Introduction

Silvain Rideau

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The subject of this text is the model theory of valued fields. Model theory studies structures and first order definable sets in these structures. We often distinguish between pure model theory, which studies abstract theories and the combinatorics of definable sets, and applied model theory, whose goal is to study specific structures that appear in other areas of mathematics; in these pages, these structures will be valued fields. But the distinction is somewhat arbitrary. Questions in pure model theory often arise from applications to specific structures and "purer" considerations often allow us to better understand the concrete cases we are interested in.
In fact, this text is clearly an example of applied model theory but some of the results presented here are of a much "purer" nature.

Valued fields and the Ax-Kochen-Eršov principle

Among the early results in the model theoretic study of valued fields, the Ax-Kochen-Eršov principle [AK65; Erš65] heralds much of the subject's later developments. Neglecting the fundamental role of Abraham Robinson's work [Rob77] on algebraically closed valued fields (ACVF) would be certainly be unreasonable, nevertheless, the Ax-Kochen-Eršov theorem contains the seed of an idea that is central in the model theoretic study of valued fields: a valued field is a structure "controlled", on the one hand, by an ordered group (its value group), and, on the other hand, by a field (its residue field). Hahn fields illustrate this idea perfectly. From any field k and any Abelian ordered group Γ, one may construct the valued field $k((t^{\Gamma}))$ whose elements are formal power series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ whose coefficients are in k and such that the set $\{\gamma \in \Gamma : a_\gamma \neq 0\}$ is well-ordered. The residue field of $k((t^{\Gamma}))$ is exactly k and its value group is Γ.
Admittedly, not every valued field K whose value group is Γ and residue field is k is isomorphic to $k((t^{\Gamma}))$. But according to the Ax-Kochen-Eršov theorem, if it is Henselian (i.e. the valuation can be extended in a unique way to any algebraic extension) and if k has characteristic zero, the first order theory of K is controlled by that of k and Γ:

Theorem 0.1 (Ax-Kochen-Eršov theorem, 1965)**:**

Let K and L be two Henselian valued fields of equicharacteristic zero. They are elementarily equivalent[1] if and only if their valued fields and their value groups, respectively, are elementarily equivalent.

Hence, if K is Henselian of equicharacteristic zero with residue field k and value group Γ, it is elementarily equivalent to $k((t^{\Gamma}))$.
The Ax-Kochen-Eršov theorem can be extended as is to unramified mixed characteristic, i.e. when K has characteristic zero, k has positive characteristic p and $\mathrm{val}(p)$ is the smallest positive element of Γ. With some more complications it also extends to finitely ramified mixed characteristic (i.e. when $\mathrm{val}(p)$ is a finite multiple of the smallest positive element in Γ). In what follows, we will say that a field is finitely ramified if its residue characteristic is zero or if its a finitely ramified mixed characteristic field.
One of the most striking corollaries of the Ax-Kochen-Eršov theorem is that for every choice of non principal ultrafilter $\mathfrak{U}$ on the set of prime numbers, the ultraproducts $\prod_p \mathbb{Q}_p/\mathfrak{U}$ and $\prod_p \mathbb{F}_p((t))/\mathfrak{U}$ are elementarily equivalent as valued fields. In other words, a (first order) sentence is true in all $\mathbb{Q}_p$ for sufficiently large p if and only if it is true in all $\mathbb{F}_p((t))$ for sufficiently large p. Ax and Kochen's motivation to prove this theorem was to answer the following question of Artin's: is it true that every homogeneous polynomial over $\mathbb{Q}_p$ of degree d with at least $d^2 + 1$ variables has a non trivial root? By a theorem of Lang's, this result is true in $\mathbb{F}_p((t))$ for all p and hence for all d there is some prime p_0 such that the answer to Artin's question is positive in all $\mathbb{Q}_p$ with $p \geqslant p_0$. We now know that this result is "optimal" as there are counter-examples for small p.
Recently this kind of results have been extended, beyond the scope of first order sentences, to equalities of p-adic integrals containing additive characters [CL10], using motivic integration. This allowed, for example, Cluckers, Hales and Loeser [CHL11] to transfer the fundamental lemma of Langlands program form positive characteristic to mixed characteristic.

The idea underlying the Ax-Kochen-Eršov theorem that an Henselian valued fields is controlled by its value group and its residue field has many avatars in the model theory of valued fields as it has been developed since. One of them is a result by Delon [Del81] (in equicharacteristic zero) and Bélair [Bél99] (in finitely ramified mixed characteristic) on the "moderation" of finitely ramified valued fields.
Before explaining that result, let us explicit was is meant here by moderation. In the seventies, while working on classification, Shelah introduced a certain number of more or less moderate classes of theories, determined by the absence of certain combinatorial configurations in their models. The first one he defined, and, by far, the most studied of moderation classes is that of *stable* theories:

Definition 0.2 (Stable theory)**:**
Let $\varphi(x, y)$ be an $\mathcal{L}$-formula and M an $\mathcal{L}$-structure. We say that φ has the order property in M if there exists tuples $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}} \in M$ such that $M \vDash \varphi(a_i, b_j)$ if and only if $i \leqslant j$.
A theory T is said to be stable if no $\mathcal{L}$-formula has the order property in any model of T.

[1] Meaning that every first order sentence in the language of valued fields is true in one of those fields if and only if it is also true in the other.

