_{\mathrm{rg}}$-embedding, $a \in M$ and $P \in A[x]$ its minimal polynomial over A.*
(1) There exists $b \in N^{\star} \succcurlyeq N$ whose minimal polynomial over $f(A)$ is $f_{\star}P$.
(2) f can be extended by sending a to b.

Fact 2.1.2. *Let $M, N \vDash$ DOAG, $A \leqslant M$, $f : A \to M$ an $\mathfrak{L}_{\mathrm{og}}$-embedding, $\gamma \in M$, n its order in M/A, $\alpha := n\gamma$[10] and $C := \{\varepsilon \in \mathbb{Q} \cdot A : \varepsilon < \gamma\}$.*
(1) There exists $\delta \in N^{\star} \succcurlyeq N$ such that $n\delta = f(\alpha)$ and, if $n = \infty$, for any $\varepsilon \in \mathbb{Q} \cdot A$, $f(\varepsilon) < \delta$ if and only if $\varepsilon \in C$.
(2) f can be extended by sending γ to δ.

In this section, we work in the language $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$. Let $M \vDash$ ACVF and $A \leqslant M$. Assume that $\mathbf{K}(A)$ is a field. We now describe various extensions by one $\mathbf{K}$-element. 14/01

Proposition 2.1.3 (Purely ramified 1-types). *Fix any $\gamma \in v(\mathbf{RV}^{\times}(A))$. Let n be its order in $\mathbf{\Gamma}^{\times}(A)/\mathrm{v}(\mathbf{K}^{\times}(A))$ and $c \in \mathbf{K}(A)$ be such that $n\gamma = \mathrm{v}(c)$ — and $c = 1$ if $n = \infty$.*
(1) For every $Q = \sum_i c_i x^i \in \mathbf{K}(A)[x]$ of degree less than n and $a \in \mathbf{K}(M)$ with $v(a) = \gamma$,
- *$\mathrm{v}(Q(a)) = \min_i(\mathrm{v}(c_i) + i\gamma)$ and the minimum is attained in exactly once;*
- *$\mathrm{rv}(Q(a)) = \mathrm{rv}(c_{i_0})\mathrm{rv}(a)^{i_0}$, where $\mathrm{v}(c_{i_0}) + i_0\gamma$ is minimal.*

(2) Assume that $\mathbf{k}(M) \subseteq \mathbf{k}(A)$. There exists $a \in \mathbf{K}(M)$ with $a^n = c$[11]*, $\mathrm{v}(a) = \gamma$ and $\mathrm{rv}(a) \in \mathbf{RV}(A)$. Moreover, for any $\xi \in \mathbf{RV}(M)$ with $\xi^n = \mathrm{rv}(c)$, there exists such an $a \in \mathbf{K}(M)$ with $\mathrm{rv}(a) = \xi$.*

[10] By convention, $\infty\gamma = 0$.
[11] By convention $a^{\infty} = 1$.

(3) Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-isomorphism, by n, c and $\xi := \mathrm{rv}(a)$: for any $N \models \mathrm{ACVF}$, any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding $f : A \to N$ and any $b \in \mathbf{K}(N)$, with $b^n = f(c)$ and $\mathrm{rv}(b) = f(\xi)$, f can be extended by sending a to b.

Proof. (1) We always have $\mathrm{v}(Q(a)) = \mathrm{v}(\sum_i(c_i a^i)) \geqslant \min_i \mathrm{v}(c_i a^i) = \min_i(\mathrm{v}(c_i) + i\gamma)$. If the inequality were strict, there would exist $i < j < n$ such that $\mathrm{v}(c_i a^i) = \mathrm{v}(r_j a^j)$, i.e. $(j-i)\mathrm{v}(a) = \mathrm{v}(c_i) - \mathrm{v}(c_j) \in \mathrm{v}(\mathbf{K}(A))$, contradicting the minimality of n. We have also proved that all the $\mathrm{v}(c_i a^i) = \mathrm{v}(c_i) + i\gamma$ are distinct — in particular the minimum i_0 is unique. It follows that $\mathrm{rv}(Q(a)) = \mathrm{rv}(c_{i_0}) + i_0\mathrm{rv}(a)$.

(2) Assume $n < \infty$ — otherwise the statement is trivial. For any a with $a^n = c$, we have $n\mathrm{v}(a) = \mathrm{v}(c) = n\gamma$ and hence $\mathrm{v}(a) = \gamma = \mathrm{v}(\xi)$, for some $\xi \in \mathbf{RV}(A)$. It follows that $\mathrm{rv}(a)\xi^{-1} \in \mathbf{k}(M) \subseteq \mathbf{k}(A)$, and hence $\mathrm{rv}(a) \in \mathrm{rv}(A)$.
Now if we fix $\xi \in \mathbf{RV}(M)$ with $\xi^n = \mathrm{rv}(c)$, then $\xi^n = \mathrm{rv}(c) = \mathrm{rv}(a)^n$ and hence $\xi\mathrm{rv}(a)^{-1}$is a root of the unit in $\mathbf{k}^\times$. Write $P := x^n - 1 = \prod_i x - e^i$, where the $e^i \in \mathbf{K}(M)$ are the n-th roots of the unit — they are in $\mathcal{O}$ since it is integrally closed. Then $x^n - 1 = \mathbf{k}_\star P = \prod_i x - \mathbf{k}(e)^i$ and hence, for some i, $\xi = \mathrm{rv}(a)\mathrm{rv}(e^i)$ and $(ae^i)^n = c$.

(3) Let C be the structure generated by Aa. By (1), the minimal polynomial of a over $\mathbf{K}(A)$ is $x^n - c$. So we have $\mathbf{K}(C) = \mathbf{K}(A)[a] \simeq \mathbf{K}(A)[x]/(x^n - c)$. Also by (1), $\mathbf{RV}(C) = \mathbf{RV}(A)$ and $\mathbf{\Gamma}(C) = \mathbf{\Gamma}(A)$. Applying (1) to $f(\gamma)$, we see that $x^n - f(c)$ is the minimal polynomial of b over $f(\mathbf{K}(A))$ and hence $f|_{\mathbf{K}}$ extends to an $\mathfrak{L}_{\mathrm{rg}}$-embedding $g|_{\mathbf{K}} : \mathbf{K}(C) \to \mathbf{K}(N)$ sending a to b. Let also $g|_{\mathbf{RV}} = f|_{\mathbf{RV}}$ and $g|_{\mathbf{\Gamma}} = f|_{\mathbf{\Gamma}}$.
For any $Q = \sum_{i<n} c_i x^i \in \mathbf{K}(A)[x]$, by (1), we have $g(\mathrm{rv}(Q(a))) = f(\mathrm{rv}(c_{i_0}\xi^{i_0})) = \mathrm{rv}(f(c_{i_0}))\mathrm{rv}(b)^{i_0} = \mathrm{rv}(g(Q(a)))$, where $\mathrm{v}(c_{i_0}) + i_0\gamma$ is minimal — and hence so is $\mathrm{v}(f(c_{i_0})) + i_0 f(\gamma)$. So $g : C \to N$ is indeed an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding sending a to b. ☐

Definition 2.1.4. An exact lift of $Q \in \mathbf{k}(M)[x]$ is $P \in \mathcal{O}[x]$ with $\mathrm{res}_\star P = Q$ and $\deg(P) = \deg(Q)$.

Proposition 2.1.5 (Purely residual 1-types). *Fix any $\alpha \in \mathbf{k}(A)$. Let $P \in \mathcal{O}(A)[x]$ be an exact lift of its minimal polynomial[12] over $\mathrm{res}(\mathcal{O}(A))$.*

(1) For every $Q = \sum_i c_i x^i \in \mathbf{K}(A)[x]$ of degree less than P and $a \in \mathbf{K}(M)$, with $\mathrm{res}(a) = \alpha$:
- *$\mathrm{v}(Q(a)) = \min_i \mathrm{v}(c_i) \neq \infty$;*
- *$\mathrm{rv}(Q(a)) = \mathrm{rv}(c_{i_0})\mathrm{res}_\star Q_0(\alpha)$, where $\mathrm{v}(c_{i_0})$ is minimal and $Q = c_{i_0}Q_0$.*

(2) There exists $a \in \mathbf{K}(M)$ with $\mathrm{res}(a) = \alpha$ and $P(a) = 0$.

(3) Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-isomorphism, by P and α: for any $N \models \mathrm{ACVF}$, any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding $f : A \to N$, and any $b \in \mathbf{K}(N)$, with $f_\star P(b) = 0$ and $\mathrm{res}(b) = f(\alpha)$, f can be extended by sending a to b.

Proof. (1) Let i_0 be such that $\mathrm{v}(c_{i_0})$ is minimal. Let $Q_0 := c_{i_0}^{-1}Q$. Then $\mathrm{res}_\star Q_0 \neq 0$. By minimality of $\mathrm{res}_\star P$, $\mathrm{res}(Q_0(a)) = \mathrm{res}_\star Q_0(\alpha) \neq 0$ and hence $\mathrm{v}(Q_0(a)) = 0$. It follows that $\mathrm{v}(Q(a)) = \mathrm{v}(c_{i_0}) = \min_i \mathrm{v}(c_i)$. Also, $\mathrm{rv}(Q(a)) = \mathrm{rv}(c_{i_0})\mathrm{rv}(Q_0(a)) = \mathrm{res}(Q_0(a))$.

(2) Let $P = \prod_j (x - e_j)$. Since $\mathcal{O}$ is integrally closed, cf. lemma 1.1.11, we have $e_j \in \mathcal{O}$, for all j. For any $a \in \mathrm{res}^{-1}(\alpha)$, $\mathrm{res}_\star P(\alpha) = \mathrm{res}(P(a)) = \prod_j \mathrm{res}(a) - \mathrm{res}(e_j) = 0$. It follows that there exists an j such that $\mathrm{res}(e_j) = \mathrm{res}(a) = \alpha$.

[12] We allow P to be 0; in which case, α is transcendental over $\mathrm{res}(\mathcal{O}(A))$

(3) Let C be the structure generated by Aa. By (1), P is the minimal polynomial of a over $\mathbf{K}(A)$. So we have $\mathbf{K}(C) = \mathbf{K}(A)[a] \simeq \mathbf{K}(A)[x]/P$. Also by (1), $\mathbf{RV}(C) = \mathbf{RV}(A)$ and $\mathbf{\Gamma}(C) = \mathbf{\Gamma}(A)$. Applying (1) to $\beta := \mathrm{res}(b)$, we see that $f_\star P$ is the minimal polynomial of b over $\mathbf{K}(f(A))$ and thus $f|_\mathbf{K}$ extends to an $\mathfrak{L}_{\mathrm{rg}}$-embedding $g|_\mathbf{K} : \mathbf{K}(C) \to \mathbf{K}(N)$ sending a to b. Let also $g|_\mathbf{RV} = f|_\mathbf{RV}$ and $g|_\mathbf{\Gamma} = f|_\mathbf{\Gamma}$.
For any $Q = \sum_{i<\deg(P)} c_i x^i \in \mathbf{K}(A)[x]$, by (1), we have:

$$g(\mathrm{rv}(Q(a))) = \mathrm{rv}(f(c_{i_0}))\mathrm{res}_\star f_\star Q_0(f(\beta)) = \mathrm{rv}(f_\star Q(b)) = \mathrm{rv}(g(Q(a))),$$

where $\mathrm{v}(c_{i_0})$ — and hence $\mathrm{v}(f(c_{i_0}))$ — is minimal. So $g : C \to N$ is indeed an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding sending a to b. □

Remark 2.1.6.
- For every $\xi, \zeta \in \mathbf{RV}$, we define $\xi \oplus \zeta := \{\mathrm{rv}(x+y) : \mathrm{rv}(x) = \xi \text{ and } \mathrm{rv}(y) = \zeta\}$. We say that $\xi \oplus \zeta$ is well-defined if it is a singleton, whose element we denote $\xi + \zeta$. The map $\oplus$ is an hypergroup law (in the sense of Kasner): is associative, commutative, with neutral element 0...
- If $P = \sum_i \zeta_i x^i \in \mathbf{RV}[x]$ — this is a purely formal notation — and $\xi \in \mathbf{RV}$, we define $P(\xi) := \bigoplus_i \zeta_i \xi^i = \{\mathrm{rv}(Q(a)) : \mathrm{rv}_\star Q = P \text{ and } \mathrm{rv}(a) = \xi\}$. We say that it is well-defined whenever it is a singleton.
- Both previous lemmas can now be subsumed as follows: fix any $\xi \in \mathrm{rv}(A)$. Let $P \in \mathbf{K}(A)[x]$ have minimal degree such that $0 \in \mathrm{rv}_\star P(\xi)$.
  (1) For every $Q = \sum_i c_i x^i \in \mathbf{K}(A)[x]$ and every $a \in \mathbf{K}(M)$, with $\mathrm{rv}(a) = \xi$, $\mathrm{rv}(Q(a)) = \mathrm{rv}_\star Q(\xi)$ which is well-defined.
  (2) There exists $a \in \mathbf{K}(M)$ with $\mathrm{rv}(a) = \xi$ and $P(a) = 0$.
  (3) Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-isomorphism, by P and ξ: for any $N \vDash \mathrm{ACVF}$, any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding $f : A \to N$, and any $b \in \mathbf{K}(N)$, with $f_\star P(b) = 0$ and $\mathrm{rv}(b) = f(\xi)$, f can be extended by sending a to b.

To deal with the last type of extension, the immediate ones, we will first need a technical lemma on the localisation of roots of polynomials with respect to pseudo-Cauchy filters. 14/01

Lemma 2.1.7. *Let $\mathfrak{B}$ be a non-principal pseudo Cauchy filter on $\mathbf{K}(A)$ and $P \in \mathbf{K}(A)[x]$. Then one (and only one) of the following holds:*
- *there is $b \in \mathfrak{B}$ such that $\mathrm{rv} \circ P|_b$ is constant;*
- *there is a root of P in $\overline{\mathfrak{B}}$ and, for every $b \in \mathfrak{B}$, $\mathrm{v} \circ P|_{b(A)}$ is non-constant.*

Proof. Let $P = c\prod_i (x - e_i)$ and $b \in \mathfrak{B}$ be a ball of $\mathbf{K}(A)$ such that $\{i : e_i \in \overline{\mathfrak{B}}\} = \{i : e_i \in b\} =: I$. For every $a_1, a_2 \in b$ and $i \notin I$, $\mathrm{v}(a_1 - a_2) > \mathrm{v}(a_1 - e_i)$ and thus $a_2 - e_i = a_1 - e_i + (a_2 - a_1) \in a_1 - e_i + (a_1 - e_i)\mathfrak{m}$ and $\mathrm{rv}(a_1 - e_i) = \mathrm{rv}(a_2 - e_j)$. It follows that $\mathrm{rv}(P(a_1))/\mathrm{rv}(P(a_2)) = \prod_{i\in I} \mathrm{rv}(a_1 - e_i)/\mathrm{rv}(a_2 - e_i)$. In particular, if $I = \varnothing$, $\mathrm{rv}(P(a_1)) = \mathrm{rv}(P(a_2))$.

If $I \neq \varnothing$ let $b_0 \subset b_1 \subseteq b$ be closed balls of $\mathbf{K}(A)$, both in $\mathfrak{B}$. Let also $a_0 \in b_0(A)$ and $a_1 \in b_1(A) \setminus b_0$ — such an a_1 exists because there are elements of $b_1(A)$ whose at distence the radius of b_1, thus both cannot be in b_0. Then, for all $i \in I$, $\mathrm{v}(a_1 - e_i) > \mathrm{rad}(b_0) \geqslant \mathrm{v}(a_0 - e_i)$ and hence $\mathrm{v}(P(a_1)) - \mathrm{v}(P(a_2)) = \sum_{i\in I}(\mathrm{v}(a_1 - e_i) - \mathrm{v}(a_2 - e_i)) > 0$. □

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Proposition 2.1.8 (Immediate 1-types). *Fix a pseudo Cauchy filter $\mathfrak{B}$ on $\mathbf{K}(A)$. Let $P \in \mathbf{K}(A)[x]$ have minimal degree among those polynomials such that $0 \in \overline{P_\star \mathfrak{B}}$.*

(1) For every $Q \in \mathbf{K}(A)[x]$ with degree smaller than P, there exists $U \in \mathfrak{B}$ with $\mathrm{rv} \circ Q|_U$ constant, equal to an element of $\mathrm{rv}(\mathbf{K}(A))$.

(2) If $P \neq 0$, there exist $a \in \mathbf{K}(M)$ with $a \in \overline{\mathfrak{B}}$ and $P(a) = 0$.

(3) Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-isomorphism, by $\mathfrak{B}$ and P : for every $N \vDash \mathrm{ACVF}$, embedding $f : A \to N$ and $b \in \mathbf{K}(N)$, with $f_\star P(b) = 0$ and $b \in \overline{f_\star \mathfrak{B}}$, f can be extended by sending a to b.

Proof. (1) By minimality of P, $0 \notin \overline{Q_\star \mathfrak{B}}$, and thus, by lemma 2.1.7, $\mathrm{rv} \circ Q$ is constant on some $U \in \mathfrak{B}$. Since we may assume that U is a ball of $\mathbf{K}(A)$, $U(A) \neq \varnothing$ and hence $\mathrm{rv}(Q(U)) \in \mathrm{rv}(\mathbf{K}(A))$.

(2) If there is no root of P in $\overline{\mathfrak{B}}$, then, by lemma 2.1.7, $\mathrm{rv} \circ P$ is eventually constant on $\mathfrak{B}$. Since $0 \in \overline{P_\star \mathfrak{B}}$, we must have that P is eventually equal to 0 on $\mathfrak{B}$. If $P \neq 0$, $\mathfrak{B}$ contains the finite set of roots of P; in particular, $\mathfrak{B}$ contains a singleton from $\mathbf{K}(A)$.

(3) Let C be the structure generated by Aa. By (1), P is the minimal polynomial of a over $\mathbf{K}(A)$. So we have $\mathbf{K}(C) = \mathbf{K}(A)[a] \simeq \mathbf{K}(A)[x]/P$. Also, by (1), we have $\mathbf{RV}(C) = \mathbf{RV}(A)$ and $\mathbf{\Gamma}(C) = \mathbf{\Gamma}(A)$. By lemma 2.1.7, $f_\star P$ is minimal with $0 \in \overline{f_\star P_\star \mathfrak{B}}$. By (1), the minimal polynomial of b over $\mathbf{K}(f(A))$ is $f_\star P$, so $f|_{\mathbf{K}}$ extends to a ring embedding $g|_{\mathbf{K}} : \mathbf{K}(C) \to \mathbf{K}(N)$ sending a to b. Let also $g|_{\mathbf{RV}} = f|_{\mathbf{RV}}$ and $g|_{\mathbf{\Gamma}} = f|_{\mathbf{\Gamma}}$.
For any $Q \in \mathbf{K}(A)[x]$ with degree smaller than P, by (1), we find $U \in \mathfrak{B}$ such that $\mathrm{rv} \circ Q|_U$ and $\mathrm{rv} \circ f_\star Q|_{f(U)}$ are constant equal to some $\mathrm{rv}(c)$, respectively $\mathrm{rv}(f(c))$ for any $c \in U(A)$. It follows that $g(\mathrm{rv}(Q(a))) = f(\mathrm{rv}(c)) = \mathrm{rv}(f(c)) = \mathrm{rv}(f_\star Q(b)) = \mathrm{rv}(g(Q(a)))$. So $g : C \to N$ is indeed an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding sending a to b. □

We will need one last case of the embedding lemma:

Proposition 2.1.9. *Fix any $\gamma \in \mathbf{\Gamma}^\times(A)$. Let n be its order in $\mathbf{\Gamma}^\times(A)/\mathrm{v}(\mathbf{RV}^\times(A))$ and $\zeta \in \mathbf{RV}(A)$ be such that $n\gamma = \mathrm{v}(\zeta)$ — and $\zeta = 1$ if $n = \infty$.*

(1) For every $\alpha \in \mathbf{RV}(A)$, $0 \leqslant i < n$ and $\xi \in \mathbf{RV}(M)$ with $\mathrm{v}(\xi) = \gamma$, $\mathrm{v}(\alpha\xi^i) = 0$ if and only if $i = 0$ and $\alpha \in \mathbf{k}^\times(A)$.

(2) There exists $\xi \in \mathbf{RV}(M)$ with $\xi^n = \zeta$ and $\mathrm{v}(\xi) = \gamma$;

(3) Such a ξ is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-isomorphism, by γ, n and ζ: for every $N \vDash \mathrm{ACVF}$, any embedding $f : A \to N$ and any $\eta \in \mathbf{K}(N)$, with $\eta^n = f(\zeta)$ and $\mathrm{v}(\eta) = f(\gamma)$, f can be extended by sending ξ to η.

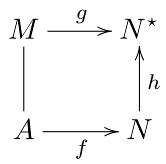
Proof. (1) We have $\mathrm{v}(\alpha\xi^i) = \mathrm{v}(\alpha) + i\mathrm{v}(\xi) = 0$ if and only $i\mathrm{v}(\xi) = -\mathrm{v}(\alpha) \in \mathrm{v}(\mathbf{RV}^\times(A))$. By minimality of n, we must have $i = 0$ and hence $\mathrm{v}(\alpha) = 0$.

(2) Let $c \in \mathbf{K}(M)$ be such that $\mathrm{rv}(x) = \zeta$. If $n < \infty$, let $a \in \mathbf{K}(M)$ be such that $a^n = c$. Then $\mathrm{rv}(a)^n = \zeta$ and $n\mathrm{v}(a) = \mathrm{v}(\zeta) = n\gamma$ and hence $\mathrm{v}(a) = \gamma$. If $n = \infty$ any $\xi \in \mathbf{RV}(M)$ with $\mathrm{v}(\xi) = \gamma$ will work.

(3) By (1), ξ is order n in $\mathbf{RV}^\times(M)/\mathbf{RV}^\times(A)$. By (1), η is also order n in $\mathbf{RV}^\times(N)/\mathbf{RV}^\times(f(A))$. So $f|_{\mathbf{RV}}$ extends to a multiplicative group embedding $g|_{\mathbf{K}} : \mathbf{RV}(A) \cdot \xi^{\mathbb{Z}} \to \mathbf{K}(N)$ sending ξ to η. Let C be the structure generated by $A\xi$. Note that, by (1) again, $\mathbf{RV}(A) \cdot \xi^{\mathbb{Z}}$ is closed under $+$ and $-$, so $\mathbf{RV}(C) = \mathbf{RV}(A) \cdot \xi^{\mathbb{Z}}$ and $g|_{\mathbf{K}}$ is an $\mathfrak{L}_{\mathrm{rg}}$-embedding. Let also

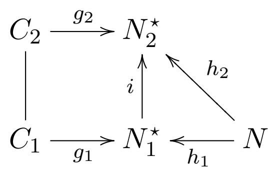
$g|_{\mathbf{\Gamma}} = f|_{\mathbf{\Gamma}}$ and $g|_{\mathbf{K}} = f|_{\mathbf{K}}$. Since v is multiplicative, it is preserved by g which is indeed an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding sending ξ to η. □

Proposition 2.1.10 (ACVF embedding lemma). *Let $M, N \vDash$ ACVF, $A \leqslant M$ and $f : A \to N$. There exists an elementary map $h : N \to N^\star$ and an embedding $g : M \to N$ such that:*



commutes.

Proof. The family of pairs of embeddings (g, h), with $g : C \to N^\star$, $C \leqslant M$ and $h : N \to N^\star$ elementary, is inductive — where (g_1, h_1) is smaller than (g_2, h_2) if $C_1 \leqslant C_2$ and there exists $i : N_1^\star \to N_2^\star$ elementary such that



commutes — and contains (f, id). By Zorn's lemma, it contains a maximal element (g, h) larger than (f, id). There remains to show that $M = C$. We proceed by proving a series of inclusions.

$\mathbf{\Gamma}(C) = \mathbf{\Gamma}(M)$ For any $\gamma \in \mathbf{k}(M)$, by fact 2.1.1, $g|_{\mathbf{\Gamma}}$ extends to $g_0 : \mathbf{\Gamma}(C)\gamma \to \mathbf{\Gamma}(N^\star)$, for some $N^\star \succcurlyeq N$. Then $g \cup g_0 : C\gamma \to N^\star$ is an embedding. By maximality, $\gamma \in \mathbf{\Gamma}(C)$.

$\mathbf{k}(C) = \mathbf{k}(M)$ For any $\alpha \in \mathbf{k}(M)$, by fact 2.1.1, $g|_{\mathbf{k}}$ extends to $g_0 : \mathbf{k}(C)\alpha \to \mathbf{k}(N^\star)$, for some $N^\star \succcurlyeq N$. We define $g_1 : \mathbf{RV}(C)\alpha \to \mathbf{RV}(N^\star)$ by $g_1(\xi P(\alpha)) = g(\xi) g_0(P(\alpha))$, for every $\xi \in \mathbf{RV}(C)$ and $P \in \mathbf{k}(C)[x]$. This is well defined, indeed, if $\xi P(\alpha) = 1$, and $\xi \neq 0$ then $\xi^{-1} = P(\alpha) \neq 0$ and hence $\xi \in \mathbf{k}(C)$, so $f(\xi) g_0(P(\alpha)) = g_0(\xi P(\alpha)) = 1$. It is obviously multiplicative. Since $\mathrm{v}(\xi P(\alpha)) \in \mathbf{k}$ if and only if $\mathrm{v}(\xi) \in \mathbf{k}$, it follows that it is also additive. Then $g \cup g_1 : C\alpha \to N^\star$ is an embedding. By maximality, $\alpha \in \mathbf{k}(C)$.

$\mathrm{v}(\mathbf{RV}(C)) = \mathbf{\Gamma}(M)$ Fix any $\gamma \in \mathbf{\Gamma}(M) = \mathbf{\Gamma}(C)$ and let n be its order in $\mathbf{\Gamma}(C)/\mathrm{v}(\mathbf{RV}(C))$ and $\zeta \in \mathbf{RV}(C)$ be such that $\gamma^n = \mathrm{v}(\zeta)$. By proposition 2.1.9.(2), there exists $\xi \in \mathbf{RV}(M)$ such that $\xi^n = \zeta$ and $\mathrm{v}(\xi) = \gamma$, and $\rho \in \mathbf{RV}(N)$ such that $\rho^n = g(\zeta)$ and $\mathrm{v}(\rho) = g(\gamma)$. By proposition 2.1.9.(3), g extends by sending ξ to ρ. By maximality, $\xi \in \mathbf{RV}(C)$ and hence $\gamma = \mathrm{v}(\xi) \in \mathrm{v}(\mathbf{RV}(C))$.

$\mathbf{RV}(C) = \mathbf{RV}(M)$ For any $\xi \in \mathbf{RV}^\times(C)$, $\mathrm{v}(\xi) \in \mathbf{\Gamma}(M) = \mathrm{v}(\mathbf{RV}(C))$ and hence there is some $\zeta \in \mathbf{RV}(C)$ such that $\mathrm{v}(\xi) = \mathrm{v}(\zeta)$. Then $\xi\zeta^{-1} \in \mathbf{k}(M) = \mathbf{k}(C)$ and hence $\xi \in \zeta\mathbf{k}(C) \subseteq \mathbf{RV}(C)$.

$\mathbf{K}(C) = \mathbf{K}(C)_{(0)}$ By the universal property of localisation, $g|_{\mathbf{K}}$ has a (unique) extension g_0 to $\mathbf{K}(C)_{(0)} \cup \mathbf{k}(C) \cup \mathbf{\Gamma}(C)$. Indeed, g_0 is a ring morphism on the sorts $\mathbf{K}$ and it is equal to g on $\mathbf{RV}$ and $\mathbf{\Gamma}$. For any $c \in \mathbf{K}(C)$ and non zero $d \in \mathbf{K}(C)$, $g_0(\mathrm{rv}(c/d)) = g(\mathrm{rv}(c)\mathrm{rv}(d)^{-1}) = \mathrm{rv}(g(c))\mathrm{rv}(g(d))^{-1} = \mathrm{rv}(g_0(c/d))$. So g_0 is an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-embedding. By maximality of g, $\mathbf{K}(C)$ is a field.

$\mathrm{res}(\mathbf{K}(C)) = \mathbf{k}(C)$ Pick any $\alpha \in \mathbf{k}(C)$. Let $P \in \mathcal{O}(C)[x]$ be an exact lift of its minimal polynomial over $\mathrm{res}(\mathcal{O}(C))$. By proposition 2.1.5.(2), we can also find $a \in \mathbf{K}(M)$ and $b \in \mathbf{K}(N^\star)$ such that $P(a) = 0 = g_\star P(b)$, $\mathrm{res}(a) = \alpha$ and $\mathrm{res}(b) = g(\alpha)$. Applying proposition 2.1.5.(3), we find a pair (g_0, h) larger than (g, h) with g_0 defined at a. By maximality, $a \in \mathbf{K}(C)$ and hence $\alpha \in \mathrm{res}(\mathbf{K}(C))$.

$\mathrm{v}(\mathbf{K}(C)) = \mathbf{\Gamma}(C)$ Pick any $\gamma \in \mathbf{\Gamma}^\times(C)$. Let n be its order in $\mathbf{\Gamma}^\times(C)/\mathrm{v}(\mathbf{K}^\times(C))$ and $c \in \mathbf{K}(C)$ such that $n \cdot \gamma = \mathrm{v}(c)$ — with $c = 1$ if $n = \infty$. By proposition 2.1.3.(2), we find $a \in \mathbf{K}(M)$ such that $a^n = c$ and $\mathrm{v}(a) = \gamma$, and $b \in \mathbf{K}(N^\star)$ such that $b^n = g(c)$ and $\mathrm{v}(b) = g(\gamma)$. Applying proposition 2.1.3.(3), we find a pair (g_0, h) larger than (g, h) with g_0 defined at a. By maximality, $a \in \mathbf{K}(C)$ and hence $\alpha \in \mathrm{v}(\mathbf{K}(C))$.

