Definable and invariant types in enrichments of NIP theories

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Let T be an NIP $\mathcal{L}$-theory and $\widetilde{T}$ be an enrichment. We give a sufficient condition on $\widetilde{T}$ for the underlying $\mathcal{L}$-type of any definable (respectively invariant) type over a model of $\widetilde{T}$ to be definable (respectively invariant). These results are then applied to Scanlon's model completion of valued differential fields.

Let T be a theory in a language $\mathcal{L}$ and consider an expansion $T \subseteq \widetilde{T}$ in a language $\widetilde{\mathcal{L}}$. In this paper, we wish to study how invariance and definability of types in T relate to invariance and definability of types in $\widetilde{T}$. More precisely, let $\mathfrak{U} \models \widetilde{T}$ be a monster model and consider some type $\widetilde{p} \in \mathcal{S}(\mathfrak{U})$ which is invariant over some small $M \models \widetilde{T}$. Then the reduct p of $\widetilde{p}$ to $\mathcal{L}$ is of course invariant under the action of the $\widetilde{\mathcal{L}}$-automorphisms of $\mathfrak{U}$ that fix M (which we will denote as $\widetilde{\mathcal{L}}(M)$-invariant), but there is, in general, no reason for it to be $\mathcal{L}(M)$-invariant. Similarly, if $\widetilde{p}$ is $\widetilde{\mathcal{L}}(M)$-definable, p might not be $\mathcal{L}(M)$-definable.

When T is stable, and $\varphi(x;y)$ is an $\mathcal{L}$-formula, φ-types are definable by Boolean combinations of instances of φ. It follows that if $\widetilde{p}$ is $\widetilde{\mathcal{L}}(M)$-invariant then p is both $\mathcal{L}(M)$-invariant and $\mathcal{L}(M)$-definable. Nevertheless, when T is only assumed to be NIP, then this is not always the case. For example one can take T to be the theory of dense linear orders and $\widetilde{\mathcal{L}} = \{\leqslant, P(x)\}$ where $P(x)$ is a new unary predicate naming a convex non-definable subset of the universe. Then there is a definable type in $\widetilde{T}$ lying at some extremity of this convex set whose reduct to $\mathcal{L} = \{\leqslant\}$ is not definable without the predicate.

In the first section of this paper, we give a sufficient condition (in the case where T is NIP) to ensure that any $\widetilde{\mathcal{L}}(M)$-invariant (resp. definable) $\mathcal{L}$-type p is also $\mathcal{L}(M)$-invariant (resp. definable). The condition is that there exists a model M of $\widetilde{T}$ whose reduct to $\mathcal{L}$ is uniformly stably embedded in every elementary extension of itself. In the case where T is o-minimal for example, this happens whenever the ordering on M is complete.

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The main technical tool developed in this first section is the notion of external separability (Definition **(1.2)**). Two sets X and Y are said to be externally separable if there exists an externally definable set Z such that $X \subseteq Z$ and $Y \cap Z = \emptyset$. In Proposition **(1.3)**, we show that in NIP theory, external separability is essentially a first order property. The results about definable and invariant sets then follow by standard methods along with a "local representation" of φ-types from [Sim15b].
The motivation for these results comes from the study of expansions of ACVF and in particular the model completion $\mathrm{VDF}_{\mathcal{EC}}$ defined by Scanlon [Sca00] of valued differential fields with a contractive derivation, i.e. a derivation ∂ such that for all x, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$. In the third section, we deduce, from the previous abstract results, a characterisation of definable (resp. invariant) types in models of $\mathrm{VDF}_{\mathcal{EC}}$ in terms of the definability (resp. invariance) of the underlying ACVF-type. This characterisation also allows us to control the canonical basis of definable types in $\mathrm{VDF}_{\mathcal{EC}}$, an essential step in proving elimination of imaginaries for that theory in [Rid].
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0.1 Notation

Let us now define some notation that will be used throughout the paper. When $\varphi(x;y)$ is a formula, we implicitly consider that y is a parameter of the formula and we define $\varphi^{\mathrm{opp}}(y;x)$ to be equal to $\varphi(x;y)$.
We write $N \prec^{+} M$ to denote that M is a $|N|^{+}$-saturated and (strongly) $|N|^{+}$-homogeneous elementary extension of N.
Let X be an $\mathcal{L}(M)$-definable set (or a union of definable sets) and $A \subseteq M$. We denote by $X(A)$ the set of realisations of X in A, i.e. the set $\{a \in A : M \vDash a \in X\}$. If $\mathcal{R}$ is a set of definable sets (in particular a set of sorts), we define $\mathcal{R}(A) \coloneqq \bigcup_{R\in\mathcal{R}} R(A)$.