This particularly combinatorial condition has, in fact, a large number of consequence that make stable theories a model theoretic "paradise" in which numerous tools where developed (definable types, forking, independence, etc.). The most classical example of a stable theory is the theory of algebraically closed fields (ACF) but there are many others. Some have very little algebraic content as the theory of infinite sets in the equality language or the theory of an equivalence relation with infinitely many infinite classes. But others, modules on a given ring, separably closed fields (SCF) or differentially closed fields in characteristic zero (DCF_0), are algebraic structures. Nevertheless, numerous structures that appear in mathematics are unstable, thus restricting the possible applications of stability theory.
For example, neither the field $\mathbb{R}$ nor any valued fields can be stable as they contain a definable order. But since the definition of stability, other notions of moderation have been introduced. In this text, we will only be interested in one of those, which is compatible with the existence of a definable order: NIP. This class was also defined by Shelah but had not been studied much until ten years ago. It has since been the target of a growing interest.

Definition 0.3 (NIP theory)**:**
Let $\varphi(x, y)$ be an $\mathcal{L}$-formula and M be an $\mathcal{L}$-structure. We say that φ has the independence property in M if there exists tuples $(a_i)_{i\in\mathbb{N}}$ and $(b_J)_{J\subseteq\mathbb{N}} \in M$ such that $M \vDash \varphi(a_i, b_J)$ if and only if $i \in J$.
An $\mathcal{L}$-theory T is said to be NIP *(not the independence property) or dependent, if no $\mathcal{L}$-formula has the independence property in any model of T.*

Among NIP theories, numerous algebraic examples can be found. First of all, all stable theories are NIP. The theory of real closed fields (RCF), or more generally every o-minimal theory like the theory of the real field equipped with the exponential map and restricted analytic functions, is also NIP, and so are many theories of valued fields: algebraically closed valued fields, the theory of the field of p-adic numbers $\mathbb{Q}_p$ and more generally the theory of any finite extension of $\mathbb{Q}_p$.
These last examples bring us back to the aforementioned "Ax-Kochen-Eršov" principle relative to moderation:

Theorem 0.4 ([Del81; Bél99])**:**
Let T be a theory of characteristic zero finitely ramified Henselian valued fields. The theory T is NIP *if and only if the theory induced by T on the residue field is* NIP*.*

This theorem is an instance of a more general principle according to which any characteristic zero finitely ramified Henselian valued field is as moderate as its value group and residue field are. The value group does not appear in the above theorem as, by a theorem of [GS84], all Abelian ordered groups are NIP.
As was mentioned earlier, stable theories are very well behaved. One of their nice properties is that every type is definable. Recall that a type p on a structure M is an ultrafilter on the Boolean algebra of sets definable with parameters in M. It is *definable* if for every formula $\varphi(x; s)$, there exists a formula $\theta(s)$ such that, for all tuple m, $\theta(m)$ holds if and only if $\varphi(x; m) \in p$. Equivalently, in every model M of a stable theory, every externally defin-

able set[2] is in fact definable in M (with parameters). This property characterises stability and therefore cannot hold in an NIP theory. But externally definable sets are still well behaved in those theories. One of their properties, introduced by Chernikov and Simon [CS13; CS], is the existence of (uniform) honest definitions which will appear in Section III.1.
It is often considered that an NIP theory is controlled by a stable part and an ordered part. In the case of valued fields, this principle, whose similarity to the Ax-Kochen-Eršov principle is striking, can be made precise. In local fields of characteristic zero[3], the residue field is finite and hence the structure is essentially controlled by the value group. We would like to say that those theories are "purely unstable" and that is formalised by Simon [Sim13] as being distal.
On the contrary, algebraically closed valued fields contain both an ordered part (the value group Γ) and an extremely stable part (the residue field which is algebraically closed). Haskell, Hrushovski and Macpherson [HHM06; HHM08] formalised the intuition that these two sets control models of ACVF by introducing the notion of *metastability*: for all tuple a, the type of a over Γ is "dominated" by the stable part. In [Hrub], Hrushovski uses metastability to deconstruct groups definable in ACVF in terms of groups internal to the value group, groups internal to the residue field and group schemes over the valuation ring.
Lastly, note that a form of the Ax-Kochen-Eršov principle is also underlying motivic integration, one of whose goal is to find additive invariants of algebraic varieties defined on valued fields. Following Hrushovski and Kazhdan [HK06], one can find such an invariant by mapping varieties to their classes in the Grothendieck semi-ring of sets definable in ACVF[4] and this semi-ring can be shown to be essentially isomorphic to the tensor product of the Grothendieck semi-ring of the residue field and that of the value group.

Quantifier elimination

The Ax-Kochen-Eršov theorem **(0.1)** is, in fact, a natural consequence of quantifier elimination results, although Ax, Kochen and Eršov never explicitly state those results.
More generally, the question of the elimination of quantifier is fundamental when studying a specific theory. Indeed, the class of definable sets in a given structure is obtained from basic definable sets, (determined by the choice of language) by finite Boolean combinations and projections. But projections are more complicated than Boolean combinations and, ideally, to get a good understanding of the class of definable sets, one would prefer that Boolean combinations suffice. That is exactly the content of a quantifier elimination theorem. For example, in the case of ACF, the basic definable sets are algebraic varieties and their Boolean combinations are sets constructible in the Zariski topology. Quantifier elimination for ACF which says that the family of constructible sets are closed under projections is therefore equivalent to a theorem of Chevalley according to which the image of a constructible set by a regular function is still constructible.

[2] That is, the intersection of M with a set definable with parameters in an elementary extension of M.
[3] Here, by a local field we mean non Archimedian local field, i.e. a finite extension of $\mathbb{Q}_p$ for some prime p.
[4] That is, the semi-ring of definable isomorphism classes of definable sets in ACVF equipped with the disjoint sum and the Cartesian product.