$\mathbf{K}(C) = \mathbf{K}(M)$ We start by proving that any pseudo Cauchy filter $\mathfrak{B}$ over $\mathbf{K}(C)$ that accumulates at some $a \in \mathbf{K}(M)$ also accumulates at some $c \in \mathbf{K}(C)$. Let $P \in K(C)[x]$ have minimal degree such that $0 \in \overline{P_\star \mathfrak{B}}$. If $P \neq 0$, by proposition 2.1.8.(2), there exists $c \in \mathbf{K}(M)$ such that $P(c) = 0$ and $c \in \overline{\mathfrak{B}}$. If $P = 0$, $c := a$ satisfies those same requirements. By compactness (corollary B.0.13) and proposition 2.1.8.(2), we also find $i : N^\star \to N^\dagger$ and $b \in \mathbf{K}(N^\dagger)$ with $b \in \overline{g_\star \mathfrak{B}}$ and $g(P)(b) = 0$ — if $P \neq 0$, some root of $g(P)$ in $N^\star$ works and we can take $i = \mathrm{id}$; if $P = 0$, the set of balls in $g_\star \mathfrak{B}$ is finitely satisfiable in N since it is a filter. Applying proposition 2.1.8.(3), we find a pair $(g_0, i \circ h)$ larger than (g, h) with g_0 defined at c. By maximality, $c \in \mathbf{K}(C)$ and, by construction, we do have $c \in \overline{\mathfrak{B}}$.

Now fix $a \in \mathbf{K}(M)$ and let $\mathfrak{B}$ be the maximal pseudo Cauchy filter over $\mathbf{K}(C)$ that accumulates at a — *i.e.* the filter generated by the balls of $\mathbf{K}(C)$ containing a. By **??**, $\mathfrak{B}$ accumulates at some $c \in \mathbf{K}(C)$. Let us assume that $a \neq c$. By **??**, there is some $d \in \mathbf{K}(C)$ such that $\mathrm{v}(a - c) = \mathrm{v}(d) \in \mathbf{\Gamma}^\times(M)$. Then $(a - c)/d \in \mathcal{O}$ and, by **??**, there is some $e \in \mathcal{O}^\times(C)$ such that $\mathrm{res}((a - c)/d) = \mathrm{res}(e)$; equivalently $(a - c)/d - e \in \mathfrak{m}$ and hence $a \in c + de + d\mathfrak{m} =: b$, the open ball of radius $\mathrm{v}(d)$ around $c + de$. By construction, $b \in \mathfrak{B}$, but $\mathrm{v}(c + de - c) = \mathrm{v}(d) + \mathrm{v}(e) = \mathrm{v}(d)$, so $c \notin b$, a contradiction. It follows that $a = c \in \mathbf{K}(C)$, and hence $\mathbf{K}(M) = \mathbf{K}(C)$.

□

Theorem 2.1.11 (Robinson, 1956). *The $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-theory* ACVF *eliminates quantifiers.*

Proof. By proposition B.0.15, this is an immediate consequence of proposition 2.1.10. □

Corollary 2.1.12. *The class of existentially closed models of* VF *coincides with* ACVF.

Proof. By remark 2.0.3.3, any model of VF embeds in a model of ACVF, so it suffices to check that embeddings between models of ACVF are existentially closed. But this is an immediate consequence of elimination of quantifiers, *cf.* theorem 2.1.11. □

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Corollary 2.1.13. *The completions of* ACVF *are the theories* $\mathrm{ACVF}_{p,q}$*, with p, q prime or zero, of non trivially valued algebraically closed valued of characteristic p and residue characteristic q.*

Note that if $p > 0$, then $q = p$.

Proof. By proposition B.0.15, it suffices to find a common substructure to any two models of $\mathrm{ACVF}_{p,q}$. If $q = p > 0$, the trivially valued field $\mathbb{F}_p$ embeds (uniquely) in any model of $\mathrm{ACVF}_{p,p}$. If $q = p = 0$, the trivially valued field $\mathbb{Q}$ embeds (uniquely) in any model of $\mathrm{ACVF}_{0,0}$. Finally, the field $\mathbb{Q}$ with the p-adic valuation embeds (uniquely) in every model of $\mathrm{ACVF}_{0,p}$. ☐

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Definition 2.1.14. Let $\mathfrak{L}_{\mathrm{div}}$ be the language with a single sort $\mathbf{K}$ with the ring language $(+,-,0,\cdot,1)$ and a predicate $|$.

Any valued field (K,v) can be made into a $\mathfrak{L}_{\mathrm{div}}$-structure by interpreting $a|b$ as $v(a) \leqslant v(b)$.

Corollary 2.1.15. *The* $\mathfrak{L}_{\mathrm{div}}$*-theory* ACVF *eliminates quantifiers.*

Proof. To do ☐

Definition 2.1.16. Let $\mathfrak{L}_{\mathbf{k},\boldsymbol{\Gamma}}$ be the three sorted language with:

- a sort $\mathbf{K}$ with the ring language $(+,-,0,\cdot,1)$;
- a sort $\boldsymbol{\Gamma}$ with the ordered group language $(+,-,0,<)$ and a constant ∞;
- a sort $\mathbf{k}$ with the ring language;
- a map $\mathrm{v} : \mathbf{RV} \to \boldsymbol{\Gamma}$;
- a map $\rho : \mathbf{K}^2 \to \mathbf{k}$.

Any valued field (K,v) can be made into a $\mathfrak{L}_{\mathbf{RV},\boldsymbol{\Gamma}}$-structure by interpreting $\rho(a,b) = \mathbf{k}(a/b)$ if $v(a) \geqslant v(b)$ and 0 otherwise.

Corollary 2.1.17. *The* $\mathfrak{L}_{\mathbf{k},\boldsymbol{\Gamma}}$*-theory* ACVF *eliminates quantifiers.*

Proof. To do ☐

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Corollary 2.1.18. *Let (K,v) be a valued field, L be a normal extension and w_1, w_2 be two valuations on L extending v, then there exists $\sigma \in \mathrm{aut}(L/K)$ such that $w_2 \circ \sigma$ is equivalent to w_1.*

Proof. Let L_i be the $\mathfrak{L}_{\mathbf{RV},\boldsymbol{\Gamma}}$-structure associated to (L,w_i). By hypothesis, the identity on K is an $\mathfrak{L}_{\mathbf{RV},\boldsymbol{\Gamma}}$-embedding from L_1 to L_2. By remark 2.0.3.3, $L_i \leqslant M_i \vDash \mathrm{ACVF}$. By elimination of quantifiers (theorem 2.1.11), there exists an $\mathfrak{L}_{\mathbf{RV},\boldsymbol{\Gamma}}(K)$-embedding $\sigma : L_1 \to M_2$. Since L is normal, $\sigma(L_1) = L_2$. So we do have $\sigma \in \mathrm{aut}(L/K)$ and $w_2 \circ \sigma$ is equivalent to w_1. ☐

2.2. Properties of definable sets

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We now give a complete description of types concentrating on $\mathbf{K}$ over algebraically closed subsets of $\mathbf{K}$ in ACVF. Fix $M \vDash \mathrm{ACVF}$ and $A = A^{\mathrm{a}} \leqslant \mathbf{K}(M)$.

Definition 2.2.1. Let $\mathfrak{B}$ be a pseudo Cauchy filter over A. An element $a \in \mathbf{K}(M)$ is said to be *generic in* $\mathfrak{B}$ *over* A if, for every ball b of A:

$$a \in b \text{ if and only if } b \in \mathfrak{B}.$$

We write $\eta_{\mathfrak{B}}|_A$ for the — *a priori* partial — type of generics of $\mathfrak{B}$ over A.

Proposition 2.2.2. *Let $a \in \mathbf{K}(M)$. Let $\mathfrak{B}$ be generated by $\{b$ ball in $A : a \in b\}$. Then:*

$$\eta_{\mathfrak{B}}|_A \vDash \mathrm{tp}(a/A).$$

In particular $\eta_{\mathfrak{B}}|_A$ is a complete type over A.

Proof. Note that, by construction, $a \vDash \eta_{\mathfrak{B}}|_A$. Let us first assume that $\mathfrak{B}$ is generated by some closed ball $b_0 = \overline{\mathrm{B}}(c, \mathrm{v}(d))$, with $c, d \in A$. If $d = 0$, then $\eta_{\mathfrak{B}}|_A(x) \vDash x = c$ which generates a complete type. Otherwise, let $a' \vDash \eta_B|_A$ and $\alpha' = \mathrm{res}((a'-c)/d)$. If $\alpha' \in \mathrm{res}(A)^{\mathrm{a}} = \mathrm{res}(A)$ — by proposition 2.1.5.(2) — then we find $e \in \mathbf{K}(A)$ with $\mathrm{res}(e) = \alpha'$, *i.e.* $\mathrm{v}(a' - (c + de)) > \mathrm{v}(d)$, so $a' \in \mathring{\mathrm{B}}(c + de, \mathrm{v}(d),)$, a contradiction to $a' \vDash \eta_B|_A$. So $\alpha' \notin \mathrm{res}(A)^{\mathrm{a}}$ and, by 2.1.5.(3), $(a'-c)/d \equiv_A (a-c)/d$, and hence $a' \equiv_A a$.

Let us now assume that $\mathfrak{B}$ is not principal — *i.e.* it is not generated by a single ball — and that $\overline{\mathfrak{B}} \cap A = \varnothing$. If the minimal P such that $0 \in \overline{P_\star \mathfrak{B}}$ is not 0, by 2.1.8.(2), $\overline{\mathfrak{B}} \cap A = \overline{\mathfrak{B}} \cap A^{\mathrm{a}} \neq \varnothing$, a contradiction. So $P = 0$ and, since any $a' \vDash \eta_B|_A$ is in $\overline{\mathfrak{B}}$, by 2.1.8.(3), we have $a' \equiv_A a$.

Let us now deal with the remaining case. We may therefore assume that $\mathfrak{B}$ is not generated a single closed ball and that there exists some $c \in \overline{\mathfrak{B}} \cap A$. Let $a' \vDash \eta_B|_A$. For every $\gamma \in \mathrm{v}(A)$, we have $\mathrm{v}(a'-c) \geqslant \gamma$ if and only if $\overline{\mathrm{B}}(c, \gamma) \in \mathfrak{B}$, which is equivalent, since $\mathfrak{B}$ is not generated by $\overline{\mathrm{B}}(c, \gamma)$, to $\mathring{\mathrm{B}}(c, \gamma) \in \mathfrak{B}$, *i.e.* $\mathrm{v}(a'-c) > \gamma$. It follows that $\mathrm{v}(a'-c) \notin \mathrm{v}(A) = \mathbb{Q} \cdot \mathrm{v}(A)$ — the equality follows from proposition 2.1.3.(2) — and that, by proposition 2.1.3.(3), $(a'-c) \equiv_A (a-c)$. So $a' \equiv_A a$. □

Remark 2.2.3. We see from the proof that there is a correspondence between the descriptions of the types over A concentrating on $\mathbf{K}$ in algebraic terms and in terms of generics of pseudo Cauchy filters:

- generics of closed balls correspond, up to translation and scaling, to residual extensions;
- generics of open balls correspond, up to translation, to ramified extensions where the cut is of the form γ^+;
- generics of non principal pseudo Cauchy filters with an accumulation point in A correspond, up to translation, to the other ramified extensions;
- generics of non principal pseudo Cauchy filters without accumulation points in A correspond to immediate extensions.

Proposition 2.2.2 states, among other thing that types in $\mathbf{K}$ are entirely determined by their restriction to the Boolean algebra generated by balls. By some abstract non-sense, every definable subset of K is, up to equivalence, in said algebra.

Definition 2.2.4.
- An A-Swiss cheese is a set of the form $b \smallsetminus \bigcup_{i<n} b_i$ where b is a ball in A and the $b_i \subset b$ are (disjoint) subballs, in A.
- A Swiss cheese $b \smallsetminus \bigcup_i b_i$ is nested inside some other Swiss cheese $d \smallsetminus \bigcup_j d_j$ if there exists a j such that $b = d_j$.

Theorem 2.2.5 (Holly, 1995). *Any $\mathfrak{L}_{\mathbf{RV},\Gamma}(A)$-definable subset of $\mathbf{K}$ has a unique decomposition as a finite disjoint union of non-nested A-Swiss cheeses.*

Proof. Let $\Delta(x)$ be the set of finite unions of A-Swiss cheeses.

Claim 2.2.5.1. *$\Delta(x)$ is stable under Boolean combinations.*

Proof. It suffices to show that the intersection of two A-Swiss cheeses is an A-Swiss cheese and that the complement of an A-Swiss cheese is a finite union of A-Swiss cheese. Let $B = b \smallsetminus \bigcup_i b_i$ and $D = d \smallsetminus \bigcup_i d_i$ be two Swiss cheeses. We have $B \cap D = (b \cap d) \smallsetminus (\bigcup_i (d \cap b_i) \cup \bigcup_j (b \cap d_j))$ where some of the intersections might be empty. Similarly $\mathbf{K} \smallsetminus B = (\mathbf{K} \smallsetminus b) \cup \bigcup_i b_i$. ◊

Proposition 2.2.2 implies that for all $p \in \mathcal{S}_x(A)$, $p|_\Delta := p \cap \Delta \vDash p$. In other terms, for any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-formula $\varphi(x)$, $a \in \varphi(M)$ and $c \in \mathbf{K}(M)$, if $c \vDash \mathrm{tp}_\Delta(a)$, then $M \vDash \varphi(c)$. By B.0.14, any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-definable subset of $\mathbf{K}$ is equivalent to a formula in $\Delta(x)$, that is a finite union of A-Swiss cheeses. Since the union of two non disjoint A-Swiss cheeses is an A-Swiss cheese: $(b \smallsetminus \bigcup_i b_i) \cup (d \smallsetminus \bigcup_i d_i) = b \smallsetminus (\bigcup_i (b_i \smallsetminus d) \cup \bigcup_{i,j} (b_i \cap d_j))$, where $d \subseteq b$; and the union of two nested swiss cheeses is a swiss cheese : $b \smallsetminus \bigcup_i b_i \cup (d \smallsetminus \bigcup_i d_i) = b \smallsetminus (\bigcup_{i>0} b_i \cup \bigcup_j d_j)$, where $d = b_0$; we may assume that it is a disjoint union of non-nested A-Swiss cheeses.

Uniqueness now follows from:

Claim 2.2.5.2. *Let $(D_i)_{i<n}$ be disjoint non nested Swiss cheeses and B be some Swiss cheese such that $B \subseteq \bigcup_i D_i$, then there is some i such that $B \subseteq D_i$.*

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Proof. We may assume that the D_i form a minimal cover. In particular, we then have that, for every i, $B \cap D_i \neq \emptyset$. If $n = 1$ then the claim is proved. So, let us assume that $n \geqslant 2$.

Let us first assume that the D_i and B are balls. Pick $c_i \in B \cap D_i$. Let b be the smallest ball containing all the c_i. It is a closed ball of radius $\gamma := \min_{i \neq j} \mathrm{v}(c_i - c_j)$. We have $b \subseteq B \subseteq \bigcup_i D_i$. If $b \subseteq D_i$, for some i, then every c_j is in $b \subseteq D_i$, contradicting that $D_i \cap D_j = \emptyset$. So we must have that, for every i, $D_i \subset b$. Let $d_i := \mathring{\mathrm{B}}(c_i, \gamma)$, it is the maximal strict subbal of b containing c_i, in particular $D_i \subseteq d_i$. So $b \subseteq \bigcup_i d_i$. Let d be some maximal open subball of b, then d intersects some d_i and by maximality $d = d_i$. It follows that $\mathbf{R}_b := \{d \subset b : \text{maximal open subball}\} = \{c + \gamma\mathfrak{m} : c \in b\}$ is finite. However, if we choose $a \in b$ and $e \in \mathrm{v}^{-1}(\gamma)$, $c \mapsto (c-a)/e$ is a bijection $b \to \mathcal{O}$ sending elements of $\mathbf{R}_b$ to elements of $\mathbf{k}$ which is infinite; a contradiction.

Let us now come back to the general case $B = b \smallsetminus \bigcup_j b_j$, $D_i = d_i \smallsetminus \bigcup_\ell d_{i,\ell}$, where the $d_{i,\ell}$ are disjoint. Since $B \subseteq \bigcup_i D_i$, we have $b \subseteq \bigcup_i d_i \cup \bigcup_j b_j$. It might happen that the d_i and b_j are not disjoint, but a subset of them is and covers b. So, by the previous case, and since $b_j \subset b$, there is some i such that $b \subseteq d_i$. If $b \cap d_{i,\ell} = \emptyset$, for all i, we indeed have $B \subseteq D_i$ and that concludes the proof. So we may assume that there is some ℓ such that $b \cap d_{i,\ell} \neq \emptyset$. If $b \subseteq d_{i,\ell}$, then $B \cap D_i = \emptyset$; a contradiction. Hence $d_i \subset b$. It follows that $d_{i,\ell} \smallsetminus \bigcup_j \subseteq B \cap d_{i,\ell} \subseteq \bigcup_{i'} D'_i \cap d_{i,\ell} \subseteq \bigcup_{i' \neq i} D_{i'}$, since $D_i \cap d_{i,\ell} = \emptyset$. In other words, $d_{i,\ell} \subseteq \bigcup_{i' \neq i} d_{i'} \cup \bigcup_j b_j$. So, by the case for balls proved above, either $d_{i,\ell} \subseteq b_j$, for some j, or $d_{i,\ell} \subseteq d_{i'}$, for some $i' \neq i$. In the latter case, since $D_i \cap D'_i = \emptyset$, $d'_i \subseteq \bigcup_{\ell'} d_{i,\ell'}$ and hence it is covered by one of the $d_{i,\ell'}$. Recall that the $d_{i,\ell'}$ are disjoint and $d_{i,\ell} \cap d_{i'} \neq \emptyset$ and hence $d_{i,\ell} = d_{i'}$, contradicting that the assumption that the D_i are not nested. It follows that any $d_{i,\ell}$ that intersects b is contained in some b_j, that is $B \subseteq D_i$. ◊

Uniqueness of the decomposition follows from the fact that whenever a Swiss cheese is included in a finite disjoint union of non-nested Swiss cheeses, then it is included in one of those Swiss cheeses. □

We will now describe the structure induced on the residue field and the value group.

Definition 2.2.6. Let T be an $\mathfrak{L}$-theory and D be a $\mathfrak{L}$-definable set. We say that D is stably embedded if for every $M \vDash T$ and every $\mathfrak{L}(M)$-definable $X \subseteq D^n$, X is $\mathfrak{L}(D(M))$-definable.

Definition 2.2.7. Let T be an $\mathfrak{L}$-theory and D be some $\mathfrak{L}'$-structure interpretable[13] in T. We say that D is a pure $\mathfrak{L}'$-structure if any $\mathfrak{L}$-definable $X \subseteq D^n$ is $\mathfrak{L}'$-definable.

In particular, if D is also stably embedded, any $\mathfrak{L}(M)$-definable subset of D^n is $\mathfrak{L}'(D(M))$-definable.

Fix $M \vDash \mathrm{ACVF}$ and $A \leqslant M$.

Proposition 2.2.8. *If $X \subseteq \mathbf{k}^n$ is $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-definable, then it is $\mathfrak{L}_{\mathrm{rg}}(\mathbf{k}(A))$-definable. In particular, the residue field $\mathbf{k}$ is a stably embedded pure ring.*

Proof. By elimination of quantifiers (theorem 2.1.11), and since $\mathfrak{L}_{\mathrm{rg}}(\mathbf{k}(A))$-definable sets are closed under Boolean combinations, it suffices to consider atomic formulas. So we may assume X is defined by $R(x, \rho(P(a), Q(a)), \alpha) = 0$ where x, y, z are tuples, $R \in \mathbb{Z}[x, y, z]$, $P, Q \in \mathbb{Z}[t]$ are y-tuples, $a \in \mathbf{K}(A)^t$ and $\alpha \in \mathbf{k}(A)^z$ is a tuple. Since $\rho(P(a), Q(a)) \in \mathbf{k}(A)$, this is indeed an $\mathfrak{L}_{\mathrm{rg}}(\mathbf{k}(A))$-formula. □

Proposition 2.2.9. *If $X \subseteq \mathbf{\Gamma}^n$ is $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(A)$-definable, then it is $\mathfrak{L}_{\mathrm{og}}(\mathbf{\Gamma}(A))$-definable. In particular, the value group $\mathbf{\Gamma}$ is a stably embedded pure ordered monoid.*

Proof. As above, it suffices to consider atomic formulas. So we may assume X is defined by $L(x, \mathrm{v}(P(a)), \gamma) < 0$ where L is a $\mathbb{Z}$-linear function, $P \in \mathbb{Z}[t]$ is a tuple, $a \in \mathbf{K}(M)^t$ and $\gamma \in \mathbf{\Gamma}(M)$ is a tuple. This is indeed an $\mathfrak{L}_{\mathrm{og}}(\mathbf{\Gamma}(A))$-formula. □

Definition 2.2.10. Let T be a $\mathfrak{L}$-theory, two $\mathfrak{L}$-definable sets D_1 and D_2 are orthogonal if for every $M \vDash T$, any $\mathfrak{L}(M)$-definable set $X \subseteq D_1^{n_1} \times D_2^{n_2}$ is a finite union of boxes of the form $Y_1 \times Y_2$ where $Y_i \subseteq D_i^{n_i}$ is $\mathfrak{L}(M)$-definable.

Proposition 2.2.11. *The value group $\mathbf{\Gamma}$ and the residue field $\mathbf{k}$ are orthogonal.*

Proof. Since finite unions of boxes are closed under Boolean combinations, it suffices to consider atomic formulas. But variables from $\mathbf{\Gamma}$ and $\mathbf{k}$ cannot both occur in the same atomic $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-formula. So subsets of $\mathbf{\Gamma}^n \times \mathbf{k}^m$ defined by atomic formulas are either of the form $\mathbf{\Gamma}^n \times Y_2$ for some $Y_2 \subseteq \mathbf{k}^m$ or $Y_1 \times \mathbf{k}^m$ for some $Y_1 \subseteq \mathbf{\Gamma}^n$. □

We continue by describing the algebraic closure in ACVF:

Proposition 2.2.12. *We have $\mathrm{acl}(A) = \mathbf{K}(A)^{\mathrm{a}} \cup C \cdot \mathbf{k}(C)^{\mathrm{a}} \cup \mathbb{Q} \cdot \mathbf{\Gamma}(A)$, where C is the divisible hull of the group generated by $\mathbf{RV}(A)$.*

[13] That is for every sort X of $\mathfrak{L}$ a choice of an $\mathfrak{L}$-definable set X^T, for every function symbol $f : X \to Y$ of $\mathfrak{L}$, an $\mathfrak{L}$-definable function $f^T : \prod_i X_i^T \to Y^T$ and for every relation symbol $R \subseteq X$ of $\mathfrak{L}$, an $\mathfrak{L}$-definable subset of $\prod_i X_i^T$.

Proof. Fix $\gamma \in \mathbf{\Gamma}^{\times}(M)$ and let n be its order in $\mathbf{\Gamma}^{\times}(M)/\mathbf{\Gamma}^{\times}(A)$. Then, if $n \neq \infty$, since $\mathbf{\Gamma}^{\times}(M)$ is torsion free, $\gamma \in \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$. If $n = \infty$, by fact 2.1.2, there is an $\mathfrak{L}(A)$-elementary embedding sending γ to any $\delta \in M^{\star} \succcurlyeq M$ realising the same cut. So $\gamma \notin \mathrm{acl}(A)$. It follows that $\mathbf{\Gamma}(\mathrm{acl}(A)) = \mathbb{Q} \cdot \mathbf{\Gamma}(A)$.

Note that the n-torsion subgroup of $\mathbf{RV}^{\times}$ is exactly the group $\mu_n(\mathbf{k})$ of n-th root of the unit. It follows that $C \subseteq \mathrm{acl}(A)$. Fix $\alpha \in \mathbf{k}(M)$ and let P be its minimal polynomial over $\mathbf{k}(C)$. Then, either $P \neq 0$, in which case, since P has finitely many roots, $\alpha \in \mathrm{acl}(A)$, or $P = 0$, in which case, by fact 2.1.1, there is an $\mathfrak{L}(A)$-elementary embedding sending α to any $\beta \in M^{\star} \succcurlyeq M$ transcendental over $\mathbf{k}(A)$ and hence $\alpha \notin \mathrm{acl}(A)$. So $\mathbf{k}(\mathrm{acl}(A)) = \mathbf{k}(C)^{\mathrm{a}}$.

In particular, $C \cdot \mathbf{k}(C)^{\mathrm{a}} \subseteq \mathbf{RV}(\mathrm{acl}(A))$. Conversely, if $\xi \in \mathbf{RV}(\mathrm{acl}(A))$, the $\mathrm{v}(\xi) \in \mathbf{\Gamma}(\mathrm{acl}(A)) = \mathbb{Q} \cdot \mathbf{\Gamma}(A) = \mathrm{v}(C)$, by proposition 2.1.9.(2). Then $\xi \in \zeta \cdot \mathbf{k}(\mathrm{acl}(A)) \subseteq C \cdot \mathbf{k}(C)^{\mathrm{a}}$, for any $\zeta \in C$ with $\mathrm{v}(\zeta) = \mathrm{v}(\xi)$.

Now, fix $a \in \mathbf{K}(M)$. By proposition 2.2.2, $\mathrm{tp}(a/\mathbf{K}(A)^{\mathrm{a}})$ is the generic of some pseudo Cauchy filter $\mathfrak{B}$. Since non trivial balls are infinite, such a generic is algebraic if and only if $\mathfrak{B}$ contains a singleton, in which case $a \in K(A)^{\mathrm{a}}$. So $\mathrm{acl}(\mathbf{K}(A)) = \mathbf{K}(A)^{\mathrm{a}}$.

Claim 2.2.12.1. *For every $\zeta \in \mathbf{RV}(M) \cup \mathbf{\Gamma}(M)$, $E := \mathbf{K}(\mathrm{acl}(\mathbf{K}(A)\zeta)) \subseteq \mathbf{K}(A)^{\mathrm{a}}$.*

Proof. Let us first assume that $\zeta \in \mathbf{RV}(M)$. Since $\mathrm{rv}^{-1}(\zeta)$ is infinite, there exists $c \in M^{\star} \succcurlyeq M$ transcendental over E, with $\mathrm{rv}(c) = \zeta$. If $\zeta \in \mathbf{\Gamma}^{\times}(M)$, we find such a c with $\mathrm{v}(c) = \zeta$. In both cases, we have $\zeta \in \mathrm{acl}(\mathbf{K}(A)c)$ and hence $E \subseteq \mathbf{K}(\mathrm{acl}(\mathbf{K}(A)c)) = \mathbf{K}(A)(c)^{\mathrm{a}}$. Now $1 = \mathrm{trdeg}(\mathbf{K}(A)(c)/\mathbf{K}(A)) = \mathrm{trdeg}(E(c)/\mathbf{K}(A)) = \mathrm{trdeg}(E/\mathbf{K}(A)) + \mathrm{trdeg}(c/E) = \mathrm{trdeg}(E/\mathbf{K}(A)) + 1$, so $E \subseteq \mathbf{K}(A)^{\mathrm{a}}$. ◊

It follows, by induction on an enumeration of $\mathbf{RV}(A) \cup \mathbf{\Gamma}(A)$, that $\mathbf{K}(\mathrm{acl}(A)) = \mathbf{K}(A)^{\mathrm{a}}$. □

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Corollary 2.2.13. *Any $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}(M)$-definable function $f : \mathbf{k}^n \times \mathbf{\Gamma}^m \to \mathbf{K}$ has finite image.*

Proof. Let $Y = f(\mathbf{k}^n \times \mathbf{\Gamma}^m)$. Then, for every elementary $h : M \to M^{\star}$, $h_{\star}Y(M^{\star}) \subseteq \mathrm{acl}(\mathbf{K}(h(M)) \cup \mathbf{k}(M^{\star}) \cup \mathbf{\Gamma}(M^{\star})) = \mathbf{K}(h(M))$, by proposition 2.2.12. If Y is infinite, then, by compactness (corollary B.0.13) there exists an elementary $h : M \to M^{\star}$ such that $h_{\star}Y(M^{\star}) \smallsetminus \mathbf{K}(h(M)) \neq \emptyset$. It follows that Y is finite. □

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Corollary 2.2.14. *Let $K \leqslant L$ be some algebraic extension and v_1, v_2 valuations on L extend a common valuation v of K. Then v_1 and v_2 are not dependent (unless they are equivalent).*

Proof. If there are dependent, by lemma 1.1.6, we may assume that there is an ordered monoid embedding $g : v_1 L \to v_2 L$. Let $\Delta = \ker(g)$. It is a convex subgroup of $v_1 L \subseteq \mathbb{Q} \cdot vK$, that last inclusion follows from proposition 2.2.12. In particular, $\Delta \subseteq \mathbb{Q} \cdot (\Delta \cap vK)$. However, since $v_1|_K = v = v_2|_K$, $\Delta \cap vK = \{0\}$ and hence $\Delta = 0$ and the v_i are equivalent. □

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3. Henselian fields

3.1. Definably closed fields

Let us now describe the definable closure in the field sort. Let $M \vDash \mathrm{ACVF}$ and $A \leqslant \mathbf{K}(M)$ be a subfield.

Remark 3.1.1. Let R be any ring $P = \sum_i a_i x^i \in R[x]$, then write $P(x+y) = \sum_i a_i (x+y)^i = \sum_i a_i \sum_j \binom{i}{j} \cdot x^i y^j = \sum_j (\sum_i \binom{i}{j} a_i x^i) y^j =: \sum_j P_j(x) y^j$. Note that $j! \cdot P_j(x) = P^{(j)}(x)$, the j-th derivative of P. In particular, $P_1 = P'$ and, in characteristic zero, $P_j = P^j/j!$.