1 External separability

Definition 1.1 (Externally φ-definable)**:**
Let M be an $\mathcal{L}$-structure, $\varphi(x;t)$ be an $\mathcal{L}$-formula and X a subset of some Cartesian power of M. We say that X is externally φ-definable if there exist $N \succcurlyeq M$ and a tuple $a \in N$ such that $X = \varphi(M;a)$.

Definition 1.2 (Externally φ-separable)**:**
Let M be an $\mathcal{L}$-structure, $\varphi(x;t)$ be an $\mathcal{L}$-formula and X, Y be subsets of some Cartesian power of M. We say that X and Y are externally φ-separable if there exist $N \succcurlyeq M$ and a tuple $a \in N$ such that $X \subseteq \varphi(M;a)$ and $Y \cap \varphi(M;a) = \emptyset$.

We will say that X and Y are φ-separable if a can be chosen in M. Note that a set X is externally φ-definable if X and its complement are externally φ-separable.

Proposition 1.3:
Let T be an $\mathcal{L}$-theory and $\varphi(x;t)$ an NIP *$\mathcal{L}$-formula. Let $U(x)$ and $V(x)$ be new predicate symbols and let $\mathcal{L}_{U,V} := \mathcal{L} \cup \{U,V\}$. Then, there is an $\mathcal{L}_{U,V}$-sentence $\theta_{U,V}$ and an $\mathcal{L}$-formula $\psi(x;s)$ such that for all $M \models T$ and any enrichment $M_{U,V}$ of M to $\mathcal{L}_{U,V}$, we have:*

$$\text{if } U \text{ and } V \text{ are externally } \varphi\text{-separable, then } M_{U,V} \models \theta_{U,V}$$

and

$$\text{if } M_{U,V} \models \theta_{U,V}, \text{ then } U \text{ and } V \text{ are externally } \psi\text{-separable.}$$

Proof. Let k_1 be the VC-dimension of $\varphi(x;t)$. By the dual version of the (p,q)-theorem (see [Mat04] and [Sim15a, Corollary 6.13]) there exists q_1 and n_1 such that for any set X, any finite $A \subseteq X$ and any $\mathcal{S} \subseteq \mathcal{P}(X)$ of VC-dimension at most k_1, if for all $A_0 \subseteq A$ of size at most q_1 there exist $S \in \mathcal{S}$ containing A_0, then there exists $S_1 \dots S_{n_1} \in \mathcal{S}$ such that $A \subseteq \bigcup_{i \leqslant n_1} S_i$. Let k_2 be the VC-dimension of $\bigcup_{i=1}^{n_1} \varphi(x;t_i)$ and q_2 and n_2 the bounds obtained by the dual (p,q)-theorem for families of VC-dimension at most k_2. Let

$$\theta_{U,V} := \forall x_1 \dots x_{q_1}, y_1 \dots y_{q_2} \bigwedge_{i \leqslant q_1} U(x_i) \wedge \bigwedge_{j \leqslant q_2} V(y_j) \Rightarrow \exists t \bigwedge_{i \leqslant q_1} \varphi(x_i;t) \wedge \bigwedge_{i \leqslant q_2} \neg\varphi(y_j;t).$$