In valued fields, there exist a great variety of quantifier elimination results. The first one is essentially due to Abraham Robinson [Rob77] and states that algebraically closed valued fields eliminate quantifiers in the most simple possible language for valued fields[5]. In that case, as in many others, this quantifier elimination result implies moderation results. Robinson's result, for example, can be used to show that the theory ACVF is C-minimal. *C-minimality* is a notion which generalises both strong minimality and o-minimality and which, as all notions of minimality, concerns the sets definable in one variable. In strongly minimal theories, one asks that all unary definable sets be the same as in the infinite set with equality (in other words, they are either finite or cofinite). In o-minimal theories, one asks that the unary definable sets are the same as those defined without quantifiers in an order (in other words, finite unions of intervals). Finally, in C-minimal theories, one asks that the unary definable sets be a Boolean combination of balls. But a notion of moderation is only as interesting as the tools it provides. For example, C-minimality implies the existence of a cell decomposition and an associated dimension.

In the case of the theory of the p-adic field $\mathbb{Q}_p$, Robinson's language does not suffice to eliminate quantifiers and new basic definable sets have to be added to obtain elimination of quantifiers. Macintyre [Mac76] showed that it suffices to add the sets of the form $P_n(K) := \{x \in K : \exists y,\, x = y^n\}$ for all n.
Since then, quantifier elimination results have been proved for a larger class of valued fields: characteristic zero (unramified) Henselian fields. These results come in two flavors according to the basic definable sets which are chosen: results with *angular component maps* and results with *leading terms*. Following the historic development of the subject, let us start with angular component maps. An *angular component map* of a valued field K is a group morphism $\mathrm{ac} : K^\star \to k^\star$ such that the restriction of ac to $\mathcal{O}^\star$ coincides with residual map $\mathrm{res} : \mathcal{O} \to k$, where $\mathcal{O} := \{x \in K : \mathrm{val}(x) \geqslant 0\}$ is the valuation ring of K. Laurent series over k (or more generally any Hahn series field), local fields and Witt vectors over a perfect field can all be equipped with an angular component map. But not every field can. Nevertheless, it suffices to assume that the valued field is $\aleph_1$-saturated to be able to construct one.
According to a theorem of Pas [Pas89], equicharacteristic zero Henselian valued fields eliminate field quantifiers in the three sorted language (whose sorts are $\mathbf{K}$, $\mathbf{k}$ and $\mathbf{\Gamma}$) of valued fields with an angular component map. This language is usually called the Denef-Pas language. If one adds "higher order angular component maps", this theorem extends to finitely ramified mixed characteristic Henselian valued fields.
The main issue with the Denef-Pas language is that, in general, even when natural angular component maps exist, they are not definable in the language of valued fields; characteristic zero local fields are the most notable exception in which "higher order angular component maps" are definable. The main goal of languages with *leading terms*, also known as amc congruences or $\mathbf{RV}$-functions, is to give an alternative to angular component maps that does not add any new definable sets of the valued field sort. Let $\mathbf{RV} := \mathbf{K}^\star/(1+\mathfrak{M})$. In any valued field, a short exact sequence $1 \to \mathbf{k}^\star \to \mathbf{RV} \to \mathbf{\Gamma} \to 0$ can be defined. By work of Basarab and Kuhlmann [Bas91; BK92], if one adds a sort for $\mathbf{RV}$, equicharacteristic zero Henselian val-

[5]Basic sets in this language are of the form $\{x : \mathrm{val}(P(x)) \geqslant \mathrm{val}(Q(x))\}$, where P and Q are polynomial in many variables

ued fields eliminate field quantifiers. As previously, if one adds "higher order leading terms", mixed characteristic (possibly infinitely ramified) Henselian valued fields also eliminate field quantifiers.

In the past twenty five years, these quantifier elimination results have been extended to all kind of enriched valued fields which appear naturally in mathematics. First of all, most valued fields we consider, be it local fields or formal series fields, are complete[6]. Hence they are naturally equipped with an analytic structure given by specialising formal power series on their convergence domain. The first quantifier elimination results for valued fields with analytic structure are due to Denef and van den Dries [DD88] for the field of p-adic numbers. They were followed by many results in broader and broader contexts, among which work of van den Dries, Haskell, Macpherson, Lipshitz, Robinson and Cluckers [Dri92; DHM99; LR00; LR05; CLR06; CL11]. In [Dri92], for example, van den Dries proves field quantifier elimination for all equicharacteristic zero Henselian valued fields with an analytic structure and an angular component map. He then deduces an analytic Ax-Kochen-Eršov theorem.

The other valued field enrichment that it is natural to consider consists in adding an operator, be it a derivation or an automorphism of valued fields[7]. The first results (construction of a model completion and elimination of field quantifiers) were proven by Scanlon in [Sca00]. In that paper, he studies, under certain technical hypothesis, valued fields equipped with a D-operator (a notion which generalises both the derivation and the automorphism case) that preserve the valuation; that is, such that for all x, $\mathrm{val}(D(x)) \geqslant \mathrm{val}(x)$. Among many other things, he proves the existence of a model completion $\mathrm{VDF}_{\mathcal{EC}}$ of the theory of valued differential fields whose derivation preserves the valuation. Studying this theory motivates Chapters III and IV of this text.

In later work, Bélair, Macintyre, Scanlon, Durhan and van den Dries [Sca03; BMS07; AD10] extended the field quantifier elimination result to isometries: field automorphisms such that for all x, $\mathrm{val}(\sigma(x)) = \mathrm{val}(x)$. Automorphisms which are not isometries also appeared in Hrushovski's work [Hrua] on twisted Lang-Weil approximations. These automorphism are such that for all x and $n \in \mathbb{N}$, if $\mathrm{val}(x) > 0$ then $\mathrm{val}(\sigma(x)) > n\mathrm{val}(x)$ (they are said to be ω-increasing). Elimination of field quantifiers for valued fields with such an automorphism was showed by Durhan in [Azg10]. Pal [Pal12] then extended this result to more general automorphisms and finally Durhan and Onay [DO] recently proved that this result also holds for any valued field automorphism.