Lemma 3.1.2. *Let b a non trivial ball of M and $P \in K(M)[x]$. The following are equivalent:*

(i) for all $x, y \in b$, $\mathrm{rv}(P(y) - P(x)) = \mathrm{rv}(y-x)\mathrm{rv}(P'(x))$;
(ii) for every $x \in b$, $\mathrm{rv}((P(x) - P(y))/(x-y))$ is constant on $b \smallsetminus \{x\}$;
(iii) $\mathrm{rv}((P(x) - P(y))/(x-y))$ is constant for $x \neq y \in b$;
(iv) $\mathrm{rv}(P'(x))$ is constant on b and if it is zero then P is constant;

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Proof. The implication (iii) ⇒(ii) is obvious.

(ii)⇒(iv) For every $x, y \in b$, we have $\mathrm{rv}(P'(x)) = \mathrm{rv}((P(x) - P(y))/(x-y)) = \mathrm{rv}(P'(y))$. If $\mathrm{rv}(P'(x)) = 0$, then for every $y \in b$, $\mathrm{rv}(P(y) - P(x)) = \mathrm{rv}(y-x)\mathrm{rv}(P'(x)) = 0$ and hence P is constant.

(iv)⇒(iii) We may assume that P is monic non constant and hence $\mathrm{rv}(P'(a)) \neq 0$ for any $a \in b$. Let $P(x) - P(a) = \prod_i (x - a_i)$ with $a_0 = a$. If a_i is a multiple root of P, then $P'(a_i) = 0$ and hence $a_i \notin b$. It follows that the $a_i \in b$ are distinct. For any x distinct from the a_i, we have $\mathrm{rv}(P'(x)/(P(x) - P(a))) = \mathrm{rv}(\sum_i \prod_{j \neq i}(x-a_j)/\prod_j (x-a_j)) = \mathrm{rv}(\sum_i (x-a_i)^{-1})$. If x is closest to a unique a_i, $\mathrm{rv}(P'(x)/(P(x) - P(a))) = \mathrm{rv}(x-a_i)^{-1}$ and hence $\mathrm{rv}(P(x) - P(a)) = \mathrm{rv}(x-a_i)\mathrm{rv}(P'(x))$.

So there only remains to show that $a = a_0$ is the only $a_i \in b$. Assume not. Let $\gamma_{a_i \neq a_j \in b} := \mathrm{v}(a_i - a_j)$ and I be such that for all $i \neq j \in I$, $\mathrm{v}(a_i - a_j) = \gamma$ and for all $\ell \neq I$, $\mathrm{v}(a_i - a_\ell) < \gamma$. For every $i \in I$, fix $e_i \in b \smallsetminus a_i$ which is closest to a_i than to any other a_j. By the above $\mathrm{rv}(P(e_i) - P(a)) = \mathrm{rv}(\prod_j (e_i - a_j)) = \mathrm{rv}(e_i - a_i)\mathrm{rv}(P'(e_i))$ and hence $\mathrm{rv}(P'(a)) = \mathrm{rv}(P'(e_i)) = \prod_{j \neq i} \mathrm{rv}(e_i - a_j) = \prod_{j\neq i} \mathrm{rv}(a_i - a_j)$. Since for every $i_1, i_2 \in I$ and $j \notin I$, $\mathrm{rv}(a_{i_1} - a_j) = \mathrm{rv}(a_{i_2} - a_j)$, it follows that $\prod_{j \in I \smallsetminus \{i\}} \mathrm{rv}(a_i - a_j)$ does not depend on $i \in I$. Fix $i_0 \in I$ and let $c_i = (a_i - a_{i_0})/c$, where $\mathrm{v}(c) = \gamma$. We have that the $\prod_{j \in I \smallsetminus \{i\}} \mathrm{rv}(c_i - c_j) = \prod_{j \in I_0 \smallsetminus \{i\}} \mathrm{res}(c_i - c_j) = Q'(\mathbf{k}(c_i))$ are equal, where $Q := \prod_{i \in I}(x - \mathrm{res}(c_i))$. So $Q' - Q'(0)$ is a degree $|I| - 1 > 0$ polynomial with $|I|$ roots. This is a contradiction and (ii) is proved.

(ii)⇒(i) We have $P(x+e) = P(x) + eP'(x) + e^2 Q(x,e)$, with $Q = \sum_i Q_i(x) e^i \in K[x,e]$. For any e sufficiently close to 0, $v(eQ(x,e)) \geqslant \min_i\{v(Q_i(x)) + (i+1)v(e)\} > \gamma$, for any $\gamma \in vK$. It follows that if y is sufficiently close to x and $P'(x) \neq 0$, $v((P(y) - P(x))/(y-x) - P'(x)) = v((y-x)Q(x, y-x))$ is arbitrarily large. Since $\mathrm{v}(P(y) - P(x)/(y-x))$ is constant, we must have $P'(x) \neq 0$ and hence, $v((P(y) - P(x))/(y-x) - P'(x)) = v((y-x)Q(x,y-x)) > v(P'(x))$, *i.e.* $\mathrm{rv}(P(y) - P(x)/(y-x)) = \mathrm{rv}(P'(x))$. By (ii), this equality holds for any $y \in b$. □

For any field K, let K^{s} denote its separable closure — that is its maximal separable extension inside K^{a}. Let also $K^{p^{-\infty}}$ denote its perfect hull — If the characteristic p of K is positive, $K^{p^{-\infty}} = \bigcup_{n>0} K^{p^{-n}}$. otherwise $K^{p^{-\infty}} = K$. We have $K^{\mathrm{a}} \simeq K^{\mathrm{s}} \otimes K^{p^{-\infty}}$.

Proposition 3.1.3. *The following are equivalent:*

(i) $A = \mathrm{dcl}(A) \cap A^{\mathrm{s}}$;

(ii) $\mathrm{dcl}(A) = A^{p^{-\infty}}$

(iii) for every $a \in A^{\mathrm{a}}$, $\mathrm{tp}_{\mathfrak{L}_{\mathrm{rg}}}(a/A) \vdash \mathrm{tp}_{\mathfrak{L}_{\mathbf{RV},\Gamma}}(a/A)$;

(iv) the valuation $\mathrm{v}|_A$ *has a unique extension to* A^{a} *(up to equivalence);*

(v) the valuation $\mathrm{v}|_A$ *has a unique extension to* A^{s} *(up to equivalence);*

(vi) for every $P = X^d + \sum_{i<d} a_i X^i$ *with* $a_{d-1} \in \mathcal{O}(A)^\times$ *and* $a_i \in \mathfrak{m}(A)$*, for* $i < d-1$*, there exists a (necessarily unique)* $c \in \mathcal{O}(A)^\times$ *with* $P(c) = 0$;

(vii) for every $P \in \mathcal{O}(A)[x]$*, with* $\mathrm{res}(P)(0) = 0$ *and* $\mathrm{res}(P')(0) \neq 0$*, there exists a (necessarily unique)* $c \in \mathfrak{m}(A)$ *such that* $P(c) = 0$;

(viii) for every $P \in \mathcal{O}(A)[x]$ *and* $a \in \mathcal{O}(A)$*, with* $\mathrm{v}(P(a)) > 2 \cdot \mathrm{v}(P'(a))$*, there exists a (necessarily unique)* $c \in A$ *with* $\mathrm{v}(c-a) > \mathrm{v}(P'(a))$ *and* $P(c) = 0$;

(ix) for every $P \in A[x]$*,* $a \in A$*,* $\gamma \in \mathrm{v}(A)$ *such that* $\mathrm{v}(P(a)) > \mathrm{v}(P'(a)) + \gamma$ *and, for every distinct* $x, y \in \mathring{\mathrm{B}}(a,\gamma)$*,* $\mathrm{rv}((P(x)-P(y))/(x-y)) = \mathrm{rv}(P'(x))$*, there exists a (necessarily unique)* $c \in A$ *with* $\mathrm{v}(c-a) > \gamma$ *and* $P(c) = 0$.

(x) for every irreducible $P \in \mathcal{O}(A)[x]$*, there exists* $\alpha \in \mathbf{k}(A)$ *and* $Q \in \mathbf{k}(A)[x]$ *irreducible such that* $\mathrm{res}(P) = \alpha$ *or* $\mathrm{res}(P) = \alpha \cdot Q^{\deg(P)/\deg(Q)}$;

(xi) for every $P \in \mathcal{O}(A)[x]$ *and* $Q_0, R_0 \in \mathbf{k}(A)[x]$ *such that* $\mathrm{res}(P) = Q_0 \cdot R_0 \neq 0$ *and* $\gcd(Q_0, R_0) = 1$*, there exists* $Q, R \in \mathcal{O}(A)[x]$ *with* $P = Q \cdot R$*,* Q *is an exact lift of* Q_0 *and* $\mathrm{res}(R) = R_0$.

Proof.

(i)⇒(ix) Let $P = c\prod_i (x - e_i)$ and $b := \mathring{\mathrm{B}}(a,\gamma)$. If, for every i, $e_i \notin b$, then $\mathrm{v}(P'(a)/P(a)) = \mathrm{v}(\sum_i (a-e_i)^{-1}) \geqslant \min_i -\mathrm{v}(a-e_i) \geqslant -\gamma$, *i.e.* $\mathrm{v}(P(a)) \leqslant \mathrm{v}(P'(a)) + \gamma$. It follows that some root c of P is in b. Since $\mathrm{v}(P(a)) > \mathrm{v}(P'(a)) + \gamma$, $\mathrm{rv}(P'(a)) = \mathrm{rv}(P'(c)) \neq 0$ — the second equality follows from lemma 3.1.2. So c is a simple root of P and hence of its minimal polynomial over A, and $c \in A^{\mathrm{s}}$. Moreover, if $c, c' \in b$ are distinct roots of P, then $0 = \mathrm{rv}((P(c) - P(c'))/(c-c')) = \mathrm{rv}(P'(c))$, a contradiction. So $c \in \mathrm{dcl}(A) \cap A^{\mathrm{s}} = A$.

(ix)⇒(viii) Since $P'(x) = P'(a) + (x-a) \cdot R(x,a)$ with $R(x,y) \in \mathcal{O}(A)[x,y]$, for every $x \in \mathring{\mathrm{B}}(a, \mathrm{v}(P'(a)))$, $\mathrm{v}(P'(x) - P'(a)) = \mathrm{v}((x-a)R(x,a)) > \mathrm{v}(P'(a))$ and hence $\mathrm{rv}(P'(x)) = \mathrm{rv}(P'(a))$. Note also that $P(y) = P(x) + (y-x) \cdot P'(x) + (y-x)^2 \cdot Q(x,y)$, where $Q \in \mathcal{O}(A)[x,y]$. So, for every $x, y \in \mathring{\mathrm{B}}(a, \mathrm{v}(P'(a)))$, since $\mathrm{v}((y-x) \cdot Q(x,y)) > \mathrm{v}(P'(a)) = \mathrm{v}(P'(x))$, $\mathrm{rv}((P(x) - P(y))/(x-y)) = \mathrm{rv}(P'(x))$[14]. By (ix), we find a (unique) c such that $P(c) = 0$ and $\mathrm{v}(c-a) > \mathrm{v}(P'(a))$.

(viii)⇒(vii) We have $\mathrm{v}(P(0)) > 0 = 2 \cdot \mathrm{v}(P'(0))$, so, by (viii), there is a (unique) c with $P(c) = 0$ and $\mathrm{v}(c) > \mathrm{v}(P'(0)) = 0$.

[14] We could use lemma 3.1.2 to deduce that, but it is very much overkill.

(vii)⇒(vi) Let $Q(x) = P(x - a_{d-1})$. Then $\mathrm{res}(Q) = (x - \alpha + \alpha)(x - \alpha)^{d-1}$ where $\alpha = \mathrm{res}(a_{d-1}) \neq 0$. So $\mathrm{res}(Q(0)) = 0$ and $\mathrm{res}(Q'(0)) = \mathrm{res}(Q)'(0) = (-\alpha)^{d-1} \neq 0$. By (vii), there exists $c \in \mathfrak{m}(A)$ such that $P(c - a_{d-1}) = Q(c) = 0$. Note that $\mathrm{res}(c - a_{d-1}) = -\alpha \neq 0$, so $c - a_{d-1} \in \mathcal{O}(A)^{\times}$.

(vi)⇒(v) Let $A \leqslant F \leqslant A^{\mathrm{s}}$ be finite Galois. It suffices to show that $\mathrm{v}|_A$ extends to F uniquely. Let $D := \{\sigma \in \mathrm{aut}(A^{\mathrm{s}}/A) : \mathrm{v} \circ \sigma$ is equivalent to $\mathrm{v}\}$ and $L := F^D$. Note that, by the conjugation theorem (corollary 2.1.18), there are at most $[F : A]$ extensions of $\mathrm{v}|_A$ to F, up to equivalence. Let us denote them $(v_i)_{i<n}$ where $v_0 = \mathrm{v}$. Note that, if $v_i|_L$ is equivalent to v, by the conjugation theorem (corollary 2.1.18) there exists $\sigma \in \mathrm{aut}(F/L) = D$ such that v_i is equivalent to $\mathrm{v} \circ \sigma$, which is equivalent, by definition of D to v. So no $v_i|_L$ is equivalent — or dependent by corollary 2.2.14 — to v. The weak approximation theorem (theorem 1.1.12) now allows us to find $b \in \mathcal{O}(L)^{\times}$ such that, for every $i > 0$, $v_i(b) > 0$.

Let $(b_j)_{j<d} \in F$ be the set of A-conjugates of $b = b_0$. For every $j > 0$, there is some $\sigma \in \mathrm{aut}(F/A) \smallsetminus D$ such that $\sigma(b) = b_j$. Since $\sigma \notin D$, $\mathrm{v} \circ \sigma$ is equivalent to some v_i, with $i > 0$, and hence $\mathrm{v}(b_i) = \mathrm{v}(\sigma(b)) > 0$. Let $P = \sum_{i<d} a_i x^i$ be the minimal (monic) polynomial of b over A. Then $\mathrm{v}(a_{d-1}) = \mathrm{v}(\sum_j b_j) = \mathrm{v}(b_0) = 0$ and, for $i < d - 1$, $\mathrm{v}(a_i) = \mathrm{v}(\sum_{J \subseteq d, |J| = i} \prod_{j \in J} b_j) > 0$. By (vi), the unique root b of P in $\mathcal{O}^{\times}$ is in A. It follows that there are no v_i that are not equivalent to v which is thus the unique extension of $\mathrm{v}|_A$ to F.

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(v)⇒(iv) Since $A^{\mathrm{a}} = \bigcup_n (A^{\mathrm{s}})^{p^{-n}}$, the valuation of any $c \in A^{\mathrm{a}}$ is uniquely determined by that of any $c^{p^n} \in A^{\mathrm{s}}$. So the unique extension of $\mathrm{v}|_A$ to A^{s} uniquely extends to A^{a}.

(iv)⇒(iii) For every $\sigma \in \mathrm{aut}(A^{\mathrm{a}}/A)$, $\mathrm{v} \circ \sigma$ extends $\mathrm{v}|_A$ and hence, by (iv), is equivalent to v. In other words, σ induces an $\mathfrak{L}_{\mathbf{RV},\mathbf{\Gamma}}$-automorphism of A^{a}, which is elementary (from M to N) by elimination of quantifiers, theorem 2.1.11. Since any two elements of A^{a} with the same $\mathfrak{L}_{\mathrm{rg}}(A)$-type have the same minimal polynomial over A and are thus $\mathrm{aut}(A^{\mathrm{a}}/A)$-conjugate, (iii) follows.

(iii)⇒(ii) Since $\mathrm{dcl}(A) \subseteq \mathrm{acl}(A) \subseteq A^{\mathrm{a}}$, (iii) implies that $a \in \mathrm{dcl}(A)$ if and only if a is the unique solution to its minimal polynomial over A, *i.e.* $a \in A^{p^{-\infty}}$.

(ii)⇒(i) We have $\mathrm{dcl}(A) \cap A^{\mathrm{s}} = A^{p^{-\infty}} \cap A^{\mathrm{s}} = A$.

(iii)⇒(x) It follows from (iii) that either all the roots of $P = c \prod_{i<d}(x - e_i)$ are in $\mathcal{O}$ or none are in $\mathcal{O}$. If none are in $\mathcal{O}$, then, for every $n < d$, $0 > \mathrm{v}(\prod_i e_i) < \mathrm{v}(\sum_{I \subseteq d, |I| = n} \prod_{i \in I} e_i)$ and hence the only coefficient of P with minimal valuation is the constant one and $\mathrm{res}(P)$ is constant. Otherwise, let Q be the minimal polynomial of some $\mathrm{res}(e_i)$ over $\mathbf{k}(A)$. By (iii), Q is the minimal polynomial of any $\mathrm{res}(e_i)$ over $\mathbf{k}(A)$. So $\mathrm{res}(\prod(x - e_i))$ is a power of Q of degree d, and $\mathrm{res}(P) = \mathrm{res}(c) Q^{d/\deg Q}$.

(x)⇒(xi) We have $P = \prod_i P_i$ where the $P_i \in \mathcal{O}(A)[x]$ are irreducible. By (x), $Q_0 \cdot R_0 = \prod_i \mathrm{res}(P_i) = \prod_i \alpha_i S_i^{m_i}$, where $\alpha_i \in \mathbf{k}(A)^{\times}$ and $S_i \in \mathcal{O}(A)[x]$ is irreducible, or 1. Reordering, we may assume there is a ℓ such that $\gcd(S_i, Q_0) \neq 1$ if and only if $i \leqslant \ell$. Since $\gcd(Q_0, R_0) = 1$, we then have $Q_0 = \beta \prod_{i \leqslant \ell} S_i^{m_i}$, where $\beta \in \mathbf{k}(A)^{\times}$. Let $Q = b \prod_{i \leqslant \ell} (a_i)^{-1} P_i$, where $\mathrm{res}(a_i) = \alpha_i$ and $\mathrm{res}(b) = \beta$, and $R = b^{-1} \prod_{i \leqslant \ell} a_i \prod_{i > \ell} P_i$. Then $Q \cdot R = P$, $\mathrm{res}(Q) = \beta \prod_i \alpha_i^{-1} \alpha_i S_i = Q_0 \neq 0$ and hence $\mathrm{res}(R) = R_0$. Since $m_i \deg(S_i) = \deg(P_i)$ for every $i \leqslant \ell$, we also have $\deg(Q) = \sum_i \deg(P_i) = \sum_i m_i \deg(S_i) = \deg(Q_0)$.

(xi)⇒(vi) Since $\mathbf{k}(P) = (X+\alpha)X^{d-1}$ where $\alpha = \mathrm{res}(a_{d-1}) \neq 0$, by (xi), there exists an exact lift $bX - a \in A[x]$ of $X + \alpha$ which divides P. So $P(a/b) = 0$, $\mathrm{res}(b) = 1$, $\mathrm{res}(a) = -\alpha$ and hence $a/b \in \mathcal{O}^\times$. □

Definition 3.1.4 (Henselian fields). A valued field (K, v) is said to be *henselian* if it the equivalent conditions of proposition 3.1.3 hold.

Proposition 3.1.5 (Hensel's lemma). *Let (K, v) be some valued field. Assume either that:*

(a) K is spherically complete;

(b) $vK \leqslant \mathbb{R}$ and K is complete.

Then (K, v) is henselian.

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Proof. Let us fix $P \in \mathcal{O}(K)[x]$ and $a \in \mathcal{O}(K)$ such that $\mathrm{res}(P(a)) = 0$ and $\mathrm{res}(P'(a)) = 0$. For every $x \in a + \mathfrak{m}(K)$, let $b_x := \overline{\mathrm{B}}(x, v(P(x)))$. Note that for every $x \in a + \mathfrak{m}$, $\mathrm{res}(P(x)) = \mathrm{res}(P(a))$. Thus $v(P(x)) > 0$ and $b_x \subseteq a + \mathfrak{m}$.

Claim 3.1.5.1. *For every $x, y \in a + \mathfrak{m}(K)$, $b_x \cap b_y \neq \varnothing$.*

Proof. We have $P(y) = P(x) + (y-x)P'(x) + (y-x)^2 Q(x,y)$, with $Q \in \mathcal{O}(K)[x,y]$. So $v(y-x) = v((y-x)P'(x) + (y-x)^2Q(x,y)) = v(P(y) - P(x)) \geqslant \min\{v(P(x)), v(P(y))\}$. If $\mathrm{v}(P(x)) \geqslant v(P(y))$, then $x \in b_y$, otherwise $y \in b_x$. ◊

Claim 3.1.5.2. *For every $x \in a + \mathfrak{m}(K)$, there exists $y \in K$ such that $v(y-x) = v(P(x))$ and $v(P(y)) \geqslant 2 \cdot v(P(x))$.*

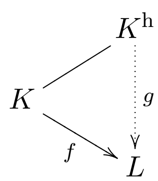
Proof. Take $y = x - P(x)/P'(x)$. Then $v(y-x) = v(-P(x)) - v(P'(x)) = v(P(x))$ and $v(P(y)) = v((-P(x)/P'(x))^2 Q(x,y)) \geqslant 2 \cdot v(P(x))$. ◊

If K is spherically complete, the pseudo Cauchy filter $\mathfrak{B}$ generated by the b_x has an accumulation point $c \in K$. If $vK \leqslant \mathbb{R}$, it follows from claim 3.1.5.2, that the b_x can have arbitrarily large radiuses in vK, so, $\mathfrak{B}$ is a Cauchy filter and, if K is complete, $\mathfrak{B}$ converges to some $c \in K$. In either cases, by claim 3.1.5.2, we find $e \in b_c$ such that $v(e-c) = v(P(c))$ and $v(P(e)) \geqslant 2 \cdot v(P(c))$. By hypothesis, $c \in b_e$ and hence $v(P(c)) = v(e-c) \geqslant v(P(e)) \geqslant 2 \cdot v(P(c))$. Since $v(P(c)) > 0$, we must have $P(c) = 0$. □

Definition 3.1.6. Let (K, v) be some valued field. We define $K^{\mathrm{h}} := \mathrm{dcl}(K) \cap K^{\mathrm{s}}$, the *henselian closure* of K — inside some fixed model of ACVF containing K.

Remark 3.1.7. Let $D := \{\sigma \in \mathrm{aut}(K^{\mathrm{s}}/K) : v \circ \sigma \simeq v\}$, for some extension v to K^{s}. Then $K^{\mathrm{h}} = (K^{\mathrm{s}})^D$.

Proposition 3.1.8 (Universal property of K^{h}). *Let (K, v) be a valued field, (L, w) be henselian and $f : K \to L$. Then there is a unique morphism $g : K^{\mathrm{h}} \to L$ such that:*



commutes.

Proof. Let $K^{\mathrm{h}} \leqslant M \vDash \mathrm{ACVF}$ and $L \leqslant N \vDash \mathrm{ACVF}$. By elimination of quantifiers, theorem 2.1.11, there exists $g : K^{\mathrm{h}} \to N$. Since $K^{\mathrm{h}} = \mathrm{dcl}(K) \cap K^{\mathrm{s}}$ and $L = \mathrm{dcl}(L) \cap L^{\mathrm{s}}$, $g(K^{\mathrm{h}}) = \mathrm{dcl}(f(K)) \cap f(K)^{\mathrm{s}} \subseteq L$. If g' is another such map, then $g'(K^{\mathrm{h}}) = \mathrm{dcl}(f(K)) \cap f(K)^{\mathrm{s}} = g(K^{\mathrm{h}})$ and $g^{-1} \circ g' \in \mathrm{aut}(K^{\mathrm{h}}/K) \leqslant \mathrm{aut}(\mathrm{dcl}(K)/K)$ is the identity. □

We now want to describe vK^{h} and $K^{\mathrm{h}}v$. But we will first need to construct spherically complete maximal extensions.

Lemma 3.1.9. *Let $M \vDash \mathrm{ACVF}$ and $A \leqslant \mathbf{K}(M)$. The following are equivalent:*

(i) A is spherically complete in M : any pseudo Cauchy filter $\mathfrak{B}$ over $\mathbf{K}(M)$ with an accumulation point in M has one in A;

(ii) A is maximally complete in M : any immediate intermediary extension $A \leqslant F \leqslant \mathbf{K}(M)$ is trivial.

Proof.

(i)⇒(ii) Fix some $c \in F$ and let $\mathfrak{B}$ be maximal pseudo Cauchy filter over A that accumulates at c. By (i) it accumulates at some $a \in A$. If $c \neq a$, then there exists $e \in A^{\times}$ such that $\mathrm{rv}(e) = \mathrm{rv}(c-a)$. So $v(c-a-e) > v(e)$ and $c \in b := \mathring{\mathrm{B}}(a+e, v(e), \in)\mathfrak{B}$. So $v(e) = v(a-(a+e)) > v(e)$, a contradiction. It follows that $c = a \in A$.

(ii)⇒(i) Let $\mathfrak{B}$ be some pseudo Cauchy filter over A that accumulates at some $c \in M$ and let $P \in A[x]$ be minimal such that $0 \in \overline{P_{\star}\mathfrak{B}}$. By proposition 2.1.8.(2) there exists some $e \in \overline{\mathfrak{B}} \cap M$ such that $P(e) = 0$ — if $P = 0$, take $e = c$. By proposition 2.1.8.(1), the extension $A \leqslant A[e]$ is immediate and $e \in A$. □

Note that (i) ⇒(ii) holds even when $M \nvDash \mathrm{ACVF}$.

Corollary 3.1.10. *Let K be a valued field. The following are equivalent:*

(i) K is spherically complete;

(ii) K is maximally complete: any immediate extension $K \leqslant L$ is trivial.

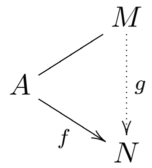
Proof. By compactness (theorem B.0.9) and the fact that every valued field embeds in a model of ACVF, K is spherically (respectively maximally) complete if and only if it is in any model of ACVF. □

Corollary 3.1.11. *Let (K, v) be spherically complete with algebraically closed residue field and divisible valued group. Then K is algebraically closed.*

Proof. Since $vK^{\mathrm{a}} = Q \cdot vK = vK$ an $K^{\mathrm{a}}v = (Kv)^{\mathrm{a}} = Kv$, the extension $K \leqslant K^{\mathrm{a}}$ is immediate and hence trivial. □

Example 3.1.12. For every $k \vDash \mathrm{ACF}_p$ and $\Gamma \vDash \mathrm{DOAG}$, $k((\Gamma)) \vDash \mathrm{ACVF}_{p,p}$. There are also examples of maximally complete models of $\mathrm{ACVF}_{0,p}$ but they are more complicated to build.

Proposition 3.1.13. *Let $M, N \vDash$ ACVF, $\mathbf{k}(M) \cup \mathbf{\Gamma}(M) \subseteq A \leqslant M$ and $f : A \to N$. If N is spherically complete, then there exists an embedding $g : M \to N$ such that:*



commutes.

Proof, cf. proposition 2.1.10. Let $C \leqslant A$ and $g : C \to N$ be maximal such that g extends f. By proposition 2.1.5, $\mathbf{k}(M) \leqslant \mathrm{res}(\mathbf{K}(C))$ and by proposition 2.1.3, $\mathbf{\Gamma}(M) \leqslant \mathrm{v}(\mathbf{K}(C))$. So $\mathbf{K}(C) \leqslant \mathbf{K}(M)$ is immediate, but, by proposition 2.1.8, $\mathbf{K}(C)$ is spherically closed in M. It follows, by lemma 3.1.9, that $C = M$. □

Proposition 3.1.14. *Let (K, v) be a valued field. There exists an embedding $f : (K, v) \to (L, w)$ with L spherically complete and $f(K) \leqslant L$ immediate. If, moreover, vK is divisible and Kv is algebraically closed, it is unique up to isomorphism.*

Proof. Let $M \vDash$ ACVF be $|K|^+$-saturated of the same characteristic and residue characteristic as K. By proposition 2.1.10, the embedding of the prime field of K into M extends to an embedding $f : K \to M$. Let $f(K) \leqslant L \leqslant \mathbf{K}(M)$ be a maximal immediate extension of K. Then L is spherically complete in M. However, any pseudo Cauchy filter over L is generated by as set of cardinal at most $|vL| = |vK| < |K|^+$, so it has an accumulation point in M. So L is spherically complete.

Now, if v trivial, then $L = K$ is indeed unique. Otherwise, if vK is divisible and Kv is algebraically closed, by corollary 3.1.11, $L \vDash$ ACVF and, given any $g : K \to F$ with F spherically complete and $g(K) \leqslant F$ immediate, by proposition 3.1.13, we find $h : F \to L$ such that $h \circ g = f$. But we have $\mathrm{rv}(L) = \mathrm{rv}(f(K)) = \mathrm{rv}(h \circ g(F))$and hence $h \circ g(F) \leqslant L$ being immediate, it is trivial. □

Corollary 3.1.15. *Let (K, v) be a valued field. The extension $K \leqslant K^{\mathrm{h}}$ is immediate.*

Proof. By proposition 3.1.14, K admits a spherically complete immediate extension L. By proposition 3.1.5, L is henselian and hence, by proposition 3.1.8, K^{h} embeds into L. It follows that $\mathrm{rv}(K) \leqslant \mathrm{rv}(K^{\mathrm{h}}) \leqslant \mathrm{rv}(L) = \mathrm{rv}(K)$. □

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3.2. Elimination of quantifiers in characteristic zero

Definition 3.2.1. Let $\mathfrak{L}_{\mathbf{RV}}$ be the language with:

- a sort $\mathbf{K}$ with the ring language $(+, -, 0, \cdot, 1)$;
- for every $n \in \mathbb{Z}_{>0}$, a sort $\mathbf{RV}_n$ with two constants $0, 1$, a binary function $\cdot$ and a ternary predicate $\oplus$;
- for every $n > 0$, a map $\mathrm{rv}_n : \mathbf{K} \to \mathbf{RV}_n$;
- for every $m, n > 0$ with $n | m$, a map $\mathrm{rv}_{n,m} : \mathbf{RV}_m \to \mathbf{RV}_n$.