Now, let $M \prec^+ N \models T$, U and V be subsets of $M^{|x|}$ and $d \in N$ be a tuple. If $U \subseteq \varphi(M;d)$ and $V \subseteq \neg\varphi(M;d)$ then for any $A \subseteq U$ and $B \subseteq V$ finite there exists $d_0 \in M$ such that $A \subseteq \varphi(M;d_0)$ and $B \subseteq \neg\varphi(M;d_0)$. In particular, $M_{U,V} \models \theta_{U,V}$.
Suppose now that $M_{U,V} \models \theta_{U,V}$. Let $B_0 \subseteq V$ have cardinality at most q_2. The family $\{\varphi(M;d) : d \in M \text{ a tuple and } B_0 \subseteq \neg\varphi(M;d)\}$ has VC-dimension at most k_1 (a subfamily always has lower VC-dimension). Because $M_{U,V} \models \theta_{U,V}$, for any $A_0 \subseteq U$ of size at most q_1, there exists $d \in M$ such that $A_0 \subseteq \varphi(M;d)$ and $B_0 \subseteq \neg\varphi(M;d)$. It follows that for any finite $A \subseteq U$ there are tuples $d_1 \dots d_{n_1} \in M$ such that $A \subseteq \bigvee_{i \leqslant n_1} \varphi(M;d_i)$ and for all $i \leqslant n_1$, $B_0 \subseteq \neg\varphi(M;d_i)$, in particular, $B_0 \subseteq \neg(\bigvee_{i \leqslant n_1} \varphi(M;d_i))$. By compactness, there exists tuple $d_1 \dots d_{n_1} \in N$ such that $U \subseteq \bigvee_{i \leqslant n_1} \varphi(M;d_i)$ and $B_0 \subseteq \neg(\bigvee_{i \leqslant n_1} \varphi(M;d_i))$.
The family $\{\neg(\bigvee_{i \leqslant n_1} \varphi(M;d_i)) : d_i \in N \text{ tuples and } U \subseteq \bigvee_{i \leqslant n_1} \varphi(M;d_i)\}$ has VC-dimension at most k_2. We have just shown that for any B_0 of size at most q_2, there is an element of that family containing B_0. It follows by the (p,q)-property and compactness that there exists tuples $d_{i,j} \in N \succcurlyeq M$ such that $V \subseteq \bigvee_{j \leqslant n_2} \neg(\bigvee_{i \leqslant n_1} \varphi(M;d_{i,j})) = \neg(\bigwedge_{j \leqslant n_2} \bigvee_{i \leqslant n_1} \varphi(M;d_{i,j}))$ and $U \subseteq \bigwedge_{j \leqslant n_2} \bigvee_{i \leqslant n_1} \varphi(M;d_{i,j})$. Hence U and V are externally $\bigwedge_{j \leqslant n_2} \bigvee_{i \leqslant n_1} \varphi(x;t_{i,j})$-separable. ∎

We would now like to characterise enrichments $\widetilde{T}$ of NIP theories that do not add new externally separable definable sets, i.e. $\widetilde{\mathcal{L}}$-definable sets that are externally $\mathcal{L}$-separable but not internally $\mathcal{L}$-separable. We show that if there is one model of $\widetilde{T}$ where this property holds uniformly, then it holds in all models of T.

Proposition 1.4:
Let T be an NIP *$\mathcal{L}$-theory (with at least two constants), $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language, $\widetilde{T} \supseteq T$ be a complete $\widetilde{\mathcal{L}}$-theory and $\chi_1(x;s)$ and $\chi_2(x;s)$ be $\widetilde{\mathcal{L}}$-formulas. The following are equivalent:*

(i) For all $\mathcal{L}$-formulas $\varphi(x;t)$, all $M \vDash \widetilde{T}$ and all $a \in M$ there exists an $\mathcal{L}$-formula $\xi(x;z)$ such that if $\chi_1(M;a)$ and $\chi_2(M;a)$ are externally φ-separated then they are ξ-separated;

(ii) For all $\mathcal{L}$-formulas $\varphi(x;t)$, there exists an $\mathcal{L}$-formula $\xi(x;z)$ such that for all $M \vDash \widetilde{T}$ and all $a \in M$, if $\chi_1(M;a)$ and $\chi_2(M;a)$ are externally φ-separated then they are ξ-separated;

(iii) For all $\mathcal{L}$-formulas $\varphi(x;t)$, there exists an $\mathcal{L}$-formula $\xi(x;z)$ and $M \vDash \widetilde{T}$ such that for all $a \in M$, if $\chi_1(M;a)$ and $\chi_2(M;a)$ are externally φ-separated then they are ξ-separated.

Proof. The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are trivial.
Let us now show that (iii) implies (ii). By Proposition **(1.3)**, there exists an $\widetilde{\mathcal{L}}$-formula $\theta(s)$ and an $\mathcal{L}$-formula $\psi(x;u)$ such that for all $N \vDash \widetilde{T}$ and $a \in N$:

$$\chi_1(N;a) \text{ and } \chi_2(N;a) \text{ externally } \varphi\text{-separated implies } N \vDash \theta(a)$$

and

$$N \vDash \theta(a) \text{ implies } \chi_1(N;a) \text{ and } \chi_2(N;a) \text{ externally } \psi\text{-separated.}$$

Let M and ξ be as in condition (iii) with respect to ψ. We have:

$$M \vDash \forall s\, \theta(s) \Rightarrow \exists u\, (\forall x\, (\chi_1(x;s) \Rightarrow \xi(x;u)) \wedge (\chi_2(x;s) \Rightarrow \neg\xi(x;u))).$$