Chapter II of this text aims at merging these two possible enrichments and studying valued fields with both an analytic structure and a valued field automorphism. The main result is a field quantifier elimination result for characteristic zero analytic σ-Henselian valued fields. The first motivation for this result is to understand the model theory of the field of Witt vectors on $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ equipped with both its analytic structure and the lifting of the Frobenius. A good understanding of this theory could, for example, allow us to give a model theoretic version of Buium's p-differential geometry and to consider certain problems in Diophantine geometry (see [Sca06]).

One last word about differential valued fields, two relatively distinct cases have also been

[6]Although completeness is not a first order property.

[7]A field automorphism σ such that $\sigma(\mathcal{O}) = \mathcal{O}$.

studied. The first one, the study of transseries fields (see, for example, [ADH13]), led people to consider valued differential fields that are not ∂-Henselian and whose derivation, albeit continuous, does not preserve the valuation. The second case is valued differential fields in which there is no interaction between the valuation and the derivation. In particular such a derivation is highly discontinuous (see, for example, [GP10]). But, apart from some techniques essentially linked to the fact that we are considering valued fields, these two other cases have very little in common with Scanlon's case or with the valued field automorphism case which are, in fact, closely related.

Elimination of imaginaries

When studying a given theory, once the question of quantifier elimination has been taken care of, another question follows: the description of the definable quotients. Indeed, for multiple reasons, definable equivalence relations, and hence the quotients by these equivalence relations, appear naturally. But these quotients, that we call interpretable sets, do not belong to the category of definable sets and most of the results we might have obtained until then on definable sets cannot be applied.

For example, definable equivalence relations appear in the following manner. Let $X \subseteq Y \times Z$ be $\varnothing$-definable sets in some structure M. We can see X as the family $(X_y)_{y\in Y}$ of its fibers above Y. One can then ask if there exists another parametrisation $(U_w)_{w\in W}$ of that family such that every set in the family X appears exactly one time in the family U. This notion, quite close to that of a moduli space, is fundamental as it allows us to identify the family (X_y) with the set W. We then say that the point in W corresponding to a fiber X_y of X is a *canonical parameter* for X_y. This canonical parameter is the "smallest" set over which X_y is defined; such a set does not exist in the structure itself in general.

The sets W and U can be constructed in a "canonical" way by considering the equivalence relation $y_1 E y_2$ on Y defined by $\forall z, z \in X_{y_1} \iff z \in X_{y_2}$ and taking $W = Y/E$ and $U = \{(z, \widehat{y}) : x \in X_y\}$ where $\widehat{y}$ is the E-class of y. But the sets W and U are interpretable and, *a priori*, not definable. Such a construction is therefore only possible in M if interpretable sets in M also happen to be definable (or, to be precise, definably isomorphic to a definable set). This is exactly what is called *elimination of imaginaries* in M, and that is equivalent to the existence in M of a smallest set of definition for every definable set.

The notion of elimination of imaginaries was introduced by Poizat in [Poi83] and is much more recent than elimination of quantifiers that has, for many years, been fundamental in every application of model theory. Thus, it is not very surprising that fewer elimination of imaginaries results are known. Among the examples of theories that eliminate imaginaries, one can find the two theories that led Poizat to define the notion: ACF and DCF_0. Elimination of imaginaries in these theories follows from the existence, for all algebraic (respectively differential) variety, of a smallest (respectively differential) field of definition and from the definability of symmetric polynomials. Among other notable examples, one should mention divisible ordered Abelian groups, real closed fields (that is, the fields that are elementary equivalent to $\mathbb{R}$) and more generally, all o-minimal theories of ordered groups.

In general, if a theory does not eliminate imaginaries, the question becomes: which canonical parameters of definable sets are sufficient to get all the others? Obviously, by adding the canonical parameters of all sets definable in a theory T, one gets a theory, named T^{eq}, defined by Shelah, which eliminates imaginaries[8]. But, although this construction is very useful from an abstract point of view, it is of little interest as far as understanding a given theory is concerned, as it does not give any information on the interpretable sets. For example, one can check that the theory of structureless infinite sets does not eliminate imaginaries; indeed except for singletons, no finite sets has a canonical parameter. But one can also check that it suffices to add, for all $n \in \mathbb{N}_{>0}$, a sort S_n whose elements are the finite subsets of cardinal n in the main sort, in order to eliminate imaginaries.

Let us now consider algebraically closed valued fields in the language with a single sort for the valued field itself. It follows from elimination of quantifiers that every infinite definable set has the same cardinality as the valued field. But there exists algebraically closed valued fields of cardinality continuum whose value group $\mathbf{\Gamma}$ is countable. It is therefore impossible that the interpretable set $\mathbf{\Gamma} = \mathbf{K}^{\star}/\mathcal{O}^{\star}$ be in bijection with a definable subset of the valued field. More elaborate counterexamples show that adding the value group, the residue field or even the set of balls of the valued field does not suffice to obtain elimination of imaginaries. The same considerations apply to characteristic zero local fields.

The first elimination of imaginaries result for valued fields is due to Haskell, Hrushovski and Macpherson in [HHM06]. They show that in algebraically closed valued fields, it suffices to add the canonical parameters of certain sets that can be seen as higher order equivalents of balls. These sets are, on the one hand, the rank n free $\mathcal{O}$-submodules of $\mathbf{K}^n$, also known as lattices, and, on the other hand, for every lattice s, the sets of the form $a + \mathfrak{M}s$ where $a \in s$. The language in which we have added canonical parameters for these two families of sets is called the *geometric language*.