Any valued field (K, v) can be made into a $\mathfrak{L}_{\mathbf{RV}}$-structure by interpreting $\mathbf{K}$ as the field K, $\mathbf{RV}_n$ as the multiplicative monoid $K/1 + n\mathfrak{m}$, the maps rv_n and $\mathrm{rv}_{n,m}$ as the canonical projections, 0 as $\mathrm{rv}_n(0)$ and $\oplus$ as the trace of the graph of addition on $\mathbf{RV}_n$.

Definition 3.2.2. Let VF denote the $\mathfrak{L}_{\mathbf{RV}}$-theory of valued fields, Hen_0 that of characteristic zero henselian fields and $\mathrm{Hen}_{0,0}$ that of residue characteristic zero henselian fields.

Remark 3.2.3. Let $M \vDash \mathrm{VF}$ with residue characteristic p and $n|m$ be positive integers. If $p = 0$ or $\gcd(p, m/n) = 1$, then $m/n \in \mathcal{O}^\times$ and $\mathrm{rv}_{n,m}$ is an isomorphism. In particular, If $p = 0$ all the $\mathbf{RV}_n$ are canonically isomorphic to $\mathbf{RV} = \mathbf{RV}_1$. If $p > 0$, then every $\mathbf{RV}_n$ is canonically isomorphic to $\mathbf{RV}_{p^{v_p(n)}}$.

Notation 3.2.4. Let $M \vDash \mathrm{VF}$ and $(\zeta_i)_{i<n} \in \mathbf{RV}(M)$. We denote by $\bigoplus_{i<n} \zeta_i = \{\mathrm{rv}(\sum_i x_i) : \mathrm{rv}(x_i) = \zeta_i\}$. If $\bigoplus_{i<n} \zeta_i$ is a singleton $\{\xi\}$, we say that $\bigoplus_{i<n} \zeta_i$ is well defined and we write $\bigoplus_{i<n} \zeta_i = \xi$. If $P = \sum_i a_i x^i \in \mathbf{K}(M)[x]$ and $\zeta \in \mathbf{RV}^x$, we write $\mathrm{rv}(P)(\zeta) = \bigoplus_i \mathrm{rv}(a_i)\zeta^i$.

Lemma 3.2.5. *Let $M \vDash \mathrm{VF}$ and $(\zeta_i)_{i<n} \in \mathbf{RV}(M)$ and $\gamma = \min_i \mathrm{v}(\zeta_i)$. Then one (and only one) of the following holds:*
- *$\bigoplus_{i<n} \zeta_i = \{\xi \in \mathbf{RV} : \mathrm{v}(\xi) > \gamma\}$;*
- *$\bigoplus_{i<n} \zeta_i = \xi$ and $\mathrm{v}(\xi) = \gamma$.*

Proof. Let us first assume that there are some x_i with $\mathrm{rv}(x_i) = \zeta_i$ and $\mathrm{v}(\sum_i x_i) = \gamma$. Then for every $m_i \in \mathfrak{m}$, $\mathrm{v}(\sum_i(x_i(1+m_i)) - \sum_i x_i) = \mathrm{v}(\sum_i x_i m_i) > \min_i \mathrm{v}(x_i) = \gamma = \mathrm{v}(\sum_i x_i)$. It follows that $\mathrm{rv}(\sum_i(x_i(1+m_i))) = \mathrm{rv}(\sum_i x_i)$ and hence $\bigoplus_i \zeta_i = \mathrm{rv}(\sum_i x_i)$.

On the other hand, if $\mathrm{v}(\sum_i x_i) > \gamma = \mathrm{v}(x_{i_0})$, then for every $m \in \gamma\mathfrak{m}$, $n := (m - \sum_i x_i)/x_{i_0} \in \mathfrak{m}$ and $\sum_{i \neq i_0} x_i + x_{i_0}(1+n) = m$ and hence $\bigoplus_i \zeta_i = \mathrm{rv}(\gamma\mathfrak{m})$. ☐

Lemma 3.2.6. *Let $M \vDash \mathrm{Hen}_{0,0}$, $P \in \mathbf{K}(M)[x]$ and $\alpha \in \mathbf{RV}(M)$. The following are equivalent:*

(i) there exists $n \in \mathbb{Z}_{\geqslant 0}$, with $n \leqslant \deg(P)$, such that $0 \in \mathrm{rv}(P^{(n)})(\alpha)$;

(ii) there exists $n \in \mathbb{Z}_{\geqslant 0}$, with $n \leqslant \deg(P)$ and $a \in \mathbf{K}(M)$ such that $P^{(n)}(a) = 0$ and $\mathrm{rv}(a) = \alpha$.

Proof. Since $\mathrm{rv}(P(a)) \in \mathrm{rv}(P)(\mathrm{rv}(a))$, (i) follows from (ii). So let us assume (i) and let $a \in \mathbf{K}(M)$ such that $\mathrm{rv}(a) = \alpha$. Let n be maximal such that $0 \in \mathrm{rv}(P^{(n)})(\alpha)$. Replacing $P = \sum_i c_i x^i$ by $P^{(n)}$, we may assume that $n = 0$. By lemma 3.2.5 and maximality of n, $\mathrm{v}(P'(a)) = \mathrm{v}(\sum_{i>0} i c_i a^{i-1}) = \min_i \mathrm{v}(i c_i a^{i-1}) = \min_i \mathrm{v}(c_i) + (i-1)\mathrm{v}(a)$. By lemma 3.2.5, we also have $\mathrm{v}(P(a)) > \min_i\{\mathrm{v}(c_i) + i\mathrm{v}(a)\} = \min\{\mathrm{v}(c_0), \mathrm{v}(P'(a)) + \mathrm{v}(a)\}$. If $\mathrm{v}(c_0) < \mathrm{v}(P'(a)) + \mathrm{v}(a)$, then $\mathrm{v}(P(a)) = \mathrm{v}(c_0) = \min_i(\mathrm{v}(c_i) + i\mathrm{v}(a))$, a contradiction. It follows that $\mathrm{v}(P(a)) > \mathrm{v}(P'(a)) + \mathrm{v}(a)$. Since, $\mathrm{rv}(P'(x))$ is constant on $\mathrm{rv}^{-1}(\alpha) = \mathring{\mathrm{B}}(a, \mathrm{v}(a))$ and $\mathbf{K}(M)$ is henselian, by lemma 3.1.2 and proposition 3.1.3.(ix), we may assume that $P(a) = 0$. ☐

Corollary 3.2.7. *Let $M \vDash \mathrm{Hen}_{0,0}$, $P \in \mathbf{K}(M)[x]$ and $\mathfrak{B}$ a pseudo Cauchy filter over $\mathbf{K}(M)$. Then one of the following holds:*
- *there exists $0 \leqslant n \leqslant \deg(P)$ and a root of $P^{(n)}$ in $\overline{\mathfrak{B}}$ — in which case $0 \in \overline{P_\star^{(n)}\mathfrak{B}}$;*
- *$\mathrm{rv} \circ P$ is eventually constant on $\mathfrak{B}$.*

Proof. Let X be the set of roots of the $P^{(n)}$ in M. Since $P^{(\deg(P)-1)}$ is linear, $X \neq \varnothing$. If $X \cap \overline{\mathfrak{B}} = \varnothing$, then there is some $b \in \mathfrak{B}$ such that $b \cap X = \varnothing$. Let $a \in X$ be such that $\gamma := \mathrm{v}(b - a)$ is maximal. We may assume that b is an open ball of radius γ. Then $b = \mathrm{rv}^{-1}(\alpha) + a$ for some $\alpha \in \mathbf{RV}(M)$. Let $Q = P(x-a)$. Since no $Q^{(n)} = P^{(n)}(x-a)$ has a root in $\mathrm{rv}^{-1}(\alpha) \cap M$, by lemmas 3.2.5 and 3.2.6, $\mathrm{rv} \circ Q$ is constant on $\mathrm{rv}^{-1}(\alpha)$. So $\mathrm{rv}(P)$ is constant on b. □

Proposition 3.2.8. *Let $M \vDash \mathrm{Hen}_{0,0}$, $A \leqslant M$ a field, $\alpha \in \mathbf{RV}(A) = \mathbf{RV}(M)$ and $P \in \mathbf{K}(A)[x]$ have minimal degree such that $0 \in \mathrm{rv}(P)(\alpha)$.*

(1) *For every $Q \in \mathbf{K}(A)[x]$ of degree smaller than P and every $a \in \mathrm{rv}^{-1}(\alpha)$, $\mathrm{rv}(Q(a)) = \mathrm{rv}(Q)(\alpha)$.*
(2) *There exists $a \in \mathbf{K}(M)$ with $\mathrm{rv}(a) = \alpha$ and $P(a) = 0$.*
(3) *Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV}}(A)$-isomorphism, by P and α: for every $N \vDash \mathrm{Hen}_{0,0}$, every embedding $f : A \to N$, every $a \in \mathbf{K}(M)$ and $b \in \mathbf{K}(N)$, if:*
   - *$P(a) = 0$ and $\mathrm{rv}(a) = \alpha$;*
   - *$f_\star P(b) = 0$ and $\mathrm{rv}(b) = f(\alpha)$;*

   then f can be extended by sending a to b.

Proof. (1) Since, by minimality, $0 \notin \mathrm{rv}(Q)(\alpha)$, $\mathrm{rv}(Q)(\alpha) = \mathrm{rv}(Q(a))$, for any $a \in \mathrm{rv}^{-1}(\alpha)$, is well-defined.

(2) By lemma 3.2.6, $P^{(n)}$ has a root in $\mathrm{rv}^{-1}(\alpha) \cap \mathbf{K}(M)$, for some $n \leqslant \deg(P)$. By minimality of P, $n = 0$.

(3) By (i) applied to b, $f_\star P$ is the minimal polynomial of b over $f(A)$. So $f|_{\mathbf{K}}$ extends to $g|_{\mathbf{K}}$ sending a to b. Let $g|_{\mathbf{RV}} = f|_{\mathbf{RV}}$. For every $Q \in \mathbf{K}(A)[x]$ of degree smaller than P, $g(\mathrm{rv}(Q(a))) = g(\mathrm{rv}(Q)(\alpha)) = \mathrm{rv}(f_\star Q)(\beta) = \mathrm{rv}(f(Q(b)))$. The second equality (and the fact that the third is well-defined) can be checked by computing the partial sums using the binary $\oplus$. So g is an $\mathfrak{L}_{\mathbf{RV}}$-embedding extending f and sending a to b. □

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Proposition 3.2.9. *Let $M \vDash \mathrm{Hen}_{0,0}$, $A \leqslant M$ a field, $\mathfrak{B}$ be a pseudo Cauchy filter $\mathbf{K}(A)$ and $P \in \mathbf{K}(A)[x]$ have minimal degree such that $0 \in \overline{P_\star \mathfrak{B}}$.*

(1) *For every $Q \in \mathbf{K}(A)[x]$ of degree smaller than P, $\mathrm{rv} \circ Q$ is eventually constant on $\mathfrak{B}$ — in particular is equal to an element of $\mathrm{rv}(\mathbf{K}(A))$;*
(2) *If $P \neq 0$, there exists $a \in \overline{\mathfrak{B}}$ with $P(a) = 0$;*
(3) *Such an a is uniquely determined, up to $\mathfrak{L}_{\mathbf{RV}}(A)$-isomorphism, by $\mathfrak{B}$ and P : for every $N \vDash \mathrm{ACVF}$, embedding $f : A \to N$, $a \in \mathbf{K}(M)$ and $b \in \mathbf{K}(N)$, if:*
   - *$P(a) = 0$ and $a \in \overline{\mathfrak{B}}$;*
   - *$f(P)(b) = 0$ and $b \in \overline{f_\star \mathfrak{B}}$;*

   then f can be extended by sending a to b.

Proof. (1) By minimality of P and corollary 3.2.7, $\mathrm{rv} \circ Q$ is eventually constant on $\mathfrak{B}$.

(2) If there is not root of P in $\mathfrak{B}$, since there are none of its derivatives either, by corollary 3.2.7, $\mathrm{rv} \circ P$ is eventually constant — and hence equal to 0 — on $\mathfrak{B}$. If $P \neq 0$, then $\mathfrak{B}$ contains a singleton $\{a\}$ and $P = x - a$.

(3) This follows immediately from (the proof of) proposition 2.1.8.(3). By (1) applied at b, $f_\star P$ is the minimal polynomial of b over $f(A)$. So let $g|_{\mathbf{K}}$ extend $f|_{\mathbf{K}}$ and send a to b. Let also $g|_{\mathbf{RV}} = f|_{\mathbf{RV}}$. For every $Q \in \mathbf{K}(A)[x]$ of degree smaller than P, let $b \in \mathfrak{B}$ be a ball of

$\mathbf{K}(A)$ such that $\mathrm{rv} \circ Q$ is constant on b and $\mathrm{rv} \circ f_\star Q$ is constant on $f(b)$ and let $c \in b(A)$. Then $g(\mathrm{rv}(Q(a))) = f(\mathrm{rv}(Q(c))) = \mathrm{rv}(f_\star(Q)(f(c))) = \mathrm{rv}(f_\star Q(b)) = \mathrm{rv}(g(Q(a)))$. So g is an $\mathfrak{L}_{\mathbf{RV}}$-embedding extending f and sending a to b. □

Proposition 3.2.10. *Let $M, N \vDash \mathrm{Hen}_{0,0}$, $\mathbf{RV}(M) \subseteq A \leqslant M$ and $f : A \to N$. There exists an embedding $h : N \to N^\star$, which is elementary for any choice of structure on N, and an embedding $g : M \to N$ such that:*

$$\begin{array}{ccc} M & \xrightarrow{g} & N^\star \\ \Big| & & \Big\uparrow h \\ A & \xrightarrow[f]{} & N \end{array}$$

commutes. If N is spherically complete, we can choose $h = \mathrm{id} : N \to N$.

Proof. Let $A \leqslant C \leqslant M$ and $g : C \to N^\star$ and $h : N \to N^\star$ elementary (for any given choice of structure on N) be maximal (as in proposition 2.1.10). If N is spherically complete, we restrict ourselves to considering tuples with $h = \mathrm{id}$. First, note that, since $\mathbf{RV}(M) \leqslant \mathbf{RV}(A)$ and rv is multiplicative, g has a unique extension to $\mathbf{K}(C)_{(0)} \cup \mathbf{RV}(M)$. So $\mathbf{K}(C)$ is a field.

Claim 3.2.10.1. $\mathbf{RV}(M) = \mathrm{rv}(\mathbf{K}(C))$

Proof. Fix some $\alpha \in \mathbf{RV}(M)$. Let $P \in \mathbf{K}(C)[x]$ be minimal such that $0 \in \mathrm{rv}(P)(\alpha)$. By proposition 3.2.8.(2), there exists $a \in \mathbf{K}(M)$ and $b \in \mathbf{K}(N^\star)$ such that $P(a) = 0$, $\mathrm{rv}(a) = \mathrm{rv}(\alpha)$, $g_\star P(b) = 0$ and $\mathrm{rv}(b) = g(\alpha)$. By proposition 3.2.8.(3), g can be extended by sending a to b. By maximality, $a \in C$. ◊

Claim 3.2.10.2. $\mathbf{K}(C)$ *is spherically complete in* $\mathbf{K}(M)$.

Proof. Let $\mathfrak{B}$ be some pseudo Cauchy filter over $\mathbf{K}(C)$ with an accumulation point in $c \in \mathbf{K}(M)$. Let $P \in \mathbf{K}(C)[x]$ be minimal such that $0 \in \overline{P_\star \mathfrak{B}}$. If $P = 0$, set $a := c$ and by compactness, we can find $i : N^\star \to N^\dagger$ and $b \in \overline{(i \circ g)_\star \mathfrak{B}}$. If N is spherically complete, we can find $b \in \overline{g_\star \mathfrak{B}} \cap N$ and we can take $N^\dagger = N^\star = N$ and $i = \mathrm{id}$. If $P \neq 0$, proposition 3.2.9.(2), there exists $a \in \mathbf{K}(M)$ and $b \in \mathbf{K}(N^\star)$ such that $P(a) = 0$, $a \in \overline{\mathfrak{B}}$, $g_\star P(b) = 0$ and $b \in \overline{g_\star \mathfrak{B}}$ — so we can also take $N^\dagger = N^\star$ and $i = \mathrm{id}$. In both cases, by proposition 3.2.9.(3), $i \circ g$ can be extended by sending a to b, so $a \in C$. ◊

By claim 3.2.10.1, the extension $\mathbf{K}(C) \leqslant \mathbf{K}(M)$ is immediate, By claim 3.2.10.2 and lemma 3.1.9, it is trivial. So $C = M$ and the proposition is proved. □

We now wish to extend that result in mixed characteristic. But first we need to introduce coarsened valuations.

Let (K, v) be a valued field and $\Delta \leqslant vK^\times$ be a convex subgroup.

Definition 3.2.11. The *coarsened* valuation associated to Δ is $w : K \to vK/\Delta$.

Let $\pi : vK \to vK/\Delta$ denote the canonical projection.

Proof. Let us check that w is a valuation. We have $w(0) = \pi(v(0)) = \{v(0)\} \neq \Delta = \pi(v(1)) = w(1)$. Also for every $x, y \in K$, $w(xy) = \pi(v(xy)) = \pi(v(x)) + \pi(v(y)) = w(x) + w(y)$ and $w(x+y) = \pi(v(x+y)) \geqslant \min\{\pi(v(x)), \pi(v(y))\} = \min\{w(x), w(y)\}$ since π is non decreasing. ☐

Remark 3.2.12. 1. We have $\mathfrak{m}_w = \bigcap_{v(a)\in\Delta} a\mathfrak{m}_v \subseteq \mathfrak{m}_v \subseteq \mathcal{O}_v \leqslant \bigcup_{v(a)\in\Delta} a\mathcal{O}_v = \mathcal{O}_w$.

2. In particular, $\mathcal{O}_v/\mathfrak{m}_w \leqslant Kw$ is a valuation ring. We also denote by $v : Kw \to \Delta$ the associated valuation. Note that, if $c \in \mathcal{O}_w$, $v(c) = v(\mathrm{res}_w(c))$ and that we have $\mathrm{res}_v \circ \mathrm{res}_w = \mathrm{res}_v$ once Kwv is canonically identified to Kv.
3. If w is not trivial, equivalently $\Delta < vK^\times$, the valuations v and w induce the same topology.

Lemma 3.2.13. *Let (K, v) be a valued field and $\Delta \leqslant vK^\times$ be a convex subgroup.*
(1) (K, v) is henselian if and only if (K, w) and (Kw, v) are.
(2) If (K, v) is spherically complete if and only if (K, w) and (Kw, v) are.

Proof. (1) Let us first assume that (K, v) is henselian. Let $P = x^d + \sum_{i<d} a_i x^i$ with $a_{d-1} \in \mathcal{O}_w^\times$ and $a_i \in \mathfrak{m}_w \subseteq \mathfrak{m}_v$, for $i < d-1$. Let $Q = a_{d-1}^d P(x/a_{d-1}) = a_{d-1}^d (x/a_{d-1})^d + \sum_{i<d} a_{d-1}^d a_i (x/a_{d-1})^i = x^d + x^{d-1} + \sum_{i<d-1} a_i a_{d-1}^{d-1-i} x^i$. So we may assume that $a_{d-1} = 1 \in \mathcal{O}_v$. By proposition 3.1.3.(vi), there is $c \in \mathcal{O}_v^\times \subseteq \mathcal{O}_w^\times$ with $P(c) = 0$.

Let now $P \in \mathcal{O}_v/\mathfrak{m}_w[x]$ such that $v(P(0)) > 0 = v(P'(0))$ and let $Q \in \mathcal{O}_v[x]$ such that $\mathrm{res}_w(Q) = P$. We have $v(Q(0)) > 0 = v(Q'(0))$ and thus, by proposition 3.1.3.item (vii), there is $c \in K$ such that $v(c) > 0$ and $Q(c) = 0$. So $\mathrm{res}_w(c) \in Kw$ is such that $v(\mathrm{res}_w(c)) > 0$ and $P(\mathrm{res}_w(c)) = 0$.

Conversely, let us assume that (K, w) and (Kw, v) are henselian and let $P \in \mathcal{O}_v[x]$ be such that $v(P(0)) > 0 = v(P'(0))$. Then $v(\mathrm{res}_w(P)(0)) > 0 = v(\mathrm{res}_w(P)'(0))$ and, by proposition 3.1.3.(vii), we find $c \in \mathfrak{m}_v/\mathfrak{m}_w$ with $\mathrm{res}_w(P)(c) = 0$. Let $a \in \mathfrak{m}_v$ be such that $\mathrm{res}_w(a) = c$. Then $\mathrm{res}_w(P(a)) = \mathrm{res}_w(P)(c) = 0 \neq \mathrm{res}_w(P')(0) = \mathrm{res}_w(P'(a))$. So $w(P(a)) > 0 = w(P'(a))$ and, by proposition 3.1.3.(viii), we find $d \in \mathfrak{m}_v$ with $P(d) = 0$.

(2) Let us first assume that (K, v) is spherically complete. Let $\mathfrak{B}$ be a non principal pseudo Cauchy filter over (K, w). Since open balls in (K, w) are intersections of open balls in (K, v), $\mathfrak{B}$ is also pseudo Cauchy over (K, v) and hence has an accumulation point in K. If $\mathfrak{B}$ is a pseudo Cauchy filter over (Kw, v), then, since the preimage by res_w of a ball of (Kw, v) is a ball of (K, v), the set $\{\mathrm{res}_w^{-1}(U) : U \in \mathfrak{B}\}$ generates a pseudo Cauchy filter $\mathfrak{F}$ over (K, v) which thus has an accumulation point $c \in \mathcal{O}_w \in \mathfrak{F}$. Then $\mathrm{res}_w(c) \in \overline{\mathfrak{B}}$.

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Conversely, let us assume that (K, w) and (Kw, v) are spherically complete and let $\mathfrak{B}$ be a non principal pseudo Cauchy filter over (F, v). Then the set of balls of (K, w) in $\mathfrak{B}$ generate a pseudo Cauchy filter $\mathfrak{F}$ over (K, w) which accumulates at some $c \in K$. If $\mathfrak{F} = \mathfrak{B}$, then $c \in \overline{\mathfrak{B}}$. If $\mathfrak{F} \neq \mathfrak{B}$, since between any two open balls of (K, v) whose radius have distinct classes modulo Δ, there is a closed ball of (K, w), $\mathfrak{F}$ is generated by some closed ball b. Translating and scaling, we my assume that $b = \mathcal{O}_w$. Then $(\mathrm{res}_w)_\star\mathfrak{B}$ is a pseudo Cauchy filter on (Kw, v) which accumulates at some $d \in Kw$. Since $\mathrm{res}_w^{-1}(d) \notin \mathfrak{F} \subseteq \mathfrak{B}$, we have $\overline{B} \supseteq \mathrm{res}_w^{-1}(d) \neq \varnothing$. ☐

Notation 3.2.14. Let (K, v) be a characteristic zero field. We define:

- $\Delta_\infty \leqslant vK$ the convex subgroup generated by $v(\mathbb{Z})$;
- $v_\infty : K \to vK/\Delta_\infty$ the coarsened valuation.

Remark 3.2.15. 1. The valuation v_∞ is the least residue characteristic zero coarsening of v.

2. We have $\mathfrak{m}_\infty := \mathfrak{m}_{v_\infty} = \bigcap_{n \in \mathbb{Z}_{>0}} n\mathfrak{m} \subseteq \mathfrak{m} \subseteq \mathcal{O} \leqslant \mathcal{O}_\infty := \mathcal{O}_{v_\infty}$.
3. There is a natural embedding $f : \mathbf{RV}_\infty := K/(1 + \mathfrak{m}_\infty) \to \varprojlim_{n>0} \mathbf{RV}_n$. It is an embedding of monoids that sends 0 to 0. Moreover, for every $\xi, \upsilon, \zeta \in \mathbf{RV}_\infty$, $\zeta \in \xi \oplus \upsilon$ if and only if, for some $x, y, z \in K$, $\mathrm{rv}_\infty(z) = \zeta = \mathrm{rv}_\infty(x+y)$, $\mathrm{rv}_\infty(x) = \xi$ and $\mathrm{rv}_\infty(y) = \upsilon$, if and only if $v_\infty(x+y-z) > v_\infty(z)$, if and only if $v(x+y-z) > v(z) + v(n)$, for every $n \in \mathbb{Z}_{>0}$, if and only if $\mathrm{rv}_{n,\infty}(\zeta) \in \mathrm{rv}_{n,\infty}(\xi) \oplus \mathrm{rv}_{n,\infty}(\upsilon)$.
4. If K is spherically complete or $\aleph_1$-saturated then f is surjective.

Proposition 3.2.16. *Let $M, N \models \mathrm{Hen}_0$, $\bigcup_n \mathbf{RV}_n(M) \subseteq A \leqslant M$ and $f : A \to N$. There exists an embedding $h : N \to N^\star$, which is elementary (for any choice of additional structure on N), and an embedding $g : M \to N$ such that:*

$$\begin{array}{ccc} M & \xrightarrow{g} & N^\star \\ \Big| & & \uparrow h \\ A & \xrightarrow[f]{} & N \end{array}$$

commutes. If N is spherically complete, we can choose $h = \mathrm{id} : N \to N$.

Proof. Let $\mathfrak{L}$ the enrichment of $\mathfrak{L}_{\mathbf{RV}}$ by a copy of itself $\mathfrak{L}_{\mathbf{RV}_\infty}$ that shares the $\mathbf{K}$ sort — we will be indexing the new symbols by by ∞ to distinguish them from the old symbols — and new symbols $\mathrm{rv}_{n,\infty} : \mathbf{RV}_\infty \to \mathbf{RV}_n$, for every $n \in \mathbb{Z}_{>0}$. Note that, writing rv_n as $\mathrm{rv}_{n,\infty} \circ \mathrm{rv}_\infty$, this is an $\mathbf{RV}_\infty$-enrichment of $\mathfrak{L}_{\mathbf{RV}_\infty}$.

Let M_∞ denote the $\mathfrak{L}$-structure associated to $(\mathbf{K}(M), \mathrm{v}_\infty, \mathrm{v})$. By lemma 3.2.13, $(M_\infty, \mathrm{v}_\infty) \models \mathrm{Hen}_{0,0}$. Let $A_\infty := A \bigcup \mathbf{RV}_\infty(M)$. Let $h_0 : N \to N^\star$ be elementary with $N^\star$ $\aleph_1$-saturated and define $f_\infty : A_\infty \to N^\star_\infty$ by extending $h_0 \circ f$ with $f_\infty|_{\mathbf{RV}_\infty} : \mathbf{RV}_\infty(M) \to \varprojlim_n \mathbf{RV}_n(M) \to \varprojlim_n \mathbf{RV}_n(N^\star) \simeq \mathbf{RV}_\infty(N^\star)$; if N is spherically complete, we may take $h = \mathrm{id} : N \to N$.

By construction, f is an $\mathfrak{L}$ morphism, f_∞ commutes with rv_∞ and $\mathrm{rv}_{n,\infty}$, $f_\infty|_{\mathbf{RV}_\infty}$ is a multiplicative morphism sending 0 to 0 and commuting with $-$. Note also that for every $\xi, \upsilon, \zeta \in \mathbf{RV}_\infty$, $\zeta \in \xi \oplus \upsilon$ if and only if $\mathrm{rv}_{n,\infty}(\zeta) \in \mathrm{rv}_{n,\infty}(\xi) \oplus \mathrm{rv}_{n,\infty}(\upsilon)$ for every $n \in \mathbb{Z}_{>0}$. So f_∞ is an $\mathfrak{L}$-morphism.

By proposition 3.2.10, there exists $h_1 : N^\star_\infty \to N^\dagger_\infty$ which is elementary for any structure on $N^\star_\infty$ — in particular, its $\mathfrak{L}$-structure — and $g_\infty : M_\infty \to N^\dagger_\infty$ such that $h_1 \circ f_\infty = g_\infty|_{A_\infty}$. If N is spherically complete, so is $N^\star_\infty = N_\infty$, by lemma 3.2.13, and we can also choose $N^\dagger_\infty = N_\infty$.

Let $g := g_\infty|_{\mathfrak{L}_{\mathbf{RV}}} : M \to N^\dagger := N_\infty|_{\mathfrak{L}_{\mathbf{RV}}}$ and $h = h_1|_{\mathfrak{L}} \circ h_0$. Then $h \circ f = g|_A$. □

Note that there is something non trivial going on. In $N^\dagger_\infty$, we might have $\bigcup_{n \in \mathbb{Z}_{>0}} n\mathcal{O} \subset \mathcal{O}_\infty$ and v_∞ might not be the least residue characteristic zero coarsening of v. However, $\mathrm{rv}_{n,\infty}$ factorises through the standard map $\varprojlim_n \mathbf{RV}_n \to \mathbf{RV}_n$.

Let $\mathfrak{L}$ be a language and S be a set of $\mathfrak{L}$-sorts. An S-enrichment of $\mathfrak{L}$ is $\mathfrak{L}' \supseteq \mathfrak{L}$ such that the symbols in $\mathfrak{L}' \smallsetminus \mathfrak{L}$ only involve the sorts in S — and potential new sorts of $\mathfrak{L}'$.