As $\widetilde{T}$ is complete, this must hold in any $N \vDash \widetilde{T}$. Thus, if $\chi_1(N;a)$ and $\chi_2(N;a)$ are externally φ-separated, we have $N \vDash \theta(a)$ and hence $\chi_1(N;a)$ and $\chi_2(N;a)$ are ξ-separated.
There remains to prove that (i) $\Rightarrow$ (iii). Pick any $M \prec^+ \mathfrak{U} \vDash \widetilde{T}$. By (i), it is impossible to find, in any elementary extension $(\mathfrak{U}^\star, M^\star)$ of the pair $(\mathfrak{U}, M)$, a tuple $a \in M^\star$ and $b \in \mathfrak{U}^\star$ such that $\chi_1(M^\star;a)$ and $\chi_2(M^\star;a)$ are separated by $\varphi(M^\star;b)$, but they are not separated by any set of the form $\xi(M^\star;c)$ where ξ is an $\mathcal{L}$-formula and $c \in M^\star$. By compactness, there exists $\xi_i(x;u_i)$ for $i \leqslant n$ such that for all $a \in M$ if $\chi_1(M;a)$ and $\chi_2(M;a)$ are externally φ-separated, there exists an i such that they are ξ_i-separated. By classic coding tricks, we can ensure that $i = 1$. ∎

Definition 1.5 (Uniform stable embeddedness)**:**
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formulas $\varphi(x;t)$ there exists a formula $\chi(x;s)$ such that for all tuples $b \in M$ there exists a tuple $a \in A$ such that $\varphi(A,b) = \chi(A,a)$.

Remark 1.6:
If there exists $M \vDash \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension, then such an M witnesses Condition **1.4.(iii)** for every choice of formulas χ_1 and χ_2.

Corollary 1.7:
Let T be an NIP $\mathcal{L}$-theory that eliminates imaginaries, $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\widetilde{T} \supseteq T$ be a complete $\widetilde{\mathcal{L}}$-theory. Suppose that there exists $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x;t)$ be an $\mathcal{L}$-formula, $N \models \widetilde{T}$, $A = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A) \subseteq N^{\mathrm{eq}}$ and $p \in \mathcal{S}^{\varphi}_x(N)$. If p is $\widetilde{\mathcal{L}}^{\mathrm{eq}}(A)$-definable, then it is in fact $\mathcal{L}(\mathcal{R}(A))$-definable where $\mathcal{R}$ denotes the set of all $\mathcal{L}$-sorts.

Proof. Let $a \models p$. Then $X := \{m \in N : \varphi(x;m) \in p\} = \{m \in N : \models \varphi(a;m)\}$ is $\mathcal{L}$-externally definable and $\widetilde{\mathcal{L}}^{\mathrm{eq}}(A)$-definable (by some $\widetilde{\mathcal{L}}$-formula χ). It follows from Remark **(1.6)** that Condition **1.4.(iii)** holds and hence, by Condition **1.4.(i)**, taking $\chi_1 = \chi$ and $\chi_2 = \neg\chi$, it follows that X is $\mathcal{L}$-definable.
Because T eliminates imaginaries, we have just shown that we can find $\ulcorner X \urcorner^{\mathcal{L}} \in \mathcal{R}$. But X is also $\widetilde{\mathcal{L}}^{\mathrm{eq}}(A)$-definable, hence any $\widetilde{\mathcal{L}}^{\mathrm{eq}}(A)$-automorphism of N^{eq} stabilises $X(N)$ globally and therefore fixes $\ulcorner X \urcorner^{\mathcal{L}}$. If we assume that N is strongly $|A|^{+}$-homogeneous (and we can), it follows that $\ulcorner X \urcorner^{\mathcal{L}} \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A) = A$. Thus $\ulcorner X \urcorner^{\mathcal{L}} \in A \cap \mathcal{R} = \mathcal{R}(A)$ and X is $\mathcal{L}(\mathcal{R}(A))$-definable. ∎

We will need the following result, which is [Sim15b, Proposition 2.11].

Proposition 1.8:
Let T be any theory, $\varphi(x;y)$ an NIP formula, $M \prec^{+} N \models T$ and $p(x)$ a global M-invariant φ-type. Let $b, b' \in \mathfrak{U} \succcurlyeq N$ such that both $\mathrm{tp}(b/N)$ and $\mathrm{tp}(b'/N)$ are finitely satisfiable in M and $\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b/N) = \mathrm{tp}_{\varphi^{\mathrm{opp}}}(b'/N)$. Then we have $p|_{\mathfrak{U}} \vdash \varphi(x;b) \iff \varphi(x;b')$.