Since then, Mellor [Mel06] proved that real closed valued fields also eliminate imaginaries in the geometric language. In Chapter I of this text, we show new elimination of imaginaries results in the geometric language: one for characteristic zero local fields and another for equicharacteristic zero pseudo-local fields (that is, ultraproducts of local fields).

As far as enriched valued fields are concerned, results are even rarer. In the case of algebraically closed valued fields with an analytic structure, it is proved in [HHM13], that they do not eliminate imaginaries in the geometric language. But no concrete description of the canonical parameters one should add is known. In the present text, we prove a result for the other family of enrichments described earlier: valued fields with operators. More precisely, Chapter III contains general considerations on the imaginaries in enrichments of ACVF which are then applied in Chapter IV to show that $\mathrm{VDF}_{\mathcal{EC}}$ eliminates imaginaries in the geometric language enriched with a symbol for the derivation.

We noted earlier that elimination of quantifiers results can be used to prove moderation results. As far as elimination of imaginaries is concerned, the relation is reversed: moderation of a theory can help understand its imaginaries. For example, stability of ACF, DCF_0 and SCF plays a considerable part in proving the respective elimination of imaginaries results. Similarly, the theory of stably dominated types and metastability was developed to study the imaginaries in ACVF. In a more recent proof of that result [Hru14; Joh], density of definable

[8] every model M of T naturally expands to a model M^{eq} of T^{eq}.

types plays a central part. Finally, in the proof of elimination of imaginaries for $\mathrm{VDF}_{\mathcal{EC}}$ in the geometric language that can be found in this text, definable types are also fundamental, but a new crucial ingredient of the proof is the good behaviour of externally definable sets in NIP theories.

Beyond elimination

Once elimination of quantifiers and imaginaries is known for a given theory, a whole array of choices opens up and there is no "canonical" next step. For example, as far as ACVF is concerned, Hrushovski and Loeser [HL] studied the space of stably dominated types and showed that not only can it be used to give an alternative definition of Berkovich spaces, but it also allows them to prove new result about their topology. More classically, one may want to study groups and fields interpretable in a theory: classify them, but also show how some geometric properties of definable sets can give rise to these algebraic structures. These considerations are central to geometric stability theory and, applied to DCF_0 and SCF, allowed Hrushovski [Hru96] to prove certain Diophantine results like the Mordell-Lang conjecture for function fields.
Let us now state some of the group classification results mentioned in the previous paragraph. In algebraically closed fields, by a result of Weil, or, more precisely, the model theoretic interpretation of this result by Hrushovski (cf. [Poi87, Section 4.e]) or van den Dries [Dri90], every definable group is definably isomorphic to an algebraic group. By a result of Bouscaren and Delon [BD01], groups definable in a separably closed field K of finite imperfection degree are definably isomorphic to the K-points of an algebraic group. In pseudo-finite fields, according to Hrushovski and Pillay [HP94],the situation is essentially the same up to certain finite groups: every group definable in a pseudo-finite field K contains a finite index definable subgroup isogenous to the K-points of an algebraic group. In differentially closed fields of characteristic zero, by a theorem of Pillay's [Pil97], all definable groups are embedded in algebraic groups. Finally, in algebraically closed fields with an automorphism, Chatzidakis and Hrushovski [CH99], in the finite rank case, and Kowalski and Pillay [KP02], in the general case, showed a similar result up to finite groups. Section IV.5 of this text contains the first steps of such a classification for groups definable in valued differential fields *à la* Scanlon.

Overview of the results

The main results contained in this text are elimination of quantifiers and imaginaries results for certain valued fields: pure valued fields as characteristic zero local fields or equicharacteristic zero pseudo-local fields, and valued fields enriched with a derivation, an automorphism or an analytic structure.
The first chapter, joint with Ehud Hrushovski and Ben Martin, contains [HMR]. Its main model theoretic results are the elimination of imaginaries in the geometric language for characteristic zero local fields (Theorem **A**) and for equicharacteristic zero pseudo-local fields (Theorem **B**). We then deduce in Corollary **(I.2.7)** that elimination of imaginaries in characteristic zero local fields is uniform when the residual characteristic grows.

These elimination results are then applied to counting classes in a family of equivalence relations parametrised by the value group and definable in a characteristic zero local field. Using techniques, developed among others by Denef, that relate these problems to the computation of certain p-adic integrals, we prove Theorem **C** which states that zeta functions associated to these counting problems are uniformly rational. This chapter also contains applications of Theorem **C** to certain zeta functions that appear in subgroup and representation growth theory.

In Chapter II, we prove field quantifier elimination (Theorem **D**) in valued fields equipped with *both* an analytic structure and an automorphism. Although the motivation was to study the field of Witt vectors on $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ equipped with its analytic structure and the lifting of the Frobenius (which is an isometry), Theorem **D** is proved for *all* automorphisms of valued fields. This chapter also attempts, in Sections II.A and II.B, to give a more systematic approach to quantifier elimination proofs in enriched valued fields, via some abstract considerations.

The subject of Chapter III is the elimination of imaginaries in certain enrichments of ACVF. The theory $\mathrm{VDF}_{\mathcal{EC}}$ is the main example. More precisely, we prove a result about the density of definable types (Theorem **E**) that implies both the elimination of imaginaries and the existence of global invariant extensions. The chapter begins, in Section III.1, by some work, joint with Pierre Simon, on more abstract considerations on externally definable sets in NIP theories. These considerations are essential in proving the other results in this chapter. I wish to thank Pierre for allowing me to present these results here.