Theorem 3.2.17 (Basarab, 1990, ...). *The $\mathfrak{L}_{\mathbf{RV}}$-theory* Hen_0 *resplendently eliminates field quantifiers: any formula in an* $(\mathbf{RV}_n)_n$*-enrichment* $\mathfrak{L}$ *of* $\mathfrak{L}_{\mathbf{RV}}$ *is equivalent (modulo* Hen_0*) to an* $\mathfrak{L}$*-formula without quantifiers on the sort* $\mathbf{K}$.

Proof. Let $\mathfrak{L}$ be some $(\mathbf{RV}_n)_n$-enrichment of $\mathfrak{L}_{\mathbf{RV}}$ and $\mathfrak{L}'$ be a further enrichment by a predicate $R_\varphi(x)$ for each formula $\varphi(x)$ without field variables (free or quantified). Let $T := \mathrm{Hen}_0 \cup \{\forall x\, R_\varphi(x) \leftrightarrow \varphi(x)\}$. It suffices to prove that T eliminates quantifiers. By proposition B.0.15, it suffices to show that given $M, N \vDash T$ and and $\mathfrak{L}'$-embedding $f : A \leqslant M \to N$, there exists an $\mathfrak{L}'$-elementary $h : N \to N^\star$ and an $\mathfrak{L}'$-embedding $g : M \to N^\star$ such that $h \circ f = g|_A$.

Let c enumerate all of $M \smallsetminus \mathbf{K}(M)$ and let $p = \text{qf-tp}_{\mathfrak{L}'}(c/A)$. Any field quantifier free $\mathfrak{L}$-formula $\varphi(xa)$ with $a \in \mathbf{K}(A)$ is equivalent to some formula $\psi(x, \mathrm{rv}_n(P(a)))$ where $\psi(xy)$ is an $\mathfrak{L}$-formula without field variables and $P \in \mathbb{Z}[z]$ is a tuple. If $\varphi(xa) \in p$, then $M \vDash \exists x \psi(x\mathrm{rv}_n(P(a)))$ and hence $M \vDash R_{\exists x\psi}(\mathrm{rv}_n(P(a)))$, which implies $N \vDash R_{\exists x\psi}(\mathrm{rv}_n(P(f(a))))$ and thus $N \vDash \exists x \varphi(xf(a))$. Since any quantifier free $\mathfrak{L}'$-formula is equivalent (modulo T) to a field quantifier free $\mathfrak{L}$-formula, we have that $f_\star p$ is finitely satisfiable in N. It is therefore realised in some elementary extension $N^\star$ of N.

So we may assume that $\bigcup_n \mathbf{RV}_n(M) \subseteq M \smallsetminus \mathbf{K}(M) \subseteq A$. By proposition 3.2.16, there exists $g|_{\mathbf{K}} : \mathbf{K}(M) \to \mathbf{K}(N^\star)$ where $N^\star$ is some $\mathfrak{L}'$-elementary extension of N, such that $h|_{\mathbf{K}} \circ f|_{\mathbf{K}} = g|_{\mathbf{K}(A)}$ and $g|_{\mathbf{K}}$ induces $f|_{\mathbf{RV}_n}$ on $\mathbf{RV}_n$. Then $g := g|_{\mathbf{K}} \cup f : M \to N^\star$ is an $\mathfrak{L}'$-embedding since none of the new symbols involve the sort $\mathbf{K}$ and we indeed have $h \circ f = g|_A$. ☐

Corollary 3.2.18. *Let* $M, N \vDash \mathrm{Hen}_0$ *and* $f : A \leqslant M \to N$ *be an* $\mathfrak{L}_{\mathbf{RV}}$*-embedding. Then*

$$f \text{ is } \mathfrak{L}_{\mathbf{RV}}\text{-elementary} \Leftrightarrow f|_{\bigcup_n \mathbf{RV}_n} \text{ is } \mathfrak{L}_{\mathbf{RV}}|_{\bigcup_n \mathbf{RV}_n}\text{-elementary}.$$

In particular,

$$M \equiv N \text{ as } \mathfrak{L}_{\mathbf{RV}}\text{-structures} \Leftrightarrow \bigcup_n \mathbf{RV}_n(M) \equiv \bigcup_n \mathbf{RV}_n(M) \text{ as } \mathfrak{L}_{\mathbf{RV}}|_{\bigcup_n \mathbf{RV}_n}\text{-structures}.$$

Proof. Exercise. ☐

3.3. Angular components

Definition 3.3.1. Let (K, v) be a valued field.

- Let $n \in \mathbb{Z}_{>0}$. An n-th angular component is a multiplicative morphism $ac_n : K^\times \to \mathbf{R}_n^\times = \mathcal{O}^\times / n\mathfrak{m}$ extending res_n on $\mathcal{O}^\times$.
- A system of n-th angular component maps $ac_n : K^\times \to \mathbf{R}_n$ is said to be compatible if for every $n|m \in \mathbb{Z}_{>0}$, $ac_n = \mathrm{res}_{n,m} \circ ac_m$, where $\mathrm{res}_{n,m} : \mathbf{R}_m \to \mathbf{R}_n$ is the canonical projection.

Remark 3.3.2. Let (K, v) be a field.

- Any n-th angular component map ac_n factorises through rv_n and gives rise to a section $s_n : \mathrm{rv}_n(x) \mapsto ac_n(x)$ of $\mathbf{R}_n^\times \to \mathbf{RV}_n^\times$.
- Conversely any section $s_n : \mathbf{RV}_n^\times \to \mathbf{R}_n^\times$ of the short exact sequence $1 \to \mathbf{R}_n^\times \to \mathbf{RV}_n^\times \to \mathbf{\Gamma}^\times \to 0$ gives rise to an n-th angular component $s_n \circ \mathrm{rv}_n$.

- Similarly compatible system of n-th angular component maps ac_n correspond to compatible system of sections $s_n : \mathbf{RV}_n^\times \to \mathbf{R}_n^\times$ with $\mathrm{res}_{n,m} \circ s_m = s_n \circ \mathrm{rv}_{n,m}$, for every $n|m \in \mathbb{Z}_{>0}$.

Proof. Let us first prove that given an n-th angular component map ac_n, the map $\mathrm{rv}_n(x) \mapsto ac_n(x)$ is well defined. For every $x \in K$ and $y \in n$:, $ac_n(x(1+y)) = ac_n(x) \cdot ac_n(1+y) = ac_n(x) \cdot \mathrm{res}_n(1+y) = ac_n(x)$. Also, for every $x \in \mathcal{O}^\times$, $s_n(\mathrm{res}_n(x)) = ac_n(x) = \mathrm{res}_n(x)$. So $s_n|_{\mathbf{R}_n^\times}$ is the identity map. Conversely, $s_n \circ \mathrm{rv}_n : K^\times \to \mathbf{R}_n^\times$ is a multiplicative morphism and for every $x \in \mathcal{O}^\times$, $s_n(\mathrm{rv}_n(x)) = s_n(\mathrm{res}_n(x)) = \mathrm{res}_n(x)$.

Finally, for every $x \in K^\times$ and every $n|m \in \mathbb{Z}_{>0}$,

$$\begin{array}{rclcl} ac_n(x) & = & s_n(\mathrm{rv}_n(x)) & = & s_n(\mathrm{rv}_{n,m}(\mathrm{rv}_m(x))) \\ \mathrm{res}_{n,m}(ac_m(x)) & = & & & \mathrm{res}_{n,m}(s_m(\mathrm{rv}_m(x))) \end{array}$$

So $ac_n = \mathrm{res}_{n,m} \circ ac_m$ if and only if $\mathrm{res}_{n,m} \circ s_m = s_n \circ \mathrm{rv}_{n,m}$. ☐

Any valued field can be endowed with a compatible system of angular components, provided we go to some elementary extension. To prove this fact, we need to introduce pure embeddings:

Lemma 3.3.3. *Let $f : A \to B$ be an Abelian group morphism. The following are equivalent:*

(i) f is injective and for every $a \in A$ and $n \in \mathbb{Z}$, if $f(a) \in n \cdot B$, then $a \in n \cdot A$;

(ii) for every finitely generated $A \leqslant C \leqslant B$, there exists $r : C \to A$ with $r \circ f = \mathrm{id}$;

(iii) for every $m := (m_{i,j})_{i<n,j<\ell} \in \mathbb{Z}$ and $a := (a_i)_{i<n} \in A$, if $mx = f(a)$ has a solution in B^ℓ then, $mx = a$ has a solution in A^ℓ;

(iv) for every $\aleph_1$-saturated Abelian group C and group morphism $g : A \to C$, there exists a group morphism $h : B \to C$ such that $h \circ f = g$;

(v) there exists an elementary embedding $g : A \to A^\star$ and a map $h : B \to A^\star$ such that $h \circ f = g$;

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Proof.

(i)⇒(ii) By the structure theory of finitely generated modules over principal ideal domains (*e.g.* [**Bou-A7**]), $C/f(A) = \bigoplus_{i<n} \mathbb{Z} \cdot c_i/f(A)$. If $c_i/f(A)$ is order $n < \infty$, then, $n \cdot c_i \in f(A)$. By (i), there exists $a \in A$ such that $n \cdot f(a) = n \cdot c_i$ and hence $n \cdot (c_i - f(a)) = 0$. So we may assume that $n \cdot c_i = 0$. Then $\pi : C \to C/f(A)$ induces an isomorphism $\sum_i \mathbb{Z} \cdot c_i \to C/f(A)$, and hence $B = f(A) \oplus (\sum_i \mathbb{Z} \cdot c_i)$, yielding the required retraction, since f is injective.

(ii)⇒(iii) Let $b \in B$ be such that $mb = f(a)$ and C be generated by Ab. By (ii), we find a retraction $r : C \to A$. Then $m \cdot r(b) = r(f(a)) = a$.

(iii)⇒(iv) By Zorn's lemma, let $h : D \to C$ be maximal such that $h \circ f = g$, $f(A) \leqslant D \leqslant B$ and the latter verifies (iii).

Claim 3.3.3.1. *For any $b \in B$, there exists $D \leqslant E \leqslant B$ such that $b \in E$, E/D is countable and $E \leqslant B$ verifies (iii).*

Proof. By downwards Lowenhein-Skolem, proposition B.0.7, we find $D \leqslant E \leqslant B$ such that, $b/D \in E/D \leqslant B/D$ is elementary and E/D is countable. Let now $m \in \mathbb{Z}^{n\ell}$, $e \in E^n$

and $b \in B^\ell$ such that $mb = e$. So $B/D \vDash \exists x\, mx = e/D$ and hence there is some $c \in E$ such that $mc - e = d \in D$. So $m(b-c) = e - e - d \in D$ and by (iii) for $D \leqslant B$, we find $a \in D^\ell$ with $ma = d$ and hence $m(c-a) = e$, where $c - a \in E$. ◊

Let e enumerate a countable set of generators of E over D. Let $\Delta(x)$ be the set of all formulas $mx = d$, where $m \in \mathbb{Z}^b$ is an almost everywhere zero tuple and $d \in D$, such that $me = d$. Note that these formulas involve at most countably many elements of D. By (iii), in $D \leqslant B$, $\Delta(x)$ is finitely satisfiable in D and hence $h_\star\Delta$ is finitely satisfiable in C. By saturation. we find $c \in C$ such that $C \vDash h_\star\Delta(c)$. The natural map $h_0 : E \to C$ extending h by $e \mapsto c$ is such that $h_0 \circ f = g$. By maximality, $D = B$.

(iv)⇒(v) Applying (iv) to an elementary embedding $g : A \to A^\star$ where $A^\star$ is $\aleph_1$-saturated yields (v).

(v)⇒(i) Since $i = h \circ f$, being elementary, is injective, f also is. If $a \in A$, $b \in B$ and $n \in \mathbb{Z}$ are such that $f(a) = n \cdot b$, then $i(a) = h(f(a)) = n \cdot h(b)$. So $A^\star \vDash \exists x\, i(a) = n \cdot x$ and hence, by elementarity, $a \in n \cdot A$. □

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Definition 3.3.4. An Abelian group morphism $f : A \to B$ is said to be pure when the above equivalent conditions hold.

Corollary 3.3.5. *Let $f : A \to B$ be a pure Abelian group morphism definable in some structure M. There exists an elementary $h : M \to M^\star$ and a retraction $r : B(M^\star) \to A(M^\star)$ of $f : A(M^\star) \to B(M^\star)$.*

Note that r is not assumed to be definable — and there is no reason it should be.

Proof. Let $h : M \to M^\star$ be elementary with $M^\star$ $\aleph_1$-saturated. Then $A(M^\star)$ is $\aleph_1$-saturated (as a group) and, by lemma 3.3.3.(iv), we find $r : B(M^\star) \to A(M^\star)$ with $r \circ f = \mathrm{id}$. □

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Corollary 3.3.6. *Any $M \vDash \mathrm{VF}$ has an elementary extension which admits a compatible system of n-th angular component maps for all n.*

Proof. The inclusion $\mathcal{O}^\times \leqslant \mathbf{K}^\times$ is pure. Indeed, if $a \in \mathcal{O}^\times(M)$ is equal to c^n for some $c \in \mathbf{K}^\times(M)$, $n\mathrm{v}(c) = \mathrm{v}(a) = 0$ and hence $c \in \mathcal{O}^\times$. By corollary 3.3.5, we find an elementary extension $h : M \to M^\star$ of K and $r : \mathbf{K}^\times(M^\star) \to \mathcal{O}^\times(M^\star)$ a retraction of the inclusion $\mathcal{O}^\times(M^\star) \leqslant \mathbf{K}^\times(M^\star)$. For every $x \in \mathbf{K}^\times(M^\star)$, let $ac_n(x) = \mathrm{res}_n(r(x))$. This is a multiplicative map and for every $x \in \mathcal{O}^\times(M^\star)$, we have $ac_n(x) = \mathrm{res}_n(r(x)) = \mathrm{res}_n(x)$. So ac_n is an angular component. Moreover, for every $m|n$ and $x \in \mathbf{K}^\star(M)$, $\mathrm{res}_{m,n}(ac_n(x)) = \mathrm{res}_{m,n}(\mathrm{res}_n(r(x))) = \mathrm{res}_m(r(x)) = ac_m(x)$. □

Definition 3.3.7. Let $\mathfrak{L}_{\mathrm{ac}}$ be the language with:

- a sort $\mathbf{K}$ with the ring language $(+, -, 0, \cdot, 1)$;
- a sort $\mathbf{\Gamma}$ with the ordered group language $(+, -, 0, <)$ and a constant ∞;
- for every $n \in \mathbb{Z}_{>0}$, a sort $\mathbf{R}_n$ with the ring language;
- a map $\mathrm{v} : K \to \mathbf{\Gamma}$;
- for every $n > 0$, a map $\mathrm{ac}_n : \mathbf{K} \to \mathbf{R}_n$;

- for every $m, n > 0$ with $n|m$, a map $\mathrm{res}_{n,m} : \mathbf{R}_m \to \mathbf{R}_n$;
- for every $n \in \mathbb{Z}_{>0}$, a map $\mathrm{s}_n : \mathbf{\Gamma} \to \mathbf{R}_n$.

Any valued field (K, v) with a (compatible) system of angular components ac_n can be made into a $\mathfrak{L}_{\mathrm{ac}}$-structure by interpreting $\mathbf{K}$ as the field K, $\mathbf{\Gamma}$ as the ordered monoid vK — with – the inverse on $vK^\times$ and $-\infty = \infty = v(0)$ —, $\mathbf{R}_n$ as the ring $\mathcal{O}/n\mathfrak{m}$, the map v as the valuation v, the maps ac_n as the n-th angular component map ac_n, the maps $\mathrm{res}_{n,m}$ as the canonical projections and the maps $\mathrm{s}_n : \mathrm{v}(x) \mapsto \mathrm{res}_n(x) \cdot \mathrm{ac}_n(x)^{-1}$ which is well defined. Note that s_n is a section of the map induced by the valuation on $\mathbf{R}_n$.

Definition 3.3.8. We denote $\mathrm{Hen}_0^{\mathrm{ac}}$ the $\mathfrak{L}_{\mathrm{ac}}$-theory of characteristic zero henselian valued fields with a compatible system of angular components.

Let $\mathbf{R} := \bigcup_n \mathbf{R}_n$.

Proposition 3.3.9. *Let $M, N \models \mathrm{Hen}_0^{\mathrm{ac}}$, $\mathbf{\Gamma}(M) \cup \mathbf{R}(M) \subseteq A \leqslant M$ and $f : A \to N$. There exists an embedding $h : N \to N^\star$, which is elementary for any structure on N, and an embedding $g : M \to N$ such that:*

$$\begin{array}{ccc} M & \xrightarrow{g} & N^\star \\ \Big| & & \Big\uparrow h \\ A & \xrightarrow[f]{} & N \end{array}$$

commutes. If N is spherically complete, we can choose $h = \mathrm{id} : N \to N$.

Proof. Let $\mathfrak{L}$ be the enrichment of $\mathfrak{L}_{\mathbf{RV}} \cup \mathfrak{L}_{\mathrm{ac}}$ by traces of the valuation $\mathrm{v} : \mathbf{RV}_n \to \mathbf{\Gamma}$ and $\mathrm{v} : \mathbf{R}_n \to \mathbf{\Gamma}$, the residue map $\mathrm{res}_n : \mathbf{RV}_n \to \mathbf{R}_n$, injections $i_n : \mathbf{R}_n^\times \to \mathbf{RV}_n$, traces of the angular components $\mathrm{ac}_n : \mathbf{RV}_n \to \mathbf{R}_n^\times$ and $\mathrm{ac}_n : \mathbf{R}_n \to \mathbf{R}_n^\times$, and sections $\mathrm{s}_n : \mathbf{\Gamma} \to \mathbf{RV}_n$. Let M_{rv} (respectively N_{rv}) denote the $\mathfrak{L}$-structure associated to M (respectively N), where $\mathrm{s}_n(\mathrm{v}(x)) = \mathrm{rv}_n(x)\mathrm{ac}_n(x)^{-1}$. This is well-defined since, for every $x \in \mathcal{O}^\times$, $\mathrm{rv}_n(x)\mathrm{ac}_n(x)^{-1} = \mathrm{res}_n(x)\mathrm{res}_n(x)^{-1} = 1$. Let $A_{\mathrm{rv}} = A \cup \bigcup_n \mathbf{RV}_n(M) \leqslant M_{\mathrm{rv}}$ and $f_{\mathrm{rv}} : A_{\mathrm{rv}} \to N_{\mathrm{rv}}$ extend f by $f_{\mathrm{rv}}(\xi) = f(\mathrm{ac}_n(\xi))\mathrm{s}_n(f(\mathrm{v}(\xi)))$, for every $n \in \mathbb{Z}_{>0}$ and $\xi \in \mathbf{RV}_n(M)$.

Claim 3.3.9.1. *f_{rv} is an $\mathfrak{L}$-morphism.*

Proof. By construction f_{rv} is an $\mathfrak{L}_{\mathrm{ac}}$-morphism and $f_{\mathrm{rv}}|_{\mathbf{RV}}$ is a multiplicative morphism preserving 0. Also, for every $x \in \mathbf{K}(M)$, $\gamma \in \mathbf{\Gamma}(M)$ and $\xi \in \mathbf{RV}_n(M)$, we have

$$\begin{array}{rcl}
f_{\mathrm{rv}}(\mathrm{rv}_n(x)) & = & f(\mathrm{ac}_n(x))\mathrm{s}_n(f(\mathrm{v}(x))) = \mathrm{ac}_n(f(x))\mathrm{s}_n(\mathrm{v}(f(x))) = \mathrm{rv}_n(f_{\mathrm{rv}}(x)) \\
f_{\mathrm{rv}}(\mathrm{s}_n(\gamma)) & = & f(\mathrm{ac}_n(\mathrm{s}_n(\gamma)))\mathrm{s}_n(f(\mathrm{v}(\mathrm{s}_n(\gamma)))) = f(1)\mathrm{s}_n(f(\gamma)) = \mathrm{s}_n(f_{\mathrm{rv}}(\gamma)) \\
\mathrm{ac}_n(f_{\mathrm{rv}}(\xi)) & = & \mathrm{ac}_n(f(\mathrm{ac}_n(\xi)))\mathrm{ac}_n(\mathrm{s}_n(f(\mathrm{v}(\xi)))) = f(\mathrm{ac}_n(\xi)) \cdot 1 = f(\mathrm{ac}_n(\xi)) \\
\mathrm{v}(f_{\mathrm{rv}}(\xi)) & = & \mathrm{v}(f(\mathrm{ac}_n(\xi))) + \mathrm{v}(\mathrm{s}_n(f(\mathrm{v}(\xi)))) = 0 + f(\mathrm{v}(\xi)) \\
f_{\mathrm{rv}}(-\xi) & = & f(\mathrm{ac}_n(-\xi))\mathrm{s}_n(f(\mathrm{v}(-\xi))) = \mathrm{res}_n(-1) \cdot f(\mathrm{ac}_n(\xi))\mathrm{s}_n(f(\mathrm{v}(\xi))) = -f_{\mathrm{rv}}(\xi) \\
f_{\mathrm{rv}}(\mathrm{rv}_{m,n}(\xi)) & = & f(\mathrm{res}_{m,n}(\mathrm{ac}_n(\xi)))\mathrm{s}_m(f(\mathrm{v}(\xi))) = \mathrm{res}_{m,n}(\mathrm{ac}_n(f_{\mathrm{rv}}(\xi)))\mathrm{s}_m(\mathrm{v}(f_{\mathrm{rv}}(\xi))) \\
 & = & \mathrm{rv}_{m,n}(\mathrm{ac}_n(f_{\mathrm{rv}}(\xi)))\mathrm{rv}_{m,n}(s_n(\mathrm{v}(f_{\mathrm{rv}}(\xi)))) = \mathrm{rv}_{m,n}(f_{\mathrm{rv}}(\xi))
\end{array}$$

Moreover, for every $\alpha \in \mathbf{R}_n(M)$, $\mathrm{ac}_n(\alpha) \in \mathbf{R}_n^\times$ and $\mathrm{v}(\alpha) \in \mathbf{\Gamma}$ are uniquely determined by $\alpha = \mathrm{ac}_n(\alpha)\mathrm{s}_n(\mathrm{v}(\alpha))$. Since $f(\alpha) = f(\mathrm{ac}_n(\alpha))\mathrm{s}_n(f(\mathrm{v}(\alpha))) = \mathrm{ac}_n(f(\alpha))\mathrm{s}_n(\mathrm{v}(f(\alpha)))$ and

$f(\mathrm{ac}_n(\alpha)) \in \mathbf{R}_n^\times$, we have $f(\mathrm{v}(\alpha)) = \mathrm{v}(f(\alpha))$ and $f(\mathrm{ac}_n(\alpha)) = \mathrm{ac}_n(f(\alpha))$. Also, if $\alpha \neq 0$, for every $\xi \in \mathbf{RV}_n(M)$, $\mathrm{res}_n(\xi) = \alpha$ if and only if $\mathrm{v}(\xi) = \mathrm{v}(\alpha)$ and $\mathrm{ac}_n(\xi) = \mathrm{ac}_n(\alpha)$. And $\mathrm{res}_n(\xi) = 0$ if and only if $\mathrm{v}(\xi) > \mathrm{v}(n)$. So $\mathrm{res}_n(\xi) = \alpha$ if and only if $\mathrm{res}_n(f(\xi)) = f(\alpha)$.

There remains to check that f_{rv} preserves $\oplus$. Let $\xi, \upsilon, \zeta \in \mathbf{RV}_n(M)$. If $\mathrm{v}(\xi) + \mathrm{v}(n) < \mathrm{v}(\upsilon)$, then $\zeta \in \xi \oplus \upsilon$ if and only if $\zeta = \xi$, if and only if $f_{\mathrm{rv}}(\zeta) = f_{\mathrm{rv}}(\xi)$ if and only if $f_{\mathrm{rv}}(\zeta) \in f_{\mathrm{rv}}(\xi) \oplus f_{\mathrm{rv}}(\upsilon)$. So, up to permutations, we may thus assume that $\mathrm{v}(\xi) = \mathrm{v}(\upsilon) \leqslant \mathrm{v}(\zeta) \leqslant \mathrm{v}(\xi) + \mathrm{v}(n)$. Diving by ξ, we may further assume that $\xi, \upsilon \in \mathbf{R}_n^\times$. We have $\zeta \in \xi \oplus \upsilon$ if and only if $\mathrm{res}_n(\zeta) = \xi + \upsilon$, if and only if, $\mathrm{v}(\zeta) = \mathrm{v}(\xi + \upsilon)$ and $\mathrm{ac}_n(\zeta) = \mathrm{ac}_n(\xi + \upsilon)$, if and only if, $\mathrm{v}(f(\zeta)) = f(\mathrm{v}(\zeta)) = f(\mathrm{v}(\xi + \upsilon)) = \mathrm{v}(f(\xi) + f(\upsilon))$ and $\mathrm{ac}_n(f(\zeta)) = f(\mathrm{ac}_n(\zeta)) = f(\mathrm{ac}_n(\xi + \upsilon)) = \mathrm{ac}_n(f(\xi) + f(\upsilon))$, if and only if $f(\zeta) \in \xi \oplus f(\upsilon)$. ◊

By proposition 3.2.16, we find $h_{\mathrm{rv}} : N_{\mathrm{rv}} \to N_{\mathrm{rv}}^\star$, which is elementary for any structure on N, and $g_{\mathrm{rv}} : M_{\mathrm{rv}} \to N_{\mathrm{rv}}^\star$ such that $h_{\mathrm{rv}} \circ f_{\mathrm{rv}} = g_{\mathrm{rv}}|_{A_{\mathrm{rv}}}$; and if N is spherically complete, we can choose $h_{\mathrm{rv}} = \mathrm{id} : N_{\mathrm{rv}} \to N_{\mathrm{rv}}$. The $\mathfrak{L}$-morphism $g|_{\mathfrak{L}}$ thus has the required properties. ☐

Theorem 3.3.10. *The $\mathfrak{L}_{\mathrm{ac}}$-theory $\mathrm{Hen}_0^{\mathrm{ac}}$ resplendently eliminates field quantifiers: any formula in a $\mathbf{\Gamma} \cup \mathbf{R}$-enrichment $\mathfrak{L}$ of $\mathfrak{L}_{\mathrm{ac}}$ is equivalent (modulo $\mathrm{Hen}_0^{\mathrm{ac}}$) to an $\mathfrak{L}$-formula without quantifiers on the sort $\mathbf{K}$.*

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Proof. We proceed as in the proof of theorem 3.2.17. It suffices, given an $\mathbf{\Gamma} \cup \mathbf{R}$-enrichment $\mathfrak{L}$ of $\mathfrak{L}_{\mathbf{RV}}$ and T an $\mathfrak{L}$-theory such that every $\mathfrak{L}$-formula without field quantifiers is equivalent to a quantifier free one, $M, N \vDash T$ and an $\mathfrak{L}$-embedding $f : A \leqslant M \to N$, to extend it to an $\mathfrak{L}$-embedding $g : M \to N^\star$ with $h \circ f = g|_A$, where $h : N \to N^\star$ is $\mathfrak{L}$-elementary.

Let c enumerate all of $M \setminus \mathbf{K}(M)$. By hypothesis, $f_\star \mathrm{qf\text{-}tp}_{\mathfrak{L}}(c/A)$ is finitely satisfiable in N and hence, enlarging N, we may assume that $\mathbf{\Gamma}(M) \cup \bigcup_n \mathbf{R}_n(M) \subseteq A$. Then we extend f by proposition 3.3.9. ☐

Corollary 3.3.11. *Let $M, N \vDash \mathrm{Hen}_0^{\mathrm{ac}}$ and $f : A \leqslant M \to N$ be an $\mathfrak{L}_{\mathrm{ac}}$-embedding. Then*

$$f \text{ is } \mathfrak{L}_{\mathrm{ac}}\text{-elementary} \Leftrightarrow f|_{\mathbf{\Gamma}\cup\mathbf{R}} \text{ is } \mathfrak{L}_{\mathrm{ac}}|_{\mathbf{\Gamma}\cup\mathbf{R}}\text{-elementary}.$$

In particular,

$$M \equiv N \text{ as } \mathfrak{L}_{\mathrm{ac}}\text{-structures} \Leftrightarrow \mathbf{\Gamma}(M) \cup \mathbf{R}(M) \equiv \mathbf{\Gamma}(M) \cup \mathbf{R}(M) \text{ as } \mathfrak{L}_{\mathrm{ac}}|_{\mathbf{\Gamma}\cup\mathbf{R}}\text{-structures}.$$

Proof. This follows from theorem 3.3.10 and the fact that any $\mathfrak{L}_{\mathrm{ac}}$-formulas without field quantifiers is of the form $\varphi(\mathrm{ac}_n(P(x)), \mathrm{v}(Q(x)), y)$ where x is a tuple of $\mathbf{K}$-variables, $P, Q \in \mathbb{Z}[x]$ are tuples, and φ is an $\mathfrak{L}_{\mathrm{ac}}|_{\mathbf{\Gamma}\cup\mathbf{R}}$-formula. The second statement is exactly the first statement applied to $f : \varnothing \leqslant M \to N$ — note that for every $n \in \mathbb{Z}_{\neq 0}$ and $m \in \mathbb{Z}_{>0}$, $\mathrm{res}_m(n) = n$ and $\mathrm{v}(n)$ is the unique element of $\mathbf{\Gamma}$ such that $s_{n^2}(\gamma) \in n\mathbf{R}_{n^2}^\times$. ☐

3.4. The Ax-Kochen-Ershov principle

Proposition 3.4.1. *Let K and L be henselian fields of residue characteristic p. Then:*

$$K \equiv L \Leftrightarrow \begin{cases} \mathrm{v}K \equiv \mathrm{v}L & \text{as ordered monoids with a constant for } \mathrm{v}(p); \\ \mathbf{R}(K) \equiv \mathbf{R}(L) & \text{as projective systems of rings.} \end{cases}$$

Proof. Note that if $K \equiv L$, then we must have $\mathrm{v}K^\times \equiv \mathrm{v}L^\times$ and $\mathbf{R}(K) \equiv \mathbf{R}(L)$. So let us prove the converse and assume that $\mathrm{v}K^\times \equiv \mathrm{v}L^\times$ and $\mathbf{R}(K) \equiv \mathbf{R}(L)$. By the Keisler–Shelah theorem, theorem B.0.18, and proposition B.0.19, we can find $\aleph_1$-saturated $M \equiv K$ and $N \equiv L$ such that $\mathbf{\Gamma}(M) \simeq \mathbf{\Gamma}(N)$ — and that isomorphism induces an isomorphism $\mathbf{\Gamma}(M)/\Delta_\infty \simeq \mathbf{\Gamma}(M)/\Delta_\infty$ — and $\mathbf{R}(M) \simeq \mathbf{R}(N)$ — which induces an isomorphism $\mathrm{res}_\infty(\mathcal{O}(M)) \simeq \varprojlim_n \mathbf{R}_n(M) \simeq \varprojlim_n \mathbf{R}_n(N) \simeq \mathrm{res}_\infty(\mathcal{O})$. It follows that (M, v_∞) and (N, v_∞) have isomorphic value groups and residue fields — even naming $\mathrm{res}_\infty(\mathcal{O}) \leqslant \mathbf{R}_\infty$.