Proposition 1.9:
Let T be an NIP $\mathcal{L}$-theory, $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\widetilde{T} \supseteq T$ be a complete $\widetilde{\mathcal{L}}$-theory. Let $\mathcal{R}$ denote the set of $\mathcal{L}$-sorts. Suppose that there exists $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\varphi(x;t)$ be an $\mathcal{L}$-formula, $N \models \widetilde{T}$ be sufficiently saturated, $A \subseteq N$ and $p \in \mathcal{S}^{\varphi}_x(N)$ be $\widetilde{\mathcal{L}}(A)$-invariant. Assume that every $\widetilde{\mathcal{L}}(A)$-definable set (in some Cartesian power of $\mathcal{R}$) is consistent with some global $\mathcal{L}(\mathcal{R}(A))$-invariant type. Then p is $\mathcal{L}(\mathcal{R}(A))$-invariant.

Proof. Let us first assume that $A \models \widetilde{T}$. Let b_1 and b_2 be such $p(x) \vdash \varphi(x,b_1) \wedge \neg\varphi(x,b_2)$. We have to show that $\mathrm{tp}_{\mathcal{L}}(b_1/A) \neq \mathrm{tp}_{\mathcal{L}}(b_2/A)$. Let $p_i = \mathrm{tp}_{\widetilde{\mathcal{L}}}(b_i/A)$, $\Sigma(t)$ be the set of $\widetilde{\mathcal{L}}(N)$-formulas $\theta(t)$ such that $\neg\theta(A) = \varnothing$ and $\Delta(t_1,t_2)$ be the set:

$$p_1(t_1) \cup p_2(t_2) \cup \Sigma(t_1) \cup \Sigma(t_2) \cup \{\varphi(n,t_1) \iff \varphi(n,t_2) : n \in N\}.$$

If Δ were consistent, there would exist $b_1^{\star}$ and $b_2^{\star}$ such that $b_i \equiv_{\widetilde{\mathcal{L}}(A)} b_i^{\star}$, $\mathrm{tp}_{\widetilde{\mathcal{L}}}(b_i^{\star}/N)$ is finitely satisfiable in A and $\mathrm{tp}_{\varphi^{\mathrm{opp}}}(b_1^{\star}/N) = \mathrm{tp}_{\varphi^{\mathrm{opp}}}(b_2^{\star}/N)$. Applying Proposition **(1.8)** it would follow that $p(x) \vdash \varphi(x;b_1^{\star}) \iff \varphi(x;b_2^{\star})$. But, because p is $\widetilde{\mathcal{L}}(A)$-invariant and $p(x) \vdash \varphi(x,b_1) \wedge \neg\varphi(x,b_2)$, we also have that $p(x) \vdash \varphi(x;b_1^{\star}) \wedge \neg\varphi(x;b_2^{\star})$, a contradiction.
By compactness, there exists $\psi_i \in p_i$, $\theta_i \in \Sigma$, $n \in \omega$ and $(c_i)_{i \in n} \in N$ such that

$$\forall t_1, t_2\, \theta_1(t_1) \wedge \theta_2(t_2) \wedge (\bigwedge_i \varphi(c_i,t_1) \iff \varphi(c_i,t_2)) \wedge \psi_1(t_1) \Rightarrow \neg\psi_2(t_2).$$

In particular, because $\neg\theta_i(A) = \varnothing$, for all m_1 and $m_2 \in A$, $(\bigwedge_i \varphi(c_i, m_1) \iff \varphi(c_i, m_2)) \wedge \psi_1(m_1) \Rightarrow \neg\psi_2(m_2)$. For all $\varepsilon : n \to 2$, let $\varphi_\varepsilon(t, c) := \bigwedge_i \varphi(c_i, t)^{\varepsilon(i)}$ where $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$. It follows that if $\varphi_\varepsilon(A, c) \cap \psi_1(A) \neq \varnothing$, then $\varphi_\varepsilon(A, c) \cap \psi_2(A) = \varnothing$. Let

$$\theta(t, c) := \bigvee_{\varphi_\varepsilon(A,c)\cap\psi_1(A)\neq\varnothing} \varphi_\varepsilon(c, t).$$

We have $\psi_1(A) \subseteq \theta(A, c)$ and $\psi_2(A) \cap \theta(A, c) = \varnothing$, i.e. $\psi_1(A)$ and $\psi_2(A)$ are externally θ-separable. By Proposition **(1.4)** and Remark **(1.6)**, $\psi_1(A)$ and $\psi_2(A)$ are in fact ξ-separable for some $\mathcal{L}(\mathcal{R}(A))$-formula ξ. It follows that $N \vDash \forall t_1, t_2\, (\psi_1(t_1) \Rightarrow \xi(t_1)) \wedge (\psi_2(t_2) \Rightarrow \neg\xi(t_2))$ and, in particular $N \vDash \xi(b_1) \wedge \neg\xi(b_2)$. So $\mathrm{tp}_{\mathcal{L}}(b_1/A) \neq \mathrm{tp}_{\mathcal{L}}(b_2/A)$.