Finally, Chapter IV contains numerous results on $\mathrm{VDF}_{\mathcal{EC}}$. The main result (Theorem **F**) is the elimination of imaginaries and the invariant extension property for $\mathrm{VDF}_{\mathcal{EC}}$ in the geometric language. This is an immediate consequence of the results in the previous chapter. We also show that the field of constants is stably embedded in models of $\mathrm{VDF}_{\mathcal{EC}}$, and we study the definable and algebraic closures. The link between types in $\mathrm{VDF}_{\mathcal{EC}}$ and those in ACVF is formalised by defining a notion similar to that of prolongations in differential algebraic geometry. The last section of this chapter is devoted to studying groups definable in $\mathrm{VDF}_{\mathcal{EC}}$ and how they relate to groups definable in ACVF.

Detailled description of the results

Chapter I. Imaginaries in p-adic fields: In addition to their interest for model theorists, our main motivation in proving those results was to better understand certain counting functions that arise in group theory. Let us begin with some historic perspective on these issues and their link to model theory.

First, let us consider the Poincaré series associated to p-adic points of varieties. Let X be an affine variety defined over $\mathbb{Z}_p$. Let a_n be the number of its $\mathbb{Z}/p^n\mathbb{Z}$-points and $\widetilde{a}_n$ the number of points in the image of the canonical map $X(\mathbb{Z}_p) \to X(\mathbb{Z}/p^n\mathbb{Z})$. To study the growth of these sequences, we define the two Poincaré series $P_X(t) := \sum_i a_i t^i$ and $\widetilde{P}_X(t) := \sum_i \widetilde{a}_i t^i$. The main question one can ask about these series is the question of their rationality. In the case of $P_X(t)$, this was showed by Igusa in the seventies. As for $\widetilde{P}_X(t)$, the use of a quantifier (coming from taking the image of a set under some function) makes this series more complicated to study. Denef proved its rationality in [Den84] using Macintyre's quantifier elimination for $\mathbb{Q}_p$ [Mac76].

In his proof, Denef considers the sets $A_n := \{x \in \mathbb{Z}_p : x \text{ is in } X \text{ modulo } p^n\mathbb{Z}_p\}$ and $\widetilde{A}_n := \{x \in \mathbb{Z}_p : \text{there exists } y \in X(\mathbb{Z}_p) \text{ such that } x \equiv y \mod p^n\mathbb{Z}_p\}$, which are definable in $\mathbb{Q}_p$, and showed that $a_n = \mu(A_n)p^{nd}$ and $\widetilde{a}_n = \mu(\widetilde{A_n})p^{nd}$ where $d \in \mathbb{N}$ depends on X and μ is the Haar measure on $\mathbb{Q}_p$ normalised so that $\mu(\mathbb{Z}_p) = 1$. He then shows that the $\mathbb{Q}$-algebra generated by functions $f : \mathbb{Q}_p \times \mathbb{Z} \to \mathbb{Z}$ and $x \mapsto p^{f(x)}$ where f is definable in $\mathbb{Q}_p$ is closed under integration. The rationality theorem then follows easily. Using a uniform quantifier elimination theorem for $\mathbb{Q}_p$, Pas [Pas91] and Macintyre [Mac90] showed that, when X is defined over $\mathbb{Z}$, rationality of these series is uniform.

Later, Grunewald, Segal and Smith [GSS88] used Denef's result on definability of p-adic integrals to study a counting problem coming from group theory. Let G be a nilpotent, finitely generated, torsion free group. The number of index n subgroup of G is finite; we denote it b_n. we then find a family of functions, definable in $\mathbb{Q}_p$ and parametrised by the value group, such that the integral of the index n function is exactly b_{p^n}. Using Denef's result, we obtain that the series $\zeta_{G,p}(t) := \sum_i b_{p^i}t^i$ is rational. By considerations similar to those of Pas and Macintyre, or more generally the p-adic specialisation of results from motivic integration, we also obtain information about the behaviour of this function as p grows. One of the interest of these uniform results is to study the global zeta function associated to G, $\zeta_G(s) = \sum_i b_i i^{-s}$. As for Riemann's zeta function, which is none other than the global zeta function of $\mathbb{Z}$, we have a Euler decomposition in local zeta functions:

$$\zeta_G(s) = \prod_p \zeta_{G,p}(p^{-s}).$$

Hence a uniform understanding of the local zeta function can lead to results on the global zeta function.

The method used to associate to each b_{p^n} a function definable in $\mathbb{Q}_p$ goes as follows. We find, uniformly in p and n, a set D_n/E_n interpretable in $\mathbb{Q}_p$ whose cardinal is b_{p^n} and we find a function f_n definable in $\mathbb{Q}_p$ such that for all $x \in D_n$, if $\widehat{x}^{E_n}$ denotes the E_n-class of x, then $\mu(\widehat{x}^{E_n}) = p^{-\mathrm{val}(f_n(x))}$. Hence Grunewald, Segal and Smith's result, as well as Denef's result on Poincaré series, can be seem as special instances of a more general problem that consists in counting classes in a family $(E_n)_{n\in\mathbb{N}}$ of equivalence relations, definable in $\mathbb{Q}_p$. In the results that we mentioned above, the equivalence classes are sufficiently simple to find the f_n explicitly. Nevertheless, in the general case, as long as we do not know exactly how interpretable sets might look like, it is impossible to find such f_n. Proving (uniform in p) elimination of imaginaries for $\mathbb{Q}_p$ thus becomes necessary.

In that chapter, we prove two elimination of imaginaries results. One about local fields:

Theorem A:

Let K be a local field of mixed characteristic $(0, p)$. If we add a constant for a generator of $K \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$ over $\mathbb{Q}_p \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$, then the theory of K in the geometric language eliminates imaginaries.