By corollary 3.3.6, (M, v_∞) and (N, v_∞) can be endowed with an angular component ac_∞, and, by lemma 3.2.13, they are both henselian. In equicharacteristic zero, the s_n maps are trivially determined: they send 0 to 1 and everything else to 0; and so are the valuation and angular component on $\mathbb{Z}$. So, by theorem 3.3.10, $M \equiv N$ as $\mathfrak{L}_{\mathrm{ac}_\infty}$-structure with $\mathrm{res}_\infty(\mathcal{O}) \leqslant \mathbf{R}_\infty$ named. Since $\mathcal{O} = \mathrm{res}_\infty^{-1}(\mathrm{res}_\infty(\mathcal{O}))$, $K \equiv M \equiv N \equiv L$ as valued fields for the initial valuation. $\square$

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Definition 3.4.2. A valued field (K, v) is said to be:

- finitely ramified if, for every $n \in \mathbb{Z}_{>0}$, $(0, v(n))$ is finite — equivalently, for every prime $p \in \mathbb{Z}$, $(0, v(p))$ is finite;
- unramified if, for every prime $p \in \mathbb{Z}$, $(0, v(p)) = \varnothing$.

A finitely ramified ramified valued of positive characteristic is trivially valued.

Theorem 3.4.3 (Ax–Kochen, 1965 — Ershov, 1965, ...). *Let (K, v) and (L, w) be two unramified henselian fields with perfect residue fields. Then:*

$$K \equiv L \text{ as valued fields} \Leftrightarrow \begin{cases} vK^\times \equiv wL^\times & \text{as ordered groups;} \\ Kv \equiv Lw & \text{as fields.} \end{cases}$$

Proof. If K (and hence L) has residue characteristic zero, then all the $\mathbf{R}_n$ are isomorphic to $\mathbf{R}_1$ and this is the same statement as proposition 3.4.1. So we may assume that K (and hence L) has residue characteristic $p > 0$. Fix some $i < n \geqslant 0$ and $c \in K$. If $p^i c \in \mathfrak{m}^n$, then $i\mathrm{v}(p) + \mathrm{v}(c) \geqslant n\mathrm{v}(p)$ and hence $\mathrm{v}(c) \geqslant (n - i) \cdot \mathrm{v}(p) > 0$, so $c \in \mathfrak{m}$. It follows, by corollary 1.3.18, that $\mathbf{R}_{p^n}(K)$ is canonically isomorphic to $\mathrm{W}_{p^n}(Kv)$; in fact $\mathbf{R}(K)$ is canonically isomorphic to the projective system of the $\mathrm{W}_{p^n}(Kv)$, which is interpretable in Kv. Since the same holds of L, we have $\mathbf{R}(K) \equiv \mathbf{R}(L)$ if and only if $Kv \equiv Lw$ and the statement now follows from proposition 3.4.1. $\square$

Corollary 3.4.4. *Let $\mathfrak{U}$ be a non principal ultrafilter on the set of primes then:*

$$\prod_{p \to \mathfrak{U}} \mathbb{Q}_p \equiv \prod_{p \to \mathfrak{U}} \mathbb{F}_p((t)).$$

Proof. We since $\mathbb{Q}_p$ and $\mathbb{F}_((t))$ are complete valued fields with value group $\mathbb{Z}$, and hence are henselian, it follows that $\prod_{p \to \mathfrak{U}} \mathbb{Q}_p, \prod_{p \to \mathfrak{U}} \mathbb{F}_p((t)) \models \mathrm{Hen}$ and that $\mathbf{\Gamma}(\prod_{p \to \mathfrak{U}} \mathbb{Q}_p) \equiv \mathbb{Z}\mathbf{\Gamma}(\prod_{p \to \mathfrak{U}} \mathbb{F}_p((t)))$. Moreover, $\mathbf{R}_1(\prod_{p \to \mathfrak{U}} \mathbb{Q}_p) = \prod_{p \to \mathfrak{U}} \mathbb{F}_p = \mathbf{R}_1(\prod_{p \to \mathfrak{U}} \mathbb{F}_p((t)))$ is a characteristic zero field. So theorem 3.4.3 applies. $\square$

One motivation behind Ax and Kochen's work was to answer a conjecture of Artin:

Conjecture 3.4.5. *Any homogenous polynomial over* $\mathbb{Q}_p$ *of degree* d *in* $n > d^2$ *variables has a non trivial zero in* $\mathbb{Q}_p$.

This conjecture is known to be false. However, it holds in $\mathbb{F}_p((t))$:

Theorem 3.4.6 (Lang, 1952). *Let* $(f_i)_{i<n}$ *be homogeneous polynomials over* $\mathbb{F}_p((t))$ *in* $n > \sum_i \deg(f_i)^2$ *variables. Then the* f_i *have a non trivial common zero in* $\mathbb{F}_p((t))$.

Corollary 3.4.7. *For every* $d \in \mathbb{Z}_{>0}$*, there exists a finite set* $A(d,n)$ *of primes such that for every* $p \notin A(d,n)$*, any* $(f_i)_{i<n} \in \mathbb{Q}_p[x]$ *homogenous of degree at most* d *with* $|x| > nd^2$ *have a common non trivial zero in* $\mathbb{Q}_p$.

Proof. There is a sentence φ in the language of rings that expresses that any family of n polynomial of degree at most d in $nd^2 + 1$ many variables has a common zero. We have $\mathbb{F}_p((t)) \vDash \varphi$ and hence, for any non principal $\mathfrak{U}$, $\prod_{p\to\mathfrak{U}} \mathbb{Q}_p \equiv \prod_{p\to\mathfrak{U}} \mathbb{F}_p((t)) \vDash \varphi$. If $A(d,n) := \{p : \mathbb{Q}_p \nvDash \varphi\}$ is infinite, there exists a non principal $\mathfrak{U}$ with $A(d,n) \in \mathfrak{U}$ and we would have $\prod_{p\to\mathfrak{U}} \mathbb{Q}_p \nvDash \varphi$, a contradiction. It follows that $A(d,n)$ is finite.

The result for $|x| > nd^2$ is an immediate consequence (setting some variables to 1) of the one for $|x| = nd^2 + 1$. □

3.5. Properties of definable sets

Lemma 3.5.1. *Let* $M \vDash \mathrm{Hen}_0$*,* $P \in \mathbf{K}(M)[x]$ *and* $C := \{c \in \mathbf{K}(M) : P^{(i)}(c) = 0$*, for some* $i \leqslant \deg(P)\}$*. For every* $c \in C$*,* $x \in \mathbf{K}(M)$ *such that* $\mathrm{v}(x-c)$ *is maximal and* $n \in \mathbb{Z}_{>0}$*, there exists an* $m \in \mathbb{Z}_{>0}$ *such that* $\mathrm{rv}_{n,m}(\mathrm{rv}_m(P_c)(\mathrm{rv}_m(x-c)))$ *is well-defined, where* $P_c(x) := P(x+c)$. 08/03

Proof. If $0 \in \mathrm{rv}_\infty(P_c)(\mathrm{rv}_\infty(x-c))$, by lemma 3.2.6, there exists, $i \in \mathbb{Z}_{>0}$ and $e \in \mathbf{K}(M)$ such that $P^{(i)}(e+c) = P_c^{(i)}(e) = 0$ and $\mathrm{rv}_\infty(e) = \mathrm{rv}_\infty(x-c)$. So $e+c \in C$ and $\mathrm{v}(x-e-c) > \mathrm{v}(x-c)$, a contradiction. It follows that $\mathrm{rv}_\infty(P_c)(\mathrm{rv}_\infty(x-c))$ is well-defined. By compactness, it follows that, for every $n \in \mathbb{Z}_{>0}$, there exists an $m \in \mathbb{Z}_{>0}$ such that $\mathrm{rv}_{n,m}(\mathrm{rv}_m(P_c)(\mathrm{rv}_m(x-c)))$ is a singleton. □

Let $\mathfrak{L}$ be an $\mathbf{RV}$-enrichment of $\mathfrak{L}_{\mathbf{RV}}$, $M \vDash \mathrm{Hen}_0$ an $\mathfrak{L}$-structure and $A \leqslant M$.

Lemma 3.5.2. *If* $X \subseteq \mathbf{RV}_n^m$ *is* $\mathfrak{L}(A)$*-definable, then it is* $\mathfrak{L}(A)|_{\mathbf{RV}}$*-definable. In particular,* $\bigcup_n \mathbf{RV}_n$ *is a pure stably embedded* $\mathfrak{L}|_{\mathbf{RV}}$*-structure.*

Proof. This is an immediate consequence of field quantifier elimination, theorem 3.2.17. □

For any ball b and $n \in \mathbb{Z}_{>0}$, let $b[n] := \{x + n^{-1}(y-x) : x, y \in b\}$. It is a ball containing b of radius $\mathrm{rad}(b) - \mathrm{v}(n)$, which is open if and on if b is.

Proposition 3.5.3. *Let* $X \subseteq \mathbf{K}$ *be* $\mathfrak{L}(A)$*-definable. Then there exists a finite* $C \subseteq \mathbf{K}(A)^{\mathrm{a}} \cap M$ *and* $n \in \mathbb{Z}_{>0}$ *such that for any ball* b *of* M *with* $b[n] \cap C \neq \varnothing$*, we either have* $b \cap X = \varnothing$ *or* $b \subseteq X$.

In other words, $\mathbb{1}_{x\in X}$ factorises through $\mathrm{rv}(x - C) := (\mathrm{rv}_n(x-c))_{c\in C}$.

Proof. By theorem 3.2.17, any $\mathfrak{L}(A)$-formula $\varphi(x)$ is equivalent, modulo Hen_0 to a formula of the form $\psi(\mathrm{rv}_n(P(x)))$, where $P \in \mathbf{K}(A)[x]$ is a tuple and ψ is an $\mathfrak{L}(A)|_{|RV}$-formula. By lemma 3.5.1, making n bigger we may assume that φ is of the form $\psi(\mathrm{rv}_n(x - C))$ where $C \subseteq \mathbf{K}(A)^{\mathrm{a}} \cap M$ is the set of roots of the non constant $P_i^{(j)}$.

Let now b be an ball of M such that $b[n] \cap C \neq \varnothing$, and $x, y \in b(M)$. For every $c \in C$, if b is open, we have $\mathrm{v}(x - c) \leqslant \mathrm{rad}(b) - \mathrm{v}(n) < \mathrm{v}(x - y) - \mathrm{v}(n)$ and if b is closed, we have $\mathrm{v}(x - c) < \mathrm{rad}(b) - \mathrm{v}(n) \leqslant \mathrm{v}(x - y) - \mathrm{v}(n)$. So, in either case, $\mathrm{rv}_n(x - c) = \mathrm{rv}_n(y - c)$ and $M \vDash \varphi(x)$ if and only if $M \vDash \varphi(y)$. □

Proposition 3.5.4. *Assume that* $\mathbf{K}(A)^{\mathrm{a}} \cap M \subseteq A$. *Let* $c \in \mathbf{K}(M)$ *and* $\mathfrak{B}$ *be the filter generated by* $\{b\ \mathrm{v}_\infty$*-ball of* $\mathbf{K}(A) : c \in b\}$.

1. *If* $\overline{\mathfrak{B}} \cap A = \varnothing$, *then* $\mathrm{tp}(c/A) = \overline{\mathfrak{B}}_\infty$ *— in particular, it is the intersection of* v*-balls of* $\mathbf{K}(A)$.
2. *If* $a \in \overline{\mathfrak{B}} \cap A \neq \varnothing$, *then* $\mathrm{tp}(\mathrm{rv}_\infty(c - a)/\mathbf{RV}(A)) \vDash \mathrm{tp}(c/A)$*; and* $\mathrm{rv}(c - a) \notin \mathrm{rv}(\mathbf{K}(A))$.

Proof. For every for every $e \in \mathbf{K}(A) \setminus \overline{\mathfrak{B}}_\infty$ and $d \in \overline{\mathfrak{B}}$, $\mathrm{v}_\infty(c - e) < \mathrm{v}_\infty(c - d)$ and hence $\mathrm{rv}_\infty(c - e) = \mathrm{rv}_\infty(d - e)$. It follows that, if $\overline{\mathfrak{B}} \cap A = \varnothing$, by lemma 3.5.1, for any $\mathfrak{L}(A)$-definable X, $d \in X$ if and only if $c \in X$ and hence $\mathrm{tp}(d/A) = \mathrm{tp}(c/A)$. Note also that $\mathfrak{B}$ is not principal in that case and hence it is generated by open v_∞-balls that are themselves intersections of v-balls.

Let us now fix $a \in \overline{\mathfrak{B}} \cap A \neq \varnothing$. Let $d \in \mathbf{K}(N)$, $N \succcurlyeq M$ be such that $\mathrm{tp}(\mathrm{rv}_\infty(d - a)/\mathbf{RV}_\infty(A)) = \mathrm{tp}(\mathrm{rv}_\infty(c - a)/\mathbf{RV}_\infty(A)$. For every $\varphi(x) \in \mathrm{tp}(d/A)$ and $n \in \mathbb{Z}_{>0}$, $\exists x \varphi(x) \wedge \mathrm{rv}_n(x - a) = \xi_n$ is equivalent to an $\mathfrak{L}(A)|_{\mathbf{RV}}$ $\psi(\xi_n)$ with $\psi \in \mathrm{tp}(\mathrm{rv}_\infty(d - a)/\mathbf{RV}_\infty(A)) = \mathrm{tp}(\mathrm{rv}_\infty(c - a)/\mathbf{RV}_\infty(A)$, so $M \vDash \exists x \varphi(x) \wedge \mathrm{rv}_n(x - a) = \mathrm{rv}_n(c - a)$. By compactness, we find $d' \in \mathbf{K}$ such that $\mathrm{rv}_\infty(d' - a) = \mathrm{rv}_\infty(c - a)$ and $\mathrm{tp}(d'/A) = \mathrm{tp}(d/A)$. So we may assume that $\mathrm{rv}_\infty(d - a) = \mathrm{rv}_\infty(c - a)$.

For any $e \in \overline{\mathfrak{B}} \cap A$ and $d \vDash \eta$

, if $\mathrm{v}_\infty(c - a) > \mathrm{v}_\infty(a - e)$, then $d \in$

Let us now assume that $\overline{\mathfrak{B}} \cap A \neq \varnothing$. For any $e \in \overline{\mathfrak{B}} \cap A$, if $\mathrm{v}_\infty(c - e) > \mathrm{v}_\infty(d - e)$, then for all $n \in \mathbb{Z}_{>0}$, $b_{\geqslant \mathrm{v}(a-e)+\mathrm{v}(n)} \in \mathfrak{B}$ and hence $\mathrm{v}_\infty(d - e) > \mathrm{v}_\infty(d - e)$

If then for any $e \in \overline{\mathfrak{B}} \cap A$, $c \notin \overline{\mathrm{B}}(a, \mathrm{v}_\infty(a - e))$ and thus $\mathrm{v}_\infty(c - a) < \mathrm{v}_\infty(a - e)$, *i.e.* $\mathrm{rv}_\infty(c - e) = \mathrm{rv}_\infty(c - a)$.

Note that if b is a v-ball of $\mathbf{K}(A)$ containing c, then $\bigcup_n b[n] \in \mathfrak{B}$ and hence $a \in b[n]$ for some a. So $\mathrm{v}(c - a) \geqslant \mathrm{rad}(b) - \mathrm{v}(n)$. Since $\mathrm{rv}_n(c - a) = \mathrm{rv}_n(c - d)$, it follows that $\mathrm{v}(c - d) > \mathrm{v}(c - a) + \mathrm{v}(n) \geqslant \mathrm{rad}(b)$ and $d \in b$. By symmetry, $d \in b$ if and only if $c \in b$. In particular the filter generated by $\{b\ \mathrm{v}_\infty$-ball of $\mathbf{K}(A) : d \in b\}$ is also $\mathfrak{B}$. So for every $e \in \mathbf{K}(A)$, we have $\mathrm{rv}_\infty(c - e) = \mathrm{rv}_\infty(d - e)$ if $e \notin \overline{\mathfrak{B}}$ and $\mathrm{rv}_\infty(c - e) = \mathrm{rv}_\infty(c - a) = \mathrm{rv}_\infty(d - a) = \mathrm{rv}_\infty(d - e)$ otherwise. As earlier, we conclude that $\mathrm{tp}(d/A) = \mathrm{tp}(c/A)$.

Note that if $\mathrm{rv}(c - a) = \mathrm{rv}(e) \in \mathbf{RV}(A)$, then $\mathrm{rv}(c - (a + e)) > \mathrm{rv}(c - a)$ and hence $a \notin \mathring{\mathrm{B}}(c, \mathrm{v}(c - a)) = \mathring{\mathrm{B}}(a + e, \mathrm{v}(c - a)) \in \mathfrak{B}$, a contradiction. □

Corollary 3.5.5. *We have* $\mathrm{acl}(A) = \mathbf{K}(A)^{\mathrm{a}} \cup \mathrm{acl}(\mathbf{RV}(A))$.

Proof. If $\alpha \in \mathbf{RV}(\mathrm{acl}(A))$, then, by lemma 3.5.2, $\alpha \in \mathrm{acl}(\mathbf{RV}(A))$. Now, fix $c \in \mathbf{K}(\mathrm{acl}(A))$ and let $\mathfrak{B}$ be the maximal pseudo Cauchy v_∞-filter over $\mathbf{K}(A)^{\mathrm{a}}$ concentrating at c. If $\overline{\mathfrak{B}} \cap \mathbf{K}(A)^{\mathrm{a}} = \varnothing$, then, by proposition 3.5.4, $\mathfrak{B} \cap N$ is finite for any $N \succcurlyeq M$ and hence $\mathfrak{B}$ contains a singleton; *i.e.* $c \in \mathbf{K}(A)^{\mathrm{a}}$. If $a \in \overline{\mathfrak{B}} \cap \mathbf{K}(A)^{\mathrm{a}} \neq \emptyset$, then the set of $c' \in N \succcurlyeq M$ with $\mathrm{rv}_\infty(c'-a) = \mathrm{rv}(c-a)$ is finite. So $\mathrm{rv}(c-a) = 0$ and $c = a \in \mathbf{K}(A)^{\mathrm{a}}$. □

Definition 3.5.6. Let $\mathfrak{L}_{\mathrm{ac,fr}}$ be the language $\mathfrak{L}_{\mathrm{ac}}$ without the s_n functions and with a constant $\pi \in \mathbf{\Gamma}$ and constants $\pi_n \in \mathbf{R}_n$, for every $n \in \mathbb{Z}_{>0}$

Any finitely ramified field with angular components can be made into an $\mathfrak{L}_{\mathrm{ac,fr}}$-structure by interpreting π has the smallest positive element of $\mathbf{\Gamma}$ if it exists and 1 otherwise and π_n as $\mathrm{s}_n(\pi) = \mathrm{ac}_n(x)$ for any $x \in \mathbf{K}$ with $\mathrm{v}(x) = \pi$.

Theorem 3.5.7 (Pas, 1989). *The $\mathfrak{L}_{\mathrm{ac,fr}}$-theory $\mathrm{Hen}_0^{\mathrm{ac,fr}}$ of finitely ramified henselian valued fields of characteristic zero resplendently eliminates field quantifiers: any formula in a $\mathbf{\Gamma}\cup\mathbf{R}$-enrichment $\mathfrak{L}$ of $\mathfrak{L}_{\mathrm{ac}}$ is equivalent to an $\mathfrak{L}$-formula without quantifiers on the sort $\mathbf{K}$.*

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Proof. This follows immediately from theorem 3.3.10 and the fact that, in finitely ramified fields, $\mathrm{v}(\mathbf{R}_n)$ id finite and only contains multiples of π and hence s_n is entirely determined (by a finite disjunction on the possible values) by $\mathrm{s}_n(\pi)$. It follows that any (field quantifier free) formula involving s_n can be rewritten as disjunction of (field quantifier free) formulas that do not involve the the s_n. □

Corollary 3.5.8. *Let $\mathfrak{L}$ be a $\mathbf{\Gamma}$-enrichment of a $\mathbf{R}$-enrichment of $\mathfrak{L}_{\mathrm{ac,fr}}$. Any $\mathfrak{L}$-formula $\varphi(x,\gamma,\alpha)$, where x is a tuple of $\mathbf{K}$-variables, γ a tuple of $\mathbf{\Gamma}$-variables and α a tuple of $\mathbf{R}$-variables, is equivalent, modulo $\mathrm{Hen}_0^{\mathrm{ac,fr}}$, to:*

$$\bigvee_i \psi_i(\mathrm{v}(P(x)),\gamma) \wedge \chi_i(\mathrm{ac}_n(P(x)),\alpha),$$

where ψ_i is an $\mathfrak{L}|_{\mathbf{\Gamma}}$-formula, χ_i is an $\mathfrak{L}|_{\mathbf{R}}$-formula and $P \in \mathbb{Z}[x]$ is a tuple.

Equivalently, for any $\mathfrak{L}$-structures $M, N \vDash \mathrm{Hen}_0^{\mathrm{ac,fr}}$ and $\mathfrak{L}_{\mathrm{ac,fr}}$-embedding $f : A \leqslant M \to N$, we have:

$$f \text{ is } \mathfrak{L}\text{-elementary} \Leftrightarrow \begin{cases} f|_{\mathbf{\Gamma}} \text{ is } \mathfrak{L}|_{\mathbf{\Gamma}}\text{-elementary} \\ f|_{\mathbf{R}} \text{ is } \mathfrak{L}|_{\mathbf{R}}\text{-elementary} \end{cases}$$

Proof. Since there are no symbols involving both $\mathbf{\Gamma}$ and $\mathbf{R}$, any atomic formula is an $\mathfrak{L}|_{\mathbf{\Gamma}}$-formula or an $\mathfrak{L}|_{\mathbf{R}}$-formula (possibly applied to terms from $\mathbf{K}$). The statement follows. □

Corollary 3.5.9. *In the theory of finitely ramified characteristic zero henselian fields:*

1. *$\mathbf{\Gamma}$ is a pure stably embedded ordered monoid;*
2. *$\mathbf{R} = \bigcup_n \mathbf{R}_n$ is a pure stably embedded projective system of rings;*
3. *$\mathbf{\Gamma}$ and $\mathbf{R}$ are orthogonal.*

Proof. Let M be any finitely ramified characteristic zero henselian field, $X \subseteq \mathbf{\Gamma}^n \times \mathbf{R}^m$ be M-definable. By corollary 3.3.6, we find $M' \succcurlyeq M$ that admits a compatible system of angular components. Then, by corollary 3.5.8, $X(M') = \bigvee_i \psi_i(\gamma', M') \wedge \chi_i(\alpha', M')$, where $\gamma \in \mathbf{\Gamma}(M')$, $\alpha \in \mathbf{R}(M')$, ψ_i are ordered monoid formulas and χ_i are projective system of rings formulas. By elementarity, we find $\gamma \in \mathbf{\Gamma}(M)$, $\alpha \in \mathbf{R}(M)$ such that $X(M) = \bigvee_i \psi_i(\gamma, M) \wedge \chi_i(\alpha, M)$. All three statements follow. □

Corollary 3.5.10. *In the theory of unramified henselian fields of characteristic zero with perfect residue field, every $\mathbf{R}_n$, for $n \in \mathbb{Z}_{>0}$, is a pure stably embedded ring.*

Proof. Any definable $X \subseteq \mathbf{R}_1^m$ is, by corollary 3.5.9, of the form $\mathrm{res}_{1,n}(Y)$ where $Y \subseteq \mathbf{R}_n^m$ is definable with parameters from $\mathbf{R}_n$. But $\mathbf{R}_n$ is interpretable in $\mathbf{R}_1$ (as a ring) — by the Witt vector construction, or by natural isomorphism in residue characteristic zero — and hence, X is definable in the ring $\mathbf{R}_1$.

Since $\mathbf{R}_n$ also interprets[15] $\mathbf{R}_1$ — quotienting by its maximal ideal — any definable subset of $\mathbf{R}_n^m$ is thus also definable in the ring $\mathbf{R}_n$. ☐

3.6. Fields of p-adic numbers

Fix p a prime.

Definition 3.6.1. A valued field (K, v) is said to be p-adically closed of ramification degree e and residual degree f if:

- it is henselian of mixed characteristic $(0, p)$;
- $vK^\times$ has a smallest element $v(\pi)$ and $v(p) = e \cdot v(\pi)$;
- $[vK^\times : vK^{\times n}] = n$;
- $[Kv : \mathbb{F}_p] = f$.

Example 3.6.2. Let $\mathbb{Q}_p \leqslant F$ be a finite extension. Then F is p-adically closed of ramification degree $[vF : v\mathbb{Q}_p]$ and residual degree $[Fv : \mathbb{F}_p]$.

Lemma 3.6.3. *Let (K, v) be a p-adically closed of ramification degree e and residual degree f.*

1. *Let $q > 1$ be prime to p. We have $v(x) \geqslant 0$ if and only if $1 + \pi x^q \in K^{\times q} := \{y^q : y \in K^\times\}$.*
2. *Let $\pi, c \in \mathcal{O}(K)$ be such that $v(p) = e \cdot v(\pi)$ and $\mathbb{F}_p[\mathrm{res}(c)] = \mathbb{F}_{p^f}$. Then, for all $n \in \mathbb{Z}_{\geqslant 0}$, $\mathbf{R}_{p^n}(K) = \sum_{i<e,j<f} \mathbb{Z} \cdot \mathrm{res}_n(\pi^i c^j)$. In particular, it is finite.*
3. *Any p^n-th angular component map factorises through $K^\times \to K^\times/K^{\times m}$, where $m := |\mathbf{R}^\times_{p^n}(K)|$.*
4. *$\mathbb{Z}[\pi, c] \to K^\times/K^{\times m}$ is surjective. In particular, $K^\times/K^{\times m}$ is finite.*

Proof.
1. If $v(x) \leqslant -v(\pi) < 0$, then $v(\pi x^q) = v(\pi) + q \cdot v(x) < 0 = v(1)$ and $v(1 + \pi x^q) \in v(\pi) + q\mathbf{\Gamma} \neq q\mathbf{\Gamma}$. Conversely, if $v(x) \geqslant 0$, then $\mathrm{res}(X^q - 1 + \pi x^q) = X^q - 1$ has a simple zero at $1 = \mathrm{res}(1 + \pi x^q)$. By henselianity, $1 + \pi x^q \in K^q$.
2. We prove by induction, that any element of $\mathbf{R}_{p^n}/(\pi^\ell)$ is of the form $\sum_{i<e,j<f,k<n} \mathrm{res}_n(c_j \pi^i) p^k$ where $c_j \in \mathbb{Z}[c]$. The statement follows.
3. For any element $\alpha \in \mathbf{R}_n^\times(K)$, we have $\alpha^m = 1$ and hence, for any $x \in K^\times$, $\mathrm{ac}_n(x^m) = \mathrm{ac}_n(x)^m = 1$.
4. Note that, by minimality of $v(\pi)$, none of the $v(\pi^i)$, $0 < i < m$ are multiples of m. So there exists $y \in K^\times m$ and i such that $v(x) = i \cdot v(\pi) + m \cdot v(y)$. Let $z = xy^{-m}\pi^{-i}$. We have $v(z) = 0$. Let $a \in \mathbb{Z}[c, \pi]$ be such that $\mathrm{res}_{m^2}(a) = \mathrm{res}_{m^2}(a)$ and $P(X) = X^m - za^{-1}$. We have $\mathrm{res}_{m^2}(P(1)) = 0$, so $v(P(1)) > 2 \cdot v(m) = 2 \cdot v(P'(1))$. By henselianity, there exists $t \in K$ such that $t^m = za^{-1}$ and hence $x = zy^m \pi^i a \pi^i y^m t^m$. ☐

[15] In fact, we are really using bi-interpretations here: each structure interprets the other one and the isomorphism between double interpretations is definable.

Definition 3.6.4. Let $\mathfrak{L}_{\mathrm{Mac}}$ be the language with one sort K with the ring language and unary predicates P_n, for all $n \in \mathbb{Z}_{>0}$.

Any field K can be made into an $\mathfrak{L}_{\mathrm{Mac}}$-structure by interpreting the P_n as $K^{\times n}$. Let $p\mathrm{CF}_{e,f}$ denote the $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$ theory of p-adically closed fields with $v(p) = e \cdot v(\pi)$ and $\mathbb{F}_p[\mathrm{res}(c)] = \mathbb{F}_{p^f}$.