Let us now conclude the proof when A is not a model. Let $M \vDash \widetilde{T}$ contain A and pick any a and $b \in N$ such that $a \equiv_{\mathcal{L}(\mathcal{R}(A))} b$.

Claim 1.10: *There exists $M^\star \equiv_{\widetilde{\mathcal{L}}(A)} M$ (in particular it is a model of $\widetilde{T}$ containing A) such that $a \equiv_{\mathcal{L}(\mathcal{R}(M^\star))} b$.*

Proof. By compactness, it suffices, given $\chi(y, z) \in \mathrm{tp}_{\widetilde{\mathcal{L}}}(M/A)$, where y is a tuple of $\mathcal{R}$-variables, and $\psi_i(t; y)$ a finite number of $\mathcal{L}$-formulas, to find tuples m, n such that $\vDash \chi(m, n) \wedge \bigwedge_i \psi(a; m) \iff \psi(b; m)$. By hypothesis on A, there exists $q \in \mathcal{S}_y(N|_{\mathcal{L}})$ which is $\mathcal{L}(\mathcal{R}(A))$-invariant and consistent with $\exists z\, \chi(y, z)$. Let $m \vDash q|_{\mathcal{R}(A)ab} \cup \{\chi(y)\}$. Then $\mathrm{tp}_{\mathcal{L}}(a/m) = \mathrm{tp}_{\mathcal{L}}(b/m)$ and $\vDash \exists z\, \chi(m, z)$. In particular, we can also find n. ♦

As p is $\widetilde{\mathcal{L}}(A)$-invariant it is in particular $\widetilde{\mathcal{L}}(M^\star)$-invariant. But, as shown above, p is then $\mathcal{L}(\mathcal{R}(M^\star))$-invariant. It follows that $p \vdash \varphi(x; a) \iff \varphi(x; b)$. ■

The assumption that all $\widetilde{\mathcal{L}}(A)$-definable sets are consistent with some global $\mathcal{L}(A)$-invariant type may seem like a surprising assumption. Nevertheless, considering a coheir (in the sense of $\widetilde{T}$, whose restriction to $\mathcal{L}$ is also a coheir in the sense of T), this assumption always holds when A is a model of $\widetilde{T}$.

2 Valued differential fields

The main motivation for the results in the previous sections was to understand definable and invariant types in valued differential fields and more specifically those with a contractive derivation, i.e. for all x, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$. In [Sca00], Scanlon showed that the theory of valued fields with a valuation preserving derivation has a model completion named $\mathrm{VDF}_{\mathcal{EC}}$. It is the theory of ∂-Henselian fields whose residue field is a model of DCF_0, whose value group is divisible and such that for all x there exists a y with $\partial(y) = 0$ and $\mathrm{val}(y) = \mathrm{val}(x)$.

The main result that we will be needing here is that the theory $\mathrm{VDF}_{\mathcal{EC}}$ eliminates quantifiers in the one sorted language $\mathcal{L}_{\partial,\mathrm{div}}$ consisting of the language of rings enriched with a symbol ∂ for the derivation and a symbol $x|y$ interpreted as $\mathrm{val}(x) \leqslant \mathrm{val}(y)$. This result implies that for all substructures $A \leqslant M \vDash \mathrm{VDF}_{\mathcal{EC}}$ the map sending $p = \mathrm{tp}_{\mathcal{L}_{\partial,\mathrm{div}}}(c/A)$ to $\nabla_\omega p := \mathrm{tp}_{\mathcal{L}_{\mathrm{div}}}((\partial^i(c))_{i\in\omega}/A)$ is injective, where $\mathcal{L}_{\mathrm{div}} := \mathcal{L}_{\partial,\mathrm{div}} \smallsetminus \{\partial\}$ denotes the one sorted language of valued fields.