And a theorem about their ultraproducts:

Theorem B:

The theory of equicharacteristic zero pseudo-local fields eliminates imaginaries in the geometric language, if one adds countably many well-chosen constants.

In both cases, as the valuation is discrete, canonical parameters of sets of the form $a + \mathfrak{M}s$, where s is a lattice, are not, in fact, necessary. From the second theorem, we deduce, in Corollary **(I.2.7)**, that elimination of imaginaries for local fields is uniform when the residual characteristic grows.
It follows that every set interpretable in a local field K of characteristic zero can be identified with definable subset of $K^n \times (\mathrm{GL}_m(K)/\mathrm{GL}_m(\mathcal{O}))$for some m and $n \in \mathbb{N}$, uniformly in p. One can then easily find functions f_n such as previously, using the fact that the Haar measure on $\mathrm{GL}_m(K)$ has a density relative to the Haar measure on K^{n^2} which is a definable function, and deduce Theorem **C**, an abstract uniform rationality result for families of equivalence relations uniformly definable in characteristic zero local fields.
This theorem not only allows us to reprove the aforementioned results on local zeta functions that arise in group theory, but also to prove new results for which the equivalence relations that come into play are more complicated, like counting representations "up to isotwist".
The elimination results themselves are proved using abstract criteria, Proposition **(I.2.11)** and Corollary **(I.2.15)**, that allow to transfer an elimination of imaginaries results from one theory, ACVF, in that case, to another. Here, the transfer is made possible by the fact that definable functions in those theories in which we wish to prove elimination of imaginaries, are covered by finite correspondences definable in ACVF. The existence of invariant extensions of types on small sets of algebraically closed parameters as well as a description of all 1-types play are central to this proof.

Chapter II. Analytic difference fields: As previously mentioned, the goal of this chapter is to study valued fields enriched with both an analytic structure and a valued field automorphism. However, studying those structures is not solely motivated by a wish to understand the interactions between two well-understood types of valued field enrichments and give a model theoretic account of more and more complex structures. It is also motivated by the fact that this is the correct setting in which to consider certain Diophantine and number theoretic problems. For example, it is the model theoretic setting of Buium's p-differential geometry. Indeed, every p-differential function defined over $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ is a function definable in the structure $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped with the lifting of the Frobenius as well as symbols for every p-adic analytic function $\sum_I a_I X^I$ where $a_I \to 0$ when $|I| \to \infty$.
In [Sca06], Scanlon shows how some model theoretic facts about analytic difference fields can be used to show that certain Diophantine results are uniform. In particular, he shows that a uniform version of the Manin-Mumford conjecture on Abelian varieties over $\mathrm{W}[k]$ is a consequence of the classical conjecture (that is, Raynaud's theorem), using on the one hand, results of Buium's on how these question relate to p-differential geometry and, on the other hand, an Ax-Kochen-Eršov theorem for analytic difference fields.

The main result in this chapter is a field quantifier elimination result for a certain class of analytic difference fields:

Theorem D:

The theory of valued σ-Henselian characteristic zero fields with an analytic structure eliminates field quantifiers in the leading term language.

Most of the model theoretic results about valued fields with analytic structure rely heavily on the fact that, via Weierstrass preparation, the valuation and, even the leading term, of an analytic function in one variable is equal to that of a polynomial. One can then show that every unary definable subset of the valued field is already definable in the language of valued fields (with leading terms). In Denef and van den Dries' original proof, one of the complications is due to the fact that they know Weierstrass preparation uniquely for analytic functions and not for all terms in the language (where inverses may occur). Here we use a version of Weierstrass preparation, proved by Cluckers and Lipshitz [CL11] that works for all terms.

In valued difference fields without analytic structure, however, the heart of the proof consists in, first of all, adapting the notion of Henselianity to difference polynomials to obtain σ-Henselianity, and generalising the results on 1-types used to prove elimination of quantifiers in pure Henselian valued fields, to very specific n-types which correspond to types of tuples $(x, \sigma(x), \ldots, \sigma^{n-1}(x))$ for well-chosen $x \in \mathbf{K}$. This last point appears relatively explicitly in Section II.6.

The proof of Theorem **D** consists in blending these two approaches. In [Sca06], Scanlon already attempted this merging in the case of an isometry, but the notion of σ-Henselianity he defined was too weak, although that was concealed by some erroneous computations. The axiomatisation and proofs had to be entirely redone (and were generalised to any valued field automorphism) but certain ideas of that article are still fundamental in this new approach.

There are a few inherent difficulties to merging the proof from the analytic case and the difference case. The first one is that the usual notion of σ-Henselianity, defined, for example, in [BMS07], is only useful if we considers terms whose Taylor expansion is finite, i.e. polynomials. The other difficulty is that, as explained above, eliminating quantifiers in a difference valued field requires one to study certain n-types of the underlying (enriched) valued field, but that the main technique in the analytic case mainly describes 1-types. The main ingredient to overcome these two obstacles is to consider the differential properties of terms. That approach allows, among other things, to define, a new notion of σ-Henselianity adapted to the difference analytic setting.

Given that this is our main example, in this chapter, we also axiomatise the theory of the field $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped with its analytic structure and the lifting of the Frobenius (from now on we will denote this theory W_p). We also prove a moderation result:

Proposition 0.5 (Proposition **(II.6.31)** and Corollary **(II.7.5)**):
The theory of W_p is axiomatised by the fact that it is a σ-Henselian unramified mixed characteristic valued field with analytic structure, whose residue field is algebraically closed, whose value group is elementarily equivalent to $\mathbb{Z}$ and such that the automorphism is an isometry and reduces to the Frobenius on the residue field.

Moreover, the theory of W_p *is* NIP.