Theorem 3.6.5 (Macintyre, ? — Prestel-Roquette, ?). *The $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-theory $p\mathrm{CF}_{e,f}$ eliminates quantifiers.*

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Proof. By lemma 3.6.3, in any $M \vDash p\mathrm{CF}_{e,f}$, we have $v(x) \leqslant v(y)$ if and only if $v(yx^{-1}) \geqslant 0$ if and only if $1 + \pi y^q x^{-q} \in \mathbf{K}^{\times q}$, if and only if $x^q + \pi y^q \in \mathbf{K}^{\times q}$. Also, $\mathrm{ac}_{p^n}(x) = \mathrm{res}_{p^n}(a)$ where $a \in \mathbb{Z}[\pi, c]$ — with coefficients at most p^n — is such that $v(a) = 0$ and $xa^{-1}\pi^i \in \mathbf{K}^{\times m}$, uniformly defines compatible angular component maps.

It follows that any $\mathfrak{L}_{\mathrm{Mac}}$-embedding $f : A \leqslant M \to N$, where $N \vDash p\mathrm{CF}_{e,f}$ induces an $\mathfrak{L}_{\mathrm{ac,fr}}$-embedding. Note that $\mathbf{R}(M) = \mathbf{R}(A) = \mathbf{R}(f(A)) = \mathbf{R}(N)$ and hence $f|_{\mathbf{R}}$ is elementary.

Claim 3.6.5.1. *Pressburger arithmetic, the theory of ordered abelian groups G with a minimal positive element and such that $[G : nG] = n$ eliminates quantifiers in the language of ordered groups with predicates for $n \cdot G$ and a constant 1 for the minimal positive element.*

Proof. Let $f : A \leqslant G \to H$ be a maximal embedding. Fix $\gamma \in G$ and let n be its order in G/A. If $n < \infty$, then, for every $\alpha \in A$ $i\gamma < \alpha$ if and only if $in\gamma < n\alpha$ and $i\gamma + \alpha \in mG$ if and only if $in\gamma + n\alpha \in nmG$. So the isomorphism type of γ over A is entirely determined by $\delta = n\gamma \in A$. Since $\delta \in nG$, we also have $f(\delta) \in nG$ and we find $\varepsilon \in H$ such that $n\varepsilon = f(\delta)$. We can extend f by sending γ to ε and hence, by maximality, $\gamma \in A$. So A is relatively divisible in M.

If $n = \infty$, the isomorphism type of γ of over A is determined by $\{\alpha \in A : \gamma < \alpha\}$ and $i_m \in \mathbb{Z}$ such that $\gamma - i_m \in mG$. Indeed, $m\gamma > \alpha$ if and only if, since $m\gamma \notin \alpha + \mathbb{Z} \subseteq A$, $m\gamma > \alpha + i = m\beta$, for some $\beta \in A$, if and only if $\gamma > \beta$; and $j\gamma + \alpha \in mG$ if and only if $ki_m + \alpha \in mG$. By compactness, f extends by sending γ into some $H^\star \succcurlyeq H$. By maximality, $\gamma \in A$ and hence $A = G$. ◊

It follows that $f|_{\Gamma}$ is elementary. Note that every element in $\mathbf{R}(M) = \mathbf{R}(N)$ is named by a constant, so, being an isomorphism, $f|_{\mathbf{R}}$ is elementary. So, by corollary 3.5.8, f is $\mathfrak{L}_{\mathrm{ac,fr}}$-elementary — in particular, it is $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-elementary. □

Corollary 3.6.6. *The class $p\mathrm{CF}_{e,f}$ is model complete in $\mathfrak{L}_{\mathrm{rg}}$.*

Proof. Let $F \leqslant K$ both models of $p\mathrm{CF}_{e,f}$ and $\pi, c \in F$ such that $v(p) = ev(\pi)$ and $\mathbf{k}(K) = \mathbf{k}(F) = \mathbb{F}_p[\mathrm{res}(c)]$. Note also that, as seen in the proof of lemma 3.6.3, for every $n \in \mathbb{Z}_{>0}$, $K^\times/K^{\times n} \simeq \mathbf{R}_{n^2}(K)/\mathbf{R}_{n^2}^{\times n}(K) \times \pi^{\mathbb{Z}}/\pi^n \simeq \mathbf{R}_{n^2}(F)/\mathbf{R}_{n^2}^{\times n}(F) \times \pi^{\mathbb{Z}}/\pi^n \simeq F^\times/F^{\times n}$. So F is an $\mathfrak{L}_{\mathrm{Mac}}$-substructure of K and, by theorem 3.6.5, $F \succcurlyeq K$ — as $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-structures, and hence as $\mathfrak{L}_{\mathrm{rg}}$-structures. □

Corollary 3.6.7. *Let K be p-adically closed and $F = F^{\mathrm{a}} \cap \mathrm{dcl}(F) \subseteq K$. Then $F \preccurlyeq K$ (as rings).*

Proof. Since $\mathbf{R}_\infty(K)$ is a p-ring, by proposition 1.3.17, $\mathbb{Q}_p = \mathrm{W}(\mathbb{F}_p) \leqslant \mathrm{W}(\mathbb{F}_{p^f}) \leqslant \mathbf{R}_\infty(K)$. In fact, as seen in lemma 3.6.3, this extension is finite. Let $\alpha = \mathrm{res}_\infty(a) \in \mathbf{R}_\infty(K)$ generate it and $P \in \mathbb{Z}_p[x]$ its minimal polynomial. We have $P'(\alpha) \neq 0$, *i.e.* $v(P'(a)) < nv(p)$ for some $n \in \mathbb{Z}_{>0}$. Let $Q \in \mathbb{Z}[x]$ be such that $\mathrm{res}_{p^{2n}}(Q) = \mathrm{res}_{p^{2n}}(P)$. Then $\mathrm{res}_{p^{2n}}(Q(a)) = \mathrm{res}_{p^{2n}}(P)(\alpha) = 0$ and $\mathrm{res}_{p^n}(Q'(a)) = \mathrm{res}_{p^n}(P')(\alpha) \neq 0$. So $v(Q(a)) > 2v(Q'(a))$ and, by henseliainity, there exists $c \in \mathbb{Q}^{\mathrm{a}} \cap \mathrm{dcl}(\mathbb{Q}) \leqslant F$ such that $\mathrm{res}_{p^n}(c) = \mathrm{res}_{p^n}(a)$. So $\mathbf{R}_{p^n}(K) = \mathbf{R}_{p^n}(F)$ and $F \vDash p\mathrm{CF}_{e,f}$. By corollary 3.6.6, $F \preccurlyeq K$. □

Corollary 3.6.8. *Any two p-adically closed fields K, F of arbitrary ramification and residual degree are elementary equivalent if and only if $K \cap \mathbb{Q}^{\mathrm{a}} \simeq F \cap \mathbb{Q}^{\mathrm{a}}$.*

Proof. If $K \equiv F$, then, by compactness, $K \cap \mathbb{Q}^{\mathrm{a}}$ embeds in $F \cap \mathbb{Q}^{\mathrm{a}}$ which embeds in $K \cap \mathbb{Q}^{\mathrm{a}}$, so they are isomorphic. Conversely, if $K \cap \mathbb{Q}^{\mathrm{a}} \simeq F \cap \mathbb{Q}^{\mathrm{a}}$, by corollary 3.6.7, we have $K \equiv K \cap \mathbb{Q}^{\mathrm{a}} \simeq F \cap \mathbb{Q}^{\mathrm{a}} \equiv F$. □

Remark 3.6.9. Any p-adically closed field is elementarily equivalent to a finite extension of $(\mathbb{Q}, v_p)^{\mathrm{h}}$ which is elementarily equivalent to its completion — a finite extension of $\mathbb{Q}_p$.

Corollary 3.6.10. *Let $K \vDash p\mathrm{CF}_{e,f}$ and $F \leqslant K$. Then $\mathrm{dcl}(F) = \mathrm{acl}(F) = F^{\mathrm{a}} \cap K \preccurlyeq K$.*

Proof. By corollary 3.6.7, $F^{\mathrm{a}} \cap \mathrm{dcl}(F) \preccurlyeq K$. So $\mathrm{dcl}(F), F^{\mathrm{a}} \cap K \subseteq \mathrm{acl}(F) \subseteq F^{\mathrm{a}} \cap \mathrm{dcl}(F)$ and the statement follows. □

Corollary 3.6.11. *The theory $pCF_{e,f}$ has definable Skolem functions: for every $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-definable family $(X_y)_{y \in Y}$ of non empty sets, there exists an $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-definable function $f : Y \to \bigcup_{y \in Y} X_y$ such that for every $y \in Y$, $f(y) \in X_y$.*

Proof. For every $N \succcurlyeq M$ and $y \in Y(N)$, by corollary 3.6.10, $\mathrm{dcl}(y) \preccurlyeq N$ and hence there exists an $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-definable f such that $f(y) \in X$. By compactness, there exists $\mathfrak{L}_{\mathrm{Mac}}(\pi, c)$-definable $(f_i)_{i<n}$ such that for every $y \in Y$, $f_i(y) \in X$, for some i. The $X_i = \{y : f_i(y) \in X \wedge f_j(y) \notin X \text{ for all } j < i\}$ are disjoint and $f := \bigcup_i f_i|_{X_i}$ has the required properties. □

4. The independence property

Let $\mathfrak{L}$ be a language, T be an $\mathfrak{L}$-theory. 22/03

Definition 4.0.1. Let $(I, <)$ be totally ordered. A sequence $(a_i)_{i \in I} \in M$ some $\mathfrak{L}$-structure is said to be $\mathfrak{L}$-indiscernible if for every tuple $i, j \in I$ with $i \equiv^{\mathrm{qf}}_< j$ then $a_i \equiv_{\mathfrak{L}} a_j$.

Lemma 4.0.2. *Let M be an $\mathfrak{L}$-structure and for every $n \in \mathbb{Z}_{>0}$ and $(a_i)_{\in \mathbb{Z}_{\geqslant 0}} \in M^x$. Then for any total order $(I, <)$, there exists $M^\star \succcurlyeq M$ and $(c_i)_{i \in I} \in M^\star$ $\mathfrak{L}$-indiscernible such that, if, for every increasing $g : n \to \mathbb{Z}_{\geqslant 0}$, $M \vDash \varphi(a_{g(n)})$, then, for any increasing $f : n \to I$, $M^\star \vDash \varphi(c_{f(n)})$.*

Proof. Let π_n be the common (partial) type of the $a_{g(n)}$ and let us consider the set of formulas:

$$\Sigma((x_i)_{i \in I}) := \bigcup_{f : n \to I \text{ inc.}} \pi_n(x_{f(n)}) \cup \bigcup_{\substack{f,g : n \to I \text{ inc.} \\ \varphi}} \varphi(x_{f(n)}) \leftrightarrow \varphi(x_{g(n)}).$$

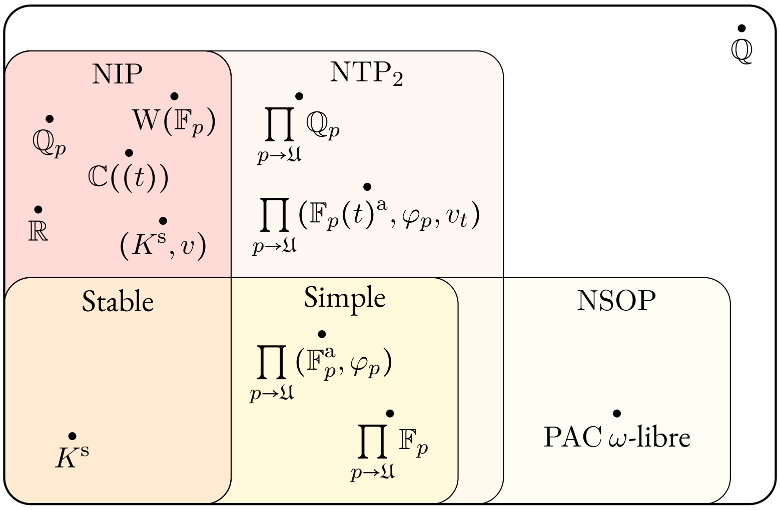


Figure 1: The universe

Claim 4.0.2.1 (Ramsey). *For any $\varphi((x_i)_{i<n})$ and $m \in \mathbb{Z}_{>0}$, there exists $J \subseteq \mathbb{Z}_{>0}$ of size m such that, for any increasing $f, g : n \to J$, $M \vDash \varphi(a_{f(n)}) \leftrightarrow \varphi(a_{g(n)})$.*

So Σ is finitely satisfiable and the lemma holds. □

For every $\mathfrak{L}$-formula φ, we define $\varphi^1 := \varphi$ and $\varphi^0 := \neg\varphi$.

Lemma 4.0.3. *Let $\varphi(x; y)$ be an $\mathfrak{L}$-formula. The following are equivalent:*

(i) for every $n \in \mathbb{Z}_{>0}$, there exists $M \vDash T$, $(a_i)_{i<n}$ and, for every $J \subseteq n$, $b_J \in M^y$ such that, $M \vDash \varphi(a_i, b_J)$ if and only if $i \in J$.

(ii) for every sets $R \subseteq I \times J$, there exists $M \vDash T$, $(a_i)_{\in I}$ and $(b_j)_{j\in J}$ with $M \vDash \varphi(a_i, b_j)$ if and only if $(i, j) \in R$, for every $i \in I$ and $j \in J$;

(iii) there exists $M \vDash T$, $A \leqslant M^x$ infinite and, for every $B \subseteq A$, $b_B \in M^y$ such that, $M \vDash \varphi(A, b_B) := \{a \in A : M \vDash \varphi(a, b_B)\} = B$;

(iv) for every total order $(I, <)$ without a largest element, there exists $M \vDash T$, $(a_i)_{i\in I} \in M^x$ $\mathfrak{L}$-indiscernible and $b \in M^y$ such that both $J_\ell := \{i \in I : M \vDash \varphi^\ell(a_i, b)\}$, for $\ell = 0, 1$ are cofinal in I;

(v) there exists $M \vDash T$, $(a_i)_{i\in\mathbb{Z}_{\geqslant 0}} \in M^x$ $\mathfrak{L}$-indiscernible and $b \in M^y$ such that $M \vDash \varphi(a_i, b)$ if and only if i is even.

(vi) $\mathrm{alt}_T(\varphi(x, y)) = \infty$, where for every $n \in \mathbb{Z}_{>0}$, $\mathrm{alt}_T(\varphi(x, y)) \geqslant n$ if there exists $M \vDash T$, $(a_i)_{i\in I} \in M^x$ $\mathfrak{L}$-indiscernible, $b \in M^y$ and $f : n+1 \to I$ increasing such that $\varphi(a_{f(i)}, b) \leftrightarrow \neg\varphi(a_{f(i+1)}, b)$, for every $i < n$;

Proof. Note that (iv) ⇒(v) ⇒(vi) are immediate.

(i)⇒(ii) By (i), the set of formulas $\Sigma((x_i)_{i\in I}, (y_j)_{j\in J}) := \{\varphi^{\mathbb{1}_{(i,j)\in R}}(x_i, y_j) : i \in I, j \in J\}$ is finitely satisfaible and hence, by compactness, satisfiable in some model of T.

(ii)⇒(iii) Consider $R = \{(i, I) \in \mathbb{Z}_{\geqslant 0} \times \mathfrak{P}(\mathbb{Z}_{\geqslant 0}) : i \in I\}$. Then (i) applied to R yields $A := \{a_i : i \geqslant 0\}$ and, for every $B = \{a_j : j \in J\} \subseteq A$, $b_B := b_J$ such that, for every $a = a_i \in A$, $M \vDash \varphi(a, b_B)$ if and only if $(i, J) \in R$, *i.e.* $i \in J$.

(iii)⇒(iv) Let $g : \mathbb{Z}_{\geqslant 0} \to A$ be some injection. By lemma 4.0.2, there exists $M^\star \succcurlyeq M$ and $(c_i)_{i \in I} \in (M^\star)^x$ which is $\mathfrak{L}$-indiscernible and such that, for any increasing $f : n \to I$, $a_{f(n)}$ realises the common partial type of the tuple $g(h(n))$, where $h : n \to \mathbb{Z}_{\geqslant 0}$ is increasing. In particular, for any $J \subseteq n$, $M^\star \vDash \exists y \bigwedge_{j<n} \varphi^{\mathbb{1}_{j \in J}}(a_{f(j)}, y)$. By compactness. for any $J \subseteq I$, we find $b \in M^\dagger \succcurlyeq M^\star$ such that $M^\dagger \vDash \varphi(a_i, b)$ if and only if $i \in J$. By induction, we can build $J \subseteq I$ cofinal such that $I \smallsetminus J$ is also cofinal.

(vi)⇒(i) Fix some n. By (v), we find $(a_i)_{i \in I} \in M^x$ $\mathfrak{L}$-indiscernible, $b \in M^y$ and $f : 2n + 1 \to I$ increasing such that $\varphi(a_{f(i)}, b) \leftrightarrow \neg\varphi(a_{f(i+1)}, b)$, for every $i < 2n$. In particular, for every $J \subseteq n$, there exists $g : n \to I$ increasing such that $M \vDash \varphi(a_{g(i)}, b)$ if and only if $i \in J$. Since $(a_i)_{i \in I}$ is $\mathfrak{L}$-indiscernible, for any increasing $g : n \to I$, we have $M \vDash \exists x \bigcup_{i<n} \varphi^{\mathbb{1}_{j \in J}}(a_{g(i)}, x)$. □

Definition 4.0.4. Let $\varphi(x; y)$ be an $\mathfrak{L}$-formula. We say that:

- φ has the independence property (in T) if it verifies the equivalent conditions of lemma 4.0.3.
- T does not have the independence property — is dependent, is NIP — if no $\mathfrak{L}$-formula has the independence property in T.
- An $\mathfrak{L}$-structure M is NIP if $\mathrm{Th}(M)$ is.

Lemma 4.0.5. *If the $\mathfrak{L}$-formulas $(\varphi_i(x, y))_{i<n}$ are* NIP *in T, then so are any boolean combinations.*

Proof. For any $\mathfrak{L}$-formulas $(\varphi_i(x, y))_{i<n}$ and ψ boolean combination of the φ_i, $\mathrm{alt}_T(\psi) \leqslant \sum_i \mathrm{alt}_T(\varphi_i)$, where ψ is any boolean combination of the φ_i. Indeed if $\varphi_i(a_1, b) \leftrightarrow \varphi_i(a_2, b)$, for every $i < n$, then $\psi(a_1, b) \leftrightarrow \psi(a_2, b)$. □

Lemma 4.0.6. *The theory T is* NIP *if and only if no $\mathfrak{L}$-formula $\varphi(x, y)$ with $|x| = 1$ has the independence property in T.*

Proof. Let us assume that no $\mathfrak{L}$-formula $\varphi(x, y)$ with $|x| = 1$ — and hence, by lemma 4.0.3, no $\mathfrak{L}$-formula $\varphi(x, y)$ with $|y| = 1$ — has the independence property in T.

Claim 4.0.6.1. *Let $M \vDash T$ and $(a_i)_{i<\omega\times|T|^+} \in M^x$ be $\mathfrak{L}$-indiscernible and $b \in M^y$ with $|y| = 1$. Then $(a_i)_i$ is eventually $\mathfrak{L}(b)$-indiscernible.*

Proof. Any $\psi((x_i)_{i<\omega}, y)$ is NIP in T and hence, by lemma 4.0.3, there exists $i_\psi < |T|^+$ and $\ell \in \{0, 1\}$ such that, for every $i \geqslant i_0$ $M \vDash \psi^\ell((a_{j,i})_{j<\omega}, b)$. Since $|T|^+$ is regular, we may assume that $i_\psi = i_0$ does not depend on ψ and hence, $(a_i)_{i \geqslant (0,i_0)}$ is $\mathfrak{L}(b)$-indiscernible. ◊

By induction, for any tuple $b \in M$, $(a_i)_i$ is eventually $\mathfrak{L}(b)$-indiscernible — *i.e.* $(a_i b)_i$ is $\mathfrak{L}$-indiscernible. Indeed, for every $c \in M^1$, by claim 4.0.6.1, $(a_i b)_i$ is eventually $\mathfrak{L}(c)$-indiscernible — *i.e.* $(a_i)_{i \geqslant i_1}$ is $\mathfrak{L}(bc)$-indiscernible. So for any $\mathfrak{L}$-formula $\varphi(x, y)$, the truth value of $\varphi(a_i, b)$ is eventually constant. So the negation of lemma 4.0.3.(iv) holds. □

Example 4.0.7. • Any stable theory is NIP : a formula $\varphi(x,y)$ is unstable if there exists $(a_i)_{i\in\omega}, (b_j)_{j\in\omega}$ such that $\models \varphi(a_i, b_j)$ if and only if $i < j$. This is a particular case of lemma 4.0.3.(ii).

- In any order, the formula $x < y$ is NIP. Indeed you cannot have $a_1 < b_{\{1\}} \leqslant a_2 < b_{\{2\}} \leqslant a_1$. In fact, any o-minimal theory is NIP.

Theorem 4.0.8 (Gurevitch-Schmidt, ?). *The $\mathfrak{L}_{\mathrm{og}}$-theory of ordered abelian groups is* NIP.

Theorem 4.0.9. *The theory* ACVF *is* NIP

Proof. By lemma 4.0.6 and theorem 2.2.5, it suffices to prove that $\varphi(x,yz) := \mathrm{v}(x-y) > \mathrm{v}(y-z)$ and $\psi(x,yz) := \mathrm{v}(x-y) \geqslant \mathrm{v}(y-z)$ are NIP. But since these sets define balls, this follows from the following observation: given three points $(a_i)_{i<3}$, if a_0, a_1 are in some ball b not containing a_2, then any ball containing a_1 and a_2 also contains a_0. □

Theorem 4.0.10. *Let M be a finitely ramified henselian field. Then*

$$M \textit{ is } \mathrm{NIP} \Leftrightarrow \mathbf{R}(M) \textit{ is } \mathrm{NIP}.$$

Proof. Let us assume that $\mathbf{R}(M)$ (and $\mathbf{\Gamma}(M)$) is NIP. Let $(a_i)_{i\in I} \in M^x$ be indiscernible.

Claim 4.0.10.1. *Increasing a_i, we may assume that each a_i enumerates an elementary substructure of M.*

Proof. Let $a_i \in M_i \preccurlyeq M$ and b_i enumerate M_i. By lemma 4.0.2, We find $(b'_i)_i$ indiscernible realising the common type of the b_i (in some elementary extension). In particular, $(a'_i)_i \equiv (a_i)_i$, where $a'_i \subseteq b'_i$ corresponds to $a_i \subseteq b_i$. By compactness, we can assume that b'_i contains a_i. ◊

Let D be $\varnothing$-definable, stably embedded, with an NIP induced structure.

Claim 4.0.10.2. *For any tuple $d \in D(M)$ the truth value of any formula $\varphi(xd)$ is eventually constant on $(a_i)_i$.*

Proof. If not, let $\varphi(xd)$ alternate along $(a_i)_i$. Since D is stably embedded, there exists $c_i \in D(M)^z$ and a formula $\psi(zy)$ such that $\varphi(a_iM) = \psi(c_iM)$. Since a_i enumerates an elementary substructure of M, we can assume that $c_i \subseteq a_i$. In particular, $(c_i)_{i\in I}$ is indiscernible and $\psi(zd)$ alternates along this sequence. This contradicts that the induced structure on D is NIP. ◊

If $|I| > \aleph_0$ is regular and the a_i and d are at most countable, then it follows form claim 4.0.10.2 that $\mathrm{tp}(a_i/d)$ is eventually constant. Let us now assume that $I = \omega$ and, by contradiction, let $c \in M$ and φ be a formula such that $M \models \varphi(a_ic)$ if and only if i is even.

Claim 4.0.10.3. *For every $i \in \omega$, let $d_{2i} \in D(M)$ be such that $(a_{2i}d_{2i})_{i\in\omega}$ and $(a_{2i}a_{2i+1})_{i\in\omega}$ are c-indiscernible. Then, we can find d'_i and a'_i such that $a'_{2i}d'_{2i} = a_{2i}d_{2i}$, $(a'_{2i+1})_{i<\omega} \equiv_c (a_{2i+1})_{i<\omega}$, the sequence $(a'_{2i}a'_{2i+1})_i$ is c-indiscernible and the sequence $(a'_id_i)_i$ is indiscernible.*

Proof. Applying lemma 4.0.2, we find $(a'_{2j,i}a'_{2j+1,i}d'_{2j,i})_{(j,i)\in\aleph_1\times\omega}$ which is c-indiscernible and realises the common type of $(a_{2i}a_{2i+1}d_{2i})_i$ over c. In particular, $(a'_j)_{j\in\aleph_1\times\omega}$ is indiscernible, $(a'_{2j,i}a'_{2j+1,i})_{(j,i)} \equiv_c (a_{2i}a_{2i+1})_i$ and $(a'_{2j,i}d'_{2j,i})_{(j,i)} \equiv_c (a_{2i}d_{2i})_i$. By compactness, we can assume that $a'_{0,i}d'_{0,i} = a_{2i}d_{2i}$, for all $i < \omega$. Let $a_{2\omega} := (a_{2i})_{i<\omega}$ and $d_{2\omega} := (d_{2i})_{i<\omega}$ and consider $b_j := (a'_{j+1,i})_{i<\omega}$. Then $(b_j)_j$ is $\mathfrak{L}(a_{2\omega})$-indiscernible, *i.e* $(b_j a_{2\omega})_j$ is $\mathfrak{L}$-indiscernible. By claim 4.0.10.2, $\mathrm{tp}(b_j a_{2\omega}/d_{2\omega})$ is eventually constant so there exists j_0 such that $(a'_{2j_0,i})_i \equiv_{a_{2\omega}d_{2\omega}} a'_{2j_0+1,i}$. By compactness, we find d'_{2i+1} such that $(a_{2j_0,i}d_{2j_0,i})_i \equiv_{a_{2\omega}d_{2\omega}} (a_{2j_0+1,i}d'_{2i+1})_i$. If we set $a'_{2i+1} = a'_{2j_0+1,i}$, we have $(a_{2i}d_{2i}a'_{2i+1}d'_{2i+1})_{i<\omega} \equiv (a'_{0,i}d'_{0,i}a'_{2j_0,i}d'_{2j_0,i})_{i<\omega}$ and hence $(a'_i d'_i)_i$ is indiscernible. ◊

By lemma 4.0.2, we may assume that $(a_{2i}a_{2i+1})_{i<\omega}$ is c-indiscernible. For every $i < \omega$, let b_{2i} enumerate $\mathbf{\Gamma}(\mathrm{dcl}(a_{2i}c))$ and d_{2i} enumerate $\mathbf{R}(\mathrm{dcl}(a_i c))$. By the claim 4.0.10.3, changing a_{2i+1} but preserving the alternation, we find b_{2i+1} and d_{2i+1} such that $(a_i b_i d_i)_{i<\omega}$ is indiscernible. By section 4, we find $(a^1_i)_i$ indiscernible that each enumerate an elementary substructures containing $a_i b_i d_i$. Iterating this construction, we find $(a^\omega_i)_{i<\omega}$ indiscernible that each enumerates a countable elementary substructures N_i containing a_i and such that $N_{2i} \leqslant N_{2i}(c)$ is immediate. Again, we can extend this sequence (preserving the alternation) to be indexed by $\aleph_1$.

Let b_0 be a ball in N_0 and b_i be the corresponding ball in N_i. Three cases are possible. Either all the b_i are disjoint and c can only be in one of them. Or they form an increasing sequence and if c is in one of them, it is in all the later ones. Or they form a decreasing sequence and if c is not in one of them it is not in any of the later ones. In all three cases, c is either eventually in b_i or outside of b_i. It follows, that the maximal pseudo Cauchy filters over each N_i accumulating at c eventually correspond via the isomorphism $a^\omega_i \mapsto a^\omega_j$ and hence, by proposition 3.5.4, eventually $a_i c \equiv a_{i+1}c$, contradicting the alternation. □

Corollary 4.0.11 (Delon, ...). *Let M be an unramified henselian field with perfect residue field. Then*

$$M \text{ is NIP} \Leftrightarrow \mathbf{k}(M) \text{ is NIP}.$$

Proof. This follows from theorem 4.0.10 and the fact that $\mathbf{R}$ is interpretable in $\mathbf{k}$. □

Example 4.0.12.
- p-adically closed fields are NIP.
- $\mathbb{C}((t))$ and $\mathbb{R}((t))$ are NIP.
- $\prod_{p\to\mathfrak{U}} \mathbb{Q}_p$ is *not* NIP, where $\mathfrak{U}$ is a non principal ultrafilter on the set of primes.

5. Perspectives

5.1. Imaginaries

We wish to consider three related questions:
- What do quotients by definable equivalence relations look like?
- Do definable families admit moduli spaces — *i.e.* a definable set whose points are in definable bijection with the definable sets that appear in the family?
- Do definable set have a smallest set of definition?

But first, let us set up some background. Let $\mathfrak{L}$ be some language, M be an $\mathfrak{L}$-structure and $X \subseteq M^x$ be $\mathfrak{L}(M)$-definable.

Proposition 5.1.1 (Poizat, 1983). *Let T be a theory such that for every finite tuples of sorts X and Y, there exists a finite tuple of sorts Z and $\mathfrak{L}$-definable injective maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$ with disjoint images*[16]*. The following are equivalent:*

(i) T eliminates imaginaries: for every $M \vDash T$ and $\mathfrak{L}(M)$-definable X set $X \subseteq M^x$, there exists an $\mathfrak{L}$-formula $\psi(xy)$ and $a \in M^y$ such that, for all $c \in M^y$, $\psi(M, c) = X$ if and only if $c = a$ — we say that a is a canonical parameter of X via ψ;

(ii) T uniformly eliminates imaginaries : for every $\mathfrak{L}$-definable $V \subseteq X \times Y$, there exists an $\mathfrak{L}$-definable $W \subseteq X \times Z$ such that for every $M \vDash T$ and $a \in Y(M)$, there exists a unique $c \in Z(M)$ such that $V_a := \{x \in X : xa \in V\} = W_c$.