Lemma 2.1:
Let $k \models \mathrm{DCF}_0$. *The Hahn field* $k((t^{\mathbb{R}}))$, *with derivation* $\partial(\sum_i a_i t^i) = \sum_i \partial(a_i) t^i$ *and its natural valuation, is a models of* $\mathrm{VDF}_{\mathcal{EC}}$ *and its reduct to* $\mathcal{L}_{\mathrm{div}}$ *is uniformly stably embedded in every elementary extension.*

Proof. The fact that $k((t^{\mathbb{R}})) \models \mathrm{VDF}_{\mathcal{EC}}$ follows from the fact that its residue field k is a model of DCF_0, its value group $\mathbb{R}$ is a divisible ordered Abelian group and that Hahn fields are spherically complete, cf. [Sca00, Proposition 6.1].
The fact that $k((t^{\mathbb{R}}))$ is uniformly stably embedded in every elementary extension is shown in [Rid, Corollary A.7]. ■

Recall that Haskell, Hrushovski and Macpherson [HHM06] showed that algebraically closed valued fields eliminate imaginaries provided the geometric sorts are added. We will be denoting by $\mathcal{G}$ the set of all geometric sorts.

Proposition 2.2:
Let $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_{\partial,\mathrm{div}}}(A) \subseteq M \models \mathrm{VDF}_{\mathcal{EC}}$. *A type* $p \in \mathcal{S}^{\mathcal{L}_{\mathrm{div}}}(M)$ *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$*-definable if and only if it is* $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$*-definable.*

Proof. If p is $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-definable then it is in particular $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable. The reciprocal implication follows immediately from Corollary **(1.7)** and Lemma **(2.1)**. ■

An immediate corollary of this proposition is an elimination of imaginaries result for canonical bases of definable types in $\mathrm{VDF}_{\mathcal{EC}}$:

Corollary 2.3:
Let $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_{\partial,\mathrm{div}}}(A) \subseteq M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $p \in \mathcal{S}^{\mathcal{L}_{\partial,\mathrm{div}}}(M)$. *The following are equivalent:*

(i) p *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$*-definable;*

(ii) $\nabla_\omega(p)$ *is* $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$*-definable;*

(iii) p *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(\mathcal{G}(A))$*-definable.*

Proof. The implication (iii) ⇒ (i) is trivial. Let us now assume (i). An $\mathcal{L}_{\mathrm{div}}(M)$-formula $\varphi(\overline{x}; m)$ is in $\nabla_\omega(p)$ if and only if $\varphi(\partial_\omega(x); m) \in p$, where $\partial_\omega(x) = (\partial^i(x))_{i\in\omega}$. It follows that $\nabla_\omega(p)$ is $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable. By Proposition **(2.2)**, $\nabla_\omega(p)$ is in fact $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-definable.
Let us now assume (ii) and let $\psi(x; m)$ be any $\mathcal{L}_{\partial,\mathrm{div}}(M)$-formula. By quantifier elimination, $\psi(x; m)$ is equivalent to $\varphi(\partial_\omega(x); \partial_\omega(m))$ for some $\mathcal{L}_{\mathrm{div}}$-formula $\varphi(\overline{x}; \overline{t})$. Therefore $\psi(x; m) \in p$ if and only if $\varphi(\overline{x}; \partial_\omega(m)) \in \nabla_\omega(p)$ and hence p is $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(\mathcal{G}(A))$-definable. ■

In [Rid], it is shown that there are enough definable types to use this partial elimination of imaginaries result to obtain elimination of imaginaries to the geometric sorts for $\mathrm{VDF}_{\mathcal{EC}}$.
Thanks to the result in Section 1 and results from [Rid], we can also characterise invariant types in $\mathrm{VDF}_{\mathcal{EC}}$. Note that, although the main results in [Rid] depend on the results

proved in the present paper, the result from [Rid] that we will be using in what follows does not.

Proposition 2.4:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_{\partial,\mathrm{div}}}(A) \subseteq M^{\mathrm{eq}}$. *A type* $p \in \mathcal{S}^{\mathcal{L}_{\mathrm{div}}}(M)$ *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$*-invariant if and only if it is* $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$*-invariant.*

Proof. To prove the non obvious implication, by Proposition **(1.9)**, we have to show that $\mathrm{VDF}_{\mathcal{EC}}$ has a model whose underlying valued field is uniformly stably embedded in any elementary extension — that is tackled in Lemma **(2.1)** — and that any $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable set (in the sort $\mathbf{K}$) is consistent with an $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-invariant $\mathcal{L}_{\mathrm{div}}$-type. It follows from [Rid, Proposition 9.7] (applied to $T = \mathrm{ACVF}$ and $\widetilde{T} = \mathrm{VDF}_{\mathcal{EC}}$) that any $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable set (in the sort $\mathbf{K}$) is consistent with an $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable $\mathcal{L}_{\mathrm{div}}$-type. But, by Proposition **(2.2)**, such a type is $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-definable. ∎