Furthermore, this chapter also tries to give a systematic approach to some well-known facts around quantifier elimination in enriched valued fields, which are usually reproved in each specific case. The first such fact is that result in mixed characteristic follow from results in equicharacteristic zero. Another one is that results with angular components follow from results with leading terms. The last one is that field quantifier elimination results are *resplendent*, that is they remain true when we enrich the sorts other than the field sort. Sections II.A and II.B contain abstract considerations that allow to show these three facts in almost all enriched valued fields that we may want to study.

Chapter III. Imaginaries in certain enrichments of ACVF**:** The main motivation for this chapter was to prove the invariant extension property in $\mathrm{VDF}_{\mathcal{EC}}$. As was already mentioned, when they were studying imaginaries in ACVF, Haskell, Hrushovski and Macpherson [HHM06] developed the notion of stably dominated type. This notion was then thoroughly studied in [HHM08] and led them to defining metastability. The main example of a metastable theory is ACVF but it would be interesting to have other examples in which the stable part is more complicated than ACF. The theory $\mathrm{VDF}_{\mathcal{EC}}$ can seem like a good candidate.
One of the problems with stably dominated types is that we only know a relatively complicated form of descent[9] which assumes the existence of global invariant extensions, a property usually referred to as the *invariant extension property*. For this reason, when defining metastability, Haskell, Hrushovski and Macpherson required not only that there are many stably dominated types but also the invariant extension property.
To prove that $\mathrm{VDF}_{\mathcal{EC}}$ is also an example of a metastable theory in order, for example, to study groups definable in $\mathrm{VDF}_{\mathcal{EC}}$ as Hrushovski [Hrub] studies groups definable in ACVF, we thus have to prove the invariant extension property for $\mathrm{VDF}_{\mathcal{EC}}$. This property has clear ties to imaginaries and hence the question of their elimination follows naturally. By analogy with DCF_0, it is reasonable to think that $\mathrm{VDF}_{\mathcal{EC}}$ has no more imaginaries than ACVF has and hence that it also eliminates imaginaries in the geometric language.
The techniques developed in this chapter to study $\mathrm{VDF}_{\mathcal{EC}}$ can be adapted to a wider setting. Thus, one of the results we obtain is an abstract criterion for the elimination of imaginaries and the invariant extension property in enrichments of ACVF. We show in Chapter IV, that this criterion applies to $\mathrm{VDF}_{\mathcal{EC}}$. The main reason we allow such a general setting is because those technique should also be useful in some other theories, for example, $\mathrm{VDF}_{\mathcal{EC}}$ with an analytic structure or Witt vectors over $\overline{\mathbb{F}_p}^{\mathrm{alg}}$.
As is the case in a recent version [Hru14; Joh] of the elimination of imaginaries in ACVF, definable types play a major part in the elaboration of this criterion. In fact, the main result in this chapter (Theorem **E**) is a density result for definable types whose canonical basis is controlled. In the case of $\mathrm{VDF}_{\mathcal{EC}}$ in the geometric language, this result can be stated as such:

[9] The fact that every global type stably dominated over B and invariant over $C \subseteq B$ is stably dominated over C.

Theorem 0.6 (Theorem **E** for $\mathrm{VDF}_{\mathcal{EC}}$):

Let $M \models \mathrm{VDF}_{\mathcal{EC}}$, $A \subseteq M^{\mathrm{eq}}$ be definably closed and X be an A-definable set, then there exists a definable type p which is $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant and consistent with X, where $\mathcal{G}(A)$ denotes the set of points from A that are in the sorts of the geometric language.

One of the crucial ingredients to control the canonical basis of the definable types built in this chapter is an abstract result proved in Section III.1 about externally definable sets in NIP-theories which implies the following fact:

Theorem 0.7 (Theorem **(III.1.4)** for $\mathrm{VDF}_{\mathcal{EC}}$):

Let $M \models \mathrm{VDF}_{\mathcal{EC}}$, $A \subseteq M$ be definably closed. If X is A-definable in $\mathrm{VDF}_{\mathcal{EC}}$ and externally definable in ACVF, then it is $\mathcal{G}(A)$-definable in ACVF.

This statement can be reformulated as follows: let p be an ACVF type and let us assume there exists a definition scheme for p made of $\mathrm{VDF}_{\mathcal{EC}}$ formulas. Then there exists a definition scheme for p made of ACVF formulas.

Chapter IV. Some model theory of valued differential fields: The last chapter of this text contains results about the theory $\mathrm{VDF}_{\mathcal{EC}}$. The most important is the following:

Theorem F:

The theory $\mathrm{VDF}_{\mathcal{EC}}$ eliminates imaginaries in the geometric language and has the invariant extension property.

In this chapter we also answer some questions that arise immediately if one wants to study $\mathrm{VDF}_{\mathcal{EC}}$. First, we show that the constant field is stably embedded and is a pure model of ACVF. The proof of this fact naturally adapts to the theory of $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped with the lifting of the Frobenius. The second question is to describe the definable and algebraic closures. We show that the definable closure is not as well-behaved as one might hope. In general, it contains more than the Henselian closure of the differential field generated by the parameters. We nevertheless show that:

Proposition 0.8 (Corollaries **(IV.3.3)** and **(IV.3.4)**):
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ and $A \subseteq \mathbf{K}(M)$. The field $\mathbf{K}(\mathrm{dcl}(A))$ is an immediate extension of the differential field generated by A and the field $\mathbf{K}(\mathrm{acl}(A))$ is an immediate extension of the algebraic closure of the field generated by A.

In this chapter, we also formalise the relationship between types in $\mathrm{VDF}_{\mathcal{EC}}$ and types in ACVF by introducing a notion of prolongation at the level of the type space which is analogous to the one defined in DCF_0. The main difference is