(iii) Any interpretable set is represented, in T, by a definable set: for every $M \vDash T$, $A \leqslant M$ and $\mathfrak{L}(A)$-definable equivalence relation $E \subseteq X \times X$, there exists and $\mathfrak{L}(A)$-definable map $f : X \to Z$ such that for all $x_1, x_2 \in X$, $x_1 E x_2$ if and only if $f(x_1) = f(x_2)$.

Sketch of proof.

(i)⇒(ii) This follows form compactness.

(ii)⇒(iii) Applying (ii) to $E \subseteq X \times X$, we define $f(x)$ to be the unique z such that $E_x = W_z$.

(iii)⇒(i) Applying (iii) to the equivalence relation $y_1 E y_2$ defined by $\forall x, \varphi(x, y_1) \Leftrightarrow \varphi(x, y_2)$, we define $\psi(x, z) := \exists y\ f(y) = z \wedge \varphi(x, y)$. □

Note that if a is a canonical parameter of X, via some φ, then $\mathrm{dcl}(a)$ does not depend on the choice of φ and is the smallest dcl-closed set of definition of X.

Theorem 5.1.2 (Poizat, 1983). *The $\mathfrak{L}_{\mathrm{rg}}$-theory* ACF *(uniformly) eliminates imaginaries.*

Sketch of proof. Let $K \vDash \mathrm{ACF}$ and $X \subseteq \mathbf{K}^n$ be $\mathfrak{L}_{\mathrm{rg}}(K)$-definable. Let p be the generic type of an irreducible components of the Zariski closure $\overline{X}^{\mathrm{Z}} \subseteq K^n$. The type p is entirely determined by $I(p) := \{P \in K[x] : p \vDash P = 0\} = \bigcup_d V_d(p)$, where $V_d(p) := \{P \in K[x] : p \vDash P = 0 \text{ and } \deg(P) \leqslant d\} \leqslant K^{m_d}$ is a sub-K-vector space of dimension r_d. Note that $\bigwedge^{r_d} V_d(p) \leqslant \bigwedge^{r_d} K^{m_d} \simeq K^{n_d}$ is dimension 1 — it is therefore an element $a_d \in \mathbb{P}(K^{n_d})$ which can be $\mathfrak{L}$-definably identified with a subset of K^{n_d+1}.

Note that since X is consistent with p, by quantifier elimination, $p \vDash X$ and, if X is a class of an $\mathfrak{L}$-definable equivalence relation, it is $\mathfrak{L}(a_d)$-definable for a sufficiently large d. Note that the orbit of p under $\mathrm{aut}_X(M) := \{\sigma \in \mathrm{aut}(M) : \sigma(X) = X\}$ is contained in the set of generic types of irreducible components of the Zariski closure $\overline{X}^{\mathrm{Z}} \subseteq K^n$, which is finite. Thus so is the orbit $C := \{c_i : i < n\}$ of a_d. Note that X is $\mathfrak{L}(c_i)$-definable for any choice of i.

Let $P(x, y) = \prod_i (x - \sum_j c_{ij} Y^j)$ and d be the tuple of its coefficients. Then C is $\mathfrak{L}(d)$-definable, and d is fixed by $\mathrm{aut}_X(M)$ — *i.e.* d is a canonical parameter of X. □

Definition 5.1.3. Let $\mathfrak{L}_{\mathcal{G}}$ be the language with:

- a sort $\mathbf{K}$ with the ring language;
- sorts S_n, for all $n \in \mathbb{Z}_{>0}$;

[16]In other words, the category of definable sets has finite coproducts.

- a map $\mathrm{s}_n : K^{n^2} \to \mathrm{S}_n$;
- sorts T_n, for all $n \in \mathbb{Z}_{>0}$;
- a map $\mathrm{t}_n : K^{n^2} \to \mathrm{T}_n$.

Any valued field can be made into an $\mathfrak{L}_{\mathcal{G}}$-structure by interpreting $\mathbf{K}$ as the field, S_n as $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$, which is the moduli space of rank n free sub-$\mathcal{O}$-modules of K^n, s_n as the canonical projection, T_n as $\bigsqcup_{s\in \mathrm{S}_n} s/\mathfrak{m}s$ and t_n as the map sending a matrix $m \in \mathrm{GL}_n(\mathbf{K})$ to the coset of $\mathrm{s}_n(m)/\mathfrak{m}\mathrm{s}_n(m)$ given by the first vector.

Theorem 5.1.4 (Haskell–Hrushovksi–Macpherson, 2006). *The $\mathfrak{L}_{\mathcal{G}}$-theory* ACVF *eliminates imaginaries.*

Sketch of proof. Let $K \vDash$ ACVF and $X \subseteq \mathbf{K}^n$ be $\mathfrak{L}(K)$-definable. A type $p \in \mathcal{S}_x(K)$ is said to be $\mathfrak{L}(K)$-definable if, for every formula $\varphi(xy)$, there exists $\theta(y)$ such that, for all $a \in K$, $p \vDash \varphi(x,a)$ if and only if $K \vDash \theta(a)$.

Claim 5.1.4.1. *There exists an $\mathfrak{L}_{\mathcal{G}}(K)$-definable type $p \in \mathcal{S}_X(K)$ with a finite* $\mathrm{aut}_X(K)$*-orbit.*

Proof. Assume first that $X \subseteq \mathbf{K}$. Then, by theorem 2.2.5, X has a unique decomposition $\bigcup_{i<n} b_i \smallsetminus b_{i,j}$ of disjoint non nested Swiss cheeses. Then $p := \eta_{b_0}$ has the required properties. If $X \subseteq \mathbf{K}^{n+1}$, by induction, we find an $\mathfrak{L}_{\mathcal{G}}(K)$-definable $q \in \mathcal{S}_{\pi(X)}(K)$ with a finite $\mathrm{aut}_X(K)$-orbit, where $\pi : \mathbf{K}^{n+1} \to \mathbf{K}^n$ is the projection on the first n coordinates. Let $a \in K^\star \succcurlyeq K$ realise q. By the dimension 1 case, we find an $\mathfrak{L}_{\mathcal{G}}(K)$-definable $r_a \in \mathcal{S}_{X_a}(K^\star)$ with a finite $\mathrm{aut}_{X_a}(K^\star)$-orbit. Let $c \vDash r_a$, then $\mathrm{tp}(ac/K)$ has the required properties. ◊

Let V be a K-vector space. A valuation on V is a map $v : V \to \Sigma$, where $(\Sigma, <)$ is ordered and $\mathrm{v}K$ acts (increasingly) on Σ and such that $v(0)$ is maximal, for all $x, y \in V$, $v(x+y) \geqslant \min\{v(x), v(y)\}$ and for all $a \in K$, $v(ax) = \mathrm{v}(a) + v(x)$. We say that a valuation v on K^n is definable if $v(x) \leqslant v(y)$ is. For very $d \in \mathbb{Z}_{\geqslant 0}$, let v_d be the (definable) valuation on K^{m_d} such that $v_d(P) \leqslant v_d(Q)$ if and only if $p(x) \vDash \mathrm{v}(P(x)) \leqslant \mathrm{v}(Q(x))$. By quantifier elimination, the v_d characterise p.

Claim 5.1.4.2. *There exists a basis $(P_i)_{i<m_d}$ of K^{m_d} such that for all $a_i \in K$, $v(\sum_i a_i P_i) = \min_i \mathrm{v}(a_i) + v(P_i)$. Moreover, the P_i can be chosen such that $v(P_i)$ is fixed by $\mathrm{aut}(K/p)$ and for all i,j, if there exists $a \in \mathrm{v}K^\times_{>0}$ such that $v(P_i) + \mathrm{v}(a) < v(P_j)$ then $v(P_i) + \mathrm{v}K^\times < v(P_j)$.*

Fix some i. Let $W_{i,d}$ be the K-vector subspace generated by the P_j with $v(P_j) > v(P_i) + \mathrm{v}K^\times$ and $S_{d,i}$ be the free $\mathcal{O}$-submodule generated by the P_j with $v(P_j) \geqslant v(P_i) = \mathrm{v}(a)$ for some $a \in K^\times$. Note that the $R_{d,i} := \{a \in K^{m_d} : v(\sum_j a_j P_j) \geqslant v(P_i)\} = S_{d,i} + W_{d,i}$ characterise v_d and that $R_{d_i} \cap S_{d,i}$ is the preimage of $R_{d_i} \cap S_{d,i}/\mathfrak{m}S_{d,i}$. Using external powers and projective spaces we can then associate a tuple of $\mathbf{K}$-points to $W_{d,i}$ and a tuple of T_n-points to $R_{d_i} \cap S_{d,i}/\mathfrak{m}S_{d,i}$.

It then remains to show that finite sets of tuples in the geometric sorts have canonical parameters — which is far from trivial. □

5.2. A classification of NIP fields

Conjecture 5.2.1. *Every infinite* NIP *field either:*
- *is separably closed;*
- *is real closed;*
- *admits a non trivial henselian valuation.*

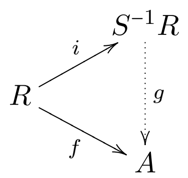
Remark 5.2.2.
1. In other terms every NIP field is elementarily equivalent (as a field) to a non trivially valued henselian field.
2. The henselian valuation (when it exists) can be assumed to be definable.
3. This conjecture implies the stable fields conjecture : every stable is separably closed.

A. The localisation of a ring

Definition A.0.1. Fix R a ring and $S \subseteq R$.
(1) The subset S is multiplicative if $1 \in S$ and for every $x, y \in S$, $xy \in S$.
(2) We then define the equivalence relation $(x, s) \sim_S (y, t)$ on $R \times S$ to hold if there exists $z \in R$ such that $zxt = zys$.
(3) We also define the ring $S^{-1}R := R \times S/\sim_S$, localised at S, where:
  - $(x, s) + (y, t) = (xt + ys, st)$;
  - $(x, s) \cdot (y, t) = (xy, st)$.
(4) Let also $i : R \to S^{-1}R$ be the ring morphism $x \mapsto (x, 1)$.

Definition A.0.2. If R is a ring and $\mathfrak{p}$ is a prime ideal, we define the localisation of R at $\mathfrak{p}$ to be $(R \smallsetminus \mathfrak{p})^{-1}R$. It is usually denoted by $R_{\mathfrak{p}}$.

Proposition A.0.3. *Let R be a ring and $S \subseteq R$ be multiplicative. The ring $S^{-1}R$ has the following universal property: given and ring A and ring morphism $f : R \to A$ such that for every $s \in S$, $f(s) \in A^{\times}$, there exists a unique $g : S^{-1}R \to A$ such that:*



commutes.

Proposition A.0.4. *Let R be a ring and $S \subseteq R$ be multiplicative. The map $\mathfrak{q} \mapsto S^{-1}R \cdot \mathfrak{q}$ is a bijection between prime ideals $\mathfrak{q} \subseteq R$ with $\mathfrak{q} \cap S = \varnothing$ and prime ideals of $S^{-1}R$.*

In particular, if $\mathfrak{p} \subseteq R$ is prime, the ring $R_{\mathfrak{p}}$ is local and its maximal ideal is $R_{\mathfrak{p}} \cdot \mathfrak{p}$.

Remark A.0.5. If R is an integral ring, $R_{(0)}$ is its fraction field and i is injective. It is the smallest field (up to unique R-isomorphism) containing R.

B. A multi-sorted model theory primer

Definition B.0.1. A language $\mathfrak{L}$ is: 11/01

- a set $\mathfrak{X}$ — sorts of $\mathfrak{L}$;
- for every tuple of sorts $X = (X_i)_{i<n}$, a set $\mathfrak{R}(X)$ — predicates on $\prod_i X_i$;
- for every tuple of sorts $X = (X_i)_{i<n}$ and sort Y, a set $\mathfrak{f}(X, Y)$ — functions $\prod_i X_i \to Y$;

Let us fix a language $\mathfrak{L}$ and disjoint sets $\mathfrak{V}(X)$, for every sort X — the variables of sort X.

Definition B.0.2. Let $x = (x_i)_{i<n}$ be a tuple of variables and Y be a sort. We define by induction:

- the set $\mathfrak{t}(x, Y)$ — terms in variables x to the sort Y:
  - if $x_i \in \mathfrak{V}(X_i)$, $x_i \in \mathfrak{t}(x, X_i)$;
  - If $t_j \in \mathfrak{t}(x, Z_j)$, for $j < m$, and $f \in \mathfrak{f}(Z, Y)$, then $f(t) \in \mathfrak{t}(x, Y)$;
- the set $\mathfrak{F}(x)$ — formulas in variables x:
  - $\bot \in \mathfrak{F}(x)$;
  - if $\varphi, \psi \in \mathfrak{F}(x)$, $\varphi \to \psi \in \mathfrak{F}(x)$;
  - if $t_1, t_2 \in \mathfrak{t}(x, Z)$, $t_1 = t_2 \in \mathfrak{F}(x)$;
  - If $t_j \in \mathfrak{t}(x, Z_j)$, for $j < m$, and $R \in \mathfrak{R}(Z)$, then $R(t) \in \mathfrak{F}(x)$;
  - If $\varphi \in \mathfrak{F}(yx)$, where y is a single variable, $\exists y \varphi \in \mathfrak{F}(x)$.

Definition B.0.3. An $\mathfrak{L}$-structure M is:

- for every sort X, a set $X(M)$;
- for every $f \in \mathfrak{f}(X, Y)$, a function $f^M : X(M) := \prod_i X_i(M) \to Y(M)$;
- for every $R \in \mathfrak{R}(X)$, a subset $R(M) \subseteq X(M)$.

Definition B.0.4. Let M be an $\mathfrak{L}$-structure, x be a tuple variables of sort X — that is, $x_i \in \mathfrak{V}_{X_i}$. We define by induction:

- for every $t \in \mathfrak{t}(x, Y)$, $t^M : X(M) \to Y(M)$:
  - $(x_i)^M : a \mapsto a_i$;
  - $f(t)^M : a \mapsto f^M((t_j^M(a))_j)$;
- for every $\varphi \in \mathfrak{F}(x)$, $\varphi(M) \subseteq X(M)$:
  - $\bot(M) = \varnothing$;
  - $(\varphi \to \psi)^M := \{a \in X(M) : a \in \varphi(M) \text{ implies } a \in \psi(M)\} = (X(M) \setminus \varphi(M)) \cup \psi(M)$;
  - $R(t)(M) := \{a \in X(M) : t^M(a) \in R(M)\}$;
  - $(\exists y \varphi)(M)$ the projection of $\varphi(M)$ into $X(M)$.

If $a \in \varphi(M)$, we usually write $M \vDash \varphi(a)$. More generally, if $\Phi \subseteq \mathfrak{F}(x)$, we write $M \vDash \Phi(a)$ if $a \in \bigcap_{\varphi \in \Phi} \varphi(M)$.

Definition B.0.5 (Morphisms)**.** Let M and N be $\mathfrak{L}$-structures, $A \subseteq M$ and $f : A \to N$ — that is, for every sort X, we have $X(A) \subseteq X(M)$ and $f : X(A) \to X(N)$.

(1) A is an $\mathfrak{L}$-substructure of M, and we write $A \leqslant M$, if for every function symbol $t : X \to Y$ and $a \in X(A) := \prod_i X_i(A)$, $t^M(a) \in Y(A)$.

(2) f is an $\mathfrak{L}$-embedding if for every quantifier free formula $\varphi(x)$ and $a \in A^x$, if $M \models \varphi(a)$ then $N \models \varphi(f(a))$.
(3) f is an $\mathfrak{L}$-existentially closed embedding if for every existential formula $\varphi(x) = \exists y \psi(xy)$, where ψ is quantifier free and y is a tuple, and $a \in A^x$, if $N \models \varphi(f(a))$ then $M \models \varphi(a)$.
(4) f is an $\mathfrak{L}$-elementary embedding if for every formula $\varphi(x)$ and $a \in A^x$, if $M \models \varphi(a)$ then $N \models \varphi(f(a))$.

Note that the implication in (2) and (4) are, in fact, equivalences. Also, any (respectively existentially closed, elementary) embedding $f : A \to N$ uniquely extends to an (respectively existentially closed, elementary) embedding of the structure generated by A.

It is often useful to specify not only the domain of definition A of the embedding f but also its domain of interpretation M. We will denote that situation by $f : A \subseteq M \to N$.

Remark B.0.6. If $A \leqslant M$, for $f : A \to N$ to be an embedding it suffices that:
(a) f is injective;
(b) for every function symbol $t : X \to Y$ and $a \in X(A)$, $f(t(a)) = t(f(a))$;
(c) for every relation symbole $R \subseteq X$ and $a \in X(A)$, $M \models R(a)$ if and only if $N \models R(A)$.

Proposition B.0.7 (Lowenheim-Skolem, ?). *Let M be some $\mathfrak{L}$-structure and $\kappa \geqslant |\mathfrak{L}|$ some cardinal.*
1. If $A \subseteq M$ and $|A| \leqslant \kappa \leqslant |M|$, there exists $A \subseteq N \preccurlyeq M$ with $|N| = \kappa$.
2. If $|M| \leqslant \kappa$, there exists $N \succcurlyeq M$ with $|N| = \kappa$.

Let $\Delta(x) \subseteq \mathfrak{F}(x)$ be closed under finite conjonctions and disjonctions. For every $\Phi, \Psi \subseteq \mathfrak{F}(x)$, we write $\Phi \models \Psi$ if for every $\mathfrak{L}$-structure M, $\Phi(M) := \bigcap_{\varphi \in \Phi} \varphi(M) \subseteq \Psi(M) = \bigcap_{\psi \in \Psi} \psi(M)$.

Definition B.0.8 (Types). (1) A *partial Δ-type* $\pi(x)$ is a filter on the semi-lattice $(\Delta, \models, \wedge, \bot)$[17].
(2) A partial Δ-type π is *complete* if there exists $c \in M$ some $\mathfrak{L}$-structure such that $\pi = \mathrm{tp}^M_\Delta(c) := \{\varphi \in \Delta(x) : M \models \varphi(c)\}$.

When $\Delta(x) = \mathfrak{F}(x)$, we usually talk about partial types and complete types in x. A complete (Δ-)type is often referred to simply as a (Δ-)type. Partial $\mathfrak{F}(\star)$-types, *i.e.* partial types without variables, are usually called theories and complete $\mathfrak{F}(\star)$-types are usually called complete theories.

Theorem B.0.9 (Compactness). *(1) Every partial Δ-type is contained in a complete Δ-type; equivalently, for every partial type $\pi(x)$, there exists an $\mathfrak{L}$-structure M and $a \in M^x$ such that $M \models \pi(a)$ — that is, for every $\varphi \in \pi$, $M \models \varphi(a)$.*
(2) For every $\Phi \subseteq \mathfrak{F}(x)$ and $\psi \in \mathfrak{F}(x)$ with $\Phi \models \psi$, there exists a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$.
(3) The topological space S_Δ — whose points are complete Δ-types and whose closed sets are generated by the $[\![\varphi]\!] := \{p \in S_\Delta : \varphi \in p\}$, for $\varphi \in \Delta$ — is compact.

[17] That is, $\pi \subseteq \Delta$ such that:
(a) $\bot \notin \pi$;
(b) for every $\varphi, \psi \in \mathfrak{F}$, $\varphi \wedge \psi \in \pi$;
(c) for every $\varphi \in \pi$ and $\psi \in \Delta$, if $\varphi \models \psi$ then $\psi \in \pi$.

The space $\mathcal{S}_\Delta$ is, in fact, spectral[18]. If Δ is closed under negation, it is Hausdorff and totally disconnected.

Corollary B.0.10. *Let π be a partial Δ-type. The following are equivalent:*

(i) π is complete;
(ii) for every $\varphi, \psi \in \Delta$ such that $\varphi \vee \psi \in \pi$, we have $\varphi \in \pi$ or $\psi \in \pi$.

Proof. Exercise. □

Notation B.0.11 (Theories and diagrams). Let M be an $\mathfrak{L}$-structure, $a \in M^x$, $A \subseteq M$. We define:

- the set of quantifier free formulas $\mathfrak{F}^{\mathrm{qf}}(x) \subseteq \mathfrak{F}(x)$;
- the quantifier free type of a in M, $\text{qf-tp}(a) := \mathrm{tp}_{\mathfrak{F}^{\mathrm{qf}}(x)}(a)$;
- the theory of M $\mathrm{Th}_{\mathfrak{L}}(M) := \mathrm{tp}^M_{\mathfrak{L}}(\star) = \{\varphi \in \mathfrak{F}(\star) : M \vDash \varphi\}$;
- the quantifier free theory of M $\text{qf-Th}_{\mathfrak{L}}(M) := \text{qf-tp}^M_{\mathfrak{L}}(\star) = \{\varphi \in \mathfrak{F}^{\mathrm{qf}}(\star) : M \vDash \varphi\}$;
- the language $\mathfrak{L}(A)$ which is $\mathfrak{L}$ with a new constant $c_a : X$, for every $a \in X(A)$ — M has a natural $\mathfrak{L}(A)$-structure by setting $c_a^M := a$;
- the diagram of A in M, $\mathfrak{D}^M_{\mathfrak{L}}(A) := \text{qf-Th}_{\mathfrak{L}(A)}(M)$;
- the elementary diagram of A in M, $\text{el-}\mathfrak{D}^M_{\mathfrak{L}}(A) := \mathrm{Th}_{\mathfrak{L}(A)}(M)$.

Lemma B.0.12. *Let M, N be $\mathfrak{L}$-structures and $A \subseteq M$. We have:*

- *$N \vDash \mathrm{Th}(M)$ if and only if $f : \varnothing \subseteq M \to N$ is an elementary embedding;*
- *$N \vDash \text{qf-Th}(M)$ if and only if $f : \varnothing \subseteq M \to N$ is an embedding.*

If N is enriched to an $\mathfrak{L}(A)$-structure:

- *$N \vDash \mathfrak{D}(A)$ if and only if $a \mapsto c_a^N$ is an embedding $f : A \subseteq M \to N$;*
- *$N \vDash \text{el-}\mathfrak{D}(A)$ if and only if $a \mapsto c_a^N$ is an elementary embedding $f : A \subseteq M \to N$.*

Corollary B.0.13. *Let M be an $\mathfrak{L}$-structure and $\pi(x)$ be a finitely satisfiable set of $\mathfrak{L}(M)$-formulas in variables x — that is, for every $(\varphi_i)_{i<n} \in \pi$, there exists $a \in M^x$ such that $M \vDash \bigwedge_{i<n} \varphi_i(a)$. Then, there exists an $\mathfrak{L}$-elementary embedding $f : M \to N$ and $a \in N^x$ such that $N \vDash \pi(a)$.*

Proposition B.0.14. *Fix T an $\mathfrak{L}$-theory and $\Delta(x)$ a set of formulas in the tuple of variables x, closed under conjonction and disjonction — up to equivalence in T — and $\varphi(x)$ an $\mathfrak{L}$-formula. The following are equivalent:*

1. *there exists $\psi \in \Delta$ such that $T \vDash \forall x\, \varphi(x) \leftrightarrow \psi(x)$;*
2. *for every $M, N \vDash T$, $a \in \varphi(M)$ and $b \in N^x$ such that $\mathrm{tp}_\Delta(a) \subseteq \mathrm{tp}_\Delta(b)$, then $N \vDash \varphi(b)$.*

This is a translation of the fact that if X is a quasi compact subset of some topological space, the closure of X is the closure of its points: $\overline{X} = \bigcup_{p \in X} \overline{p}$.

Proposition B.0.15 (Criterion for elimination of quantifiers). *Let T be an $\mathfrak{L}$-theory. The following are equivalent:*

(i) Any formula $\varphi \in \mathfrak{F}(x)$ is equivalent, modulo T, to a quantifier free formula $\psi \in \mathfrak{F}(x)$;

[18] It is compact, Kolmogorov, sober — closed irreducible subsets are the closure of a single point — and compact open subsets are closed under finite intersection and generated the open sets; equivalently it is homeomorphic to the spectrum of a ring

(ii) *for every* $M, N \vDash T$*, any* $\mathfrak{L}$*-embedding* $f : A \subseteq M \to N$ *is existentially closed;*

(iii) *for every* $M, N \vDash T$*, any* $\mathfrak{L}$*-embedding* $f : A \subseteq M \to N$ *and* $b \in M$*, there exists an* $\mathfrak{L}$*-elementary embedding* $h : N \to N^\star$ *and an* $\mathfrak{L}$*-embedding* $g : Ab \subseteq M \to N^\star$ *with* $h \circ f = g|_A$*;*

(iv) *for every* $M, N \vDash T$ *and any* $\mathfrak{L}$*-embedding* $f : A \subseteq M \to N$*, there exists an* $\mathfrak{L}$*-elementary embedding* $h : N \to N^\star$ *and an* $\mathfrak{L}$*-embedding* $g : M \to N^\star$ *with* $h \circ f = g|_A$*;*

(v) *for every* $M, N \vDash T$*, any* $\mathfrak{L}$*-embedding* $f : A \subseteq M \to N$ *is* $\mathfrak{L}$*-elementary;*

(vi) *for every* $M \vDash T$ *and* $A \subseteq M$*, the theory generated by* $T \cup \mathfrak{D}(A)$ *is complete.*

We say that T eliminates quantifiers when these equivalent conditions hold.

Lemma B.0.16. *Let* M *be some* $\mathcal{L}$*-structure* $A \subseteq M$*,* $a \in M$ *and* $p := \mathrm{tp}(a/A)$*. The following are equivalent:*

(i) *there exists an* $\mathfrak{L}(A)$*-formula* $\varphi(x)$ *such that* $\varphi(M)$ *is finite and* $M \vDash \varphi(a)$*;*

(ii) *for every* $M^\star \succcurlyeq M$*,* $p(M^\star) \subseteq M$*;*

(iii) *for every* $M^\star \succcurlyeq M$*,* $p(M^\star)$ *is finite;*

(iv) *for every* $M^\star \succcurlyeq M$*,* $\mathrm{aut}(M^\star/A) \cdot a$ *is finite;*

We say that a is algebraic over A, and we write $a \in \mathrm{acl}(A)$.

Lemma B.0.17. *Let* M *be some* $\mathcal{L}$*-structure* $A \subseteq M$*,* $a \in M$ *and* $p := \mathrm{tp}(a/A)$*. The following are equivalent:*

(i) *there exists an* $\mathfrak{L}(A)$*-formula* $\varphi(x)$ *such that* $\varphi(M) = \{a\}$*;*

(ii) *there exists* $\mathfrak{L}$*-definable function on some* X *and* $c \in X(A)$ *such that* $a = f(c)$*;*

(iii) *for every* $M^\star \succcurlyeq M$*,* $p(M^\star) = \{a\}$*;*

(iv) *for every* $M^\star \succcurlyeq M$*,* $\mathrm{aut}(M^\star/A) \cdot a = \{a\}$*;*

We say that a is definable over A, and we write $a \in \mathrm{dcl}(A)$.

Theorem B.0.18 (Keisler-Shelah,?). *For any cardinal* κ*, there exists an ultrafilter* $\mathfrak{U}$ *(on some set* X*) such that for any language* $\mathfrak{L}$ *of cardinality at most* κ *and any* $\mathfrak{L}$*-structures* M *and* N*,* $M \equiv N$ *if and only if* $M^{\mathfrak{U}} \simeq N^{\mathfrak{U}}$*.*

Proposition B.0.19. *Let* $\mathfrak{U}$ *be some non principal ultrafilter (on some set* X*) and* M *be an* $\mathfrak{L}$*-structure. Then* $M^{\mathfrak{U}}$ *is* $\aleph_1$*-saturated.*

C. Projective and inductive limits

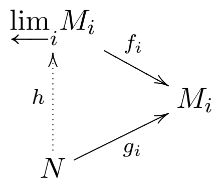
Fix $\mathfrak{L}$ some language.

Definition C.0.1. Let $(I, <)$ be a preorder, and for every $i < j \in I$, a homomorphism[19] of $\mathfrak{L}$-structures $f_{i,j} : M_j \to M_i$ such that, if $i < j < k \in I$, $f_{i,k} = f_{i,j} \circ f_{j,k}$. We define the $\mathfrak{L}$-structure $M := \varprojlim_i M_i$ by $X(M) = \{a \in \prod_i X(M_i) : \text{for all } i < j,\ f_{i,j}(a_j) = a_i\}$. Any function symbol $t : X \to Y$ is interpreted by $t(a) = (t(a_i))_i$, where $a \in X(M)$, and for any function symbol $R \subseteq X$, $M \vDash R(a)$ if and only if, for all i, $M \vDash R(a_i)$. We also define the $\mathfrak{L}$-homomorphism $f_i : M \to M_i$ by $a \mapsto a_i$.

[19] $f : M \to N$ is said to be an $\mathfrak{L}$-homomorphism if for every function symbol $t : X \to Y$ and $a \in X(M)$, f(t(a)) = t(f(a)) and, for every predicate symbol $R \subseteq A$, if $M \vDash R(a)$ then $N \vDash R(f(a))$.

For every $i < j \in I$, we have $f_{i,j} \circ f_j = f_i$.

Proposition C.0.2. *The $\mathfrak{L}$-structure $\varprojlim_i M_i$ has the following universal property: given N and $g_i : N \to M_i$ with $f_{i,j} \circ g_j = g_i$, for all $i < j \in I$, there exists a unique $h : N \to \varprojlim_i M_i$ such that*



commutes, for all $i \in I$.

References