Corollary 2.5:
Let $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_{\partial,\mathrm{div}}}(A) \subseteq M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $p \in \mathcal{S}^{\mathcal{L}_{\partial,\mathrm{div}}}(M)$. *The following are equivalent:*

(i) p *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$*-invariant;*

(ii) $\nabla_\omega(p)$ *is* $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$*-invariant;*

(iii) p *is* $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(\mathcal{G}(A))$*-invariant.*

Proof. This is proved as in Corollary **(2.3)**, except that Proposition **(2.4)** is used instead of Proposition **(2.2)**. ∎

We can now give a characterisation of forking in $\mathrm{VDF}_{\mathcal{EC}}$.

Corollary 2.6:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *be* $|A|^+$*-saturated,* $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_{\partial,\mathrm{div}}}(A) \subseteq M$ *and* $\varphi(x)$ *be an* $\mathcal{L}_{\partial,\mathrm{div}}(M)$*-formula. Then* $\varphi(x)$ *does not fork over* A *if and only if for all* $\mathcal{L}_{\mathrm{div}}(M)$*-formulas such that* $\varphi(x)$ *is equivalent to* $\psi(\partial_\omega(x))$, $\psi(\overline{x})$ *does not fork over* $\mathcal{G}(A)$ *(in* ACVF*).*

Proof. Let us first assume that $\varphi(x)$ does not fork over A and let p be a global non forking extension of $\varphi(x)$. As $\mathrm{VDF}_{\mathcal{EC}}$ is NIP, by [HP11, Proposition 2.1], p is invariant under all automorphisms that fix Lascar strong type over A. But, because $\mathrm{VDF}_{\mathcal{EC}}$ has the invariant extension property (cf. [Rid, Theorem 2.14]), Lascar strong type and strong type coincide in $\mathrm{VDF}_{\mathcal{EC}}$ (see [HP11, Proposition 2.13]), hence p is $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-invariant. It follows from Corollary **(2.5)** that $\nabla_\omega(p)$ is $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-invariant and hence $\psi(\overline{x})$ does not fork over $\mathcal{G}(A)$.
Let us now assume that no $\psi(\overline{x})$ such that $\varphi(x)$ is equivalent to $\psi(\partial_\omega(x))$ forks over $\mathcal{G}(A)$. Then there exists $q \in \mathcal{S}^{\mathcal{L}_{\mathrm{div}}}_{\overline{x}}(M)$ which is $\mathcal{L}^{\mathrm{eq}}_{\mathrm{div}}(\mathcal{G}(A))$-invariant and consistent with all such formulas $\psi(\overline{x})$. Now, the image of the continuous map $\nabla_\omega : \mathcal{S}^{\mathcal{L}_{\partial,\mathrm{div}}}_{x}(M) \to \mathcal{S}^{\mathcal{L}_{\mathrm{div}}}_{\overline{x}}(M)$ is closed and if $\chi(\overline{x})$ is an $\mathcal{L}_{\mathrm{div}}(M)$-formula containing the image of ∇_ω and $\psi(\overline{x})$ is as above, $\chi(\partial_\omega(x)) \wedge \psi(\partial_\omega(x))$ is also equivalent to $\varphi(x)$. Therefore, $q = \nabla_\omega(p)$

for some $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-invariant $p \in \mathcal{S}^{\mathcal{L}_{\partial,\mathrm{div}}}_x(M)$. This type p implies $\varphi(x)$ and hence $\varphi(x)$ does not fork over A. ∎

Remark 2.7:
The previous corollary is somewhat unsatisfying as one needs to consider all possible ways of describing $\varphi(x)$ as the prolongation points of an $\mathcal{L}_{\mathrm{div}}$-formula ψ (with parameters in a saturated model) to conclude whether φ forks or not.
Considering only one such ψ cannot be enough. For example, consider any definable set $\varphi(x)$ forking (in $\mathrm{VDF}_{\mathcal{EC}}$) over A and let $\psi(x_0, x_1) = (\mathrm{val}(x_0) \geqslant 0 \wedge \mathrm{val}(x_1) < 0) \vee \varphi(x_0)$. Then the set $\{x \in M : M \models \psi(x, \partial(x))\} = \varphi(M)$ but ψ does not fork over A (in ACVF). The obstruction here might seem frivolous, but it is the core of the problem. Indeed, it is not clear if there is a way, given φ to find a formula ψ as above that does not contain "large" subsets with no prolongation points.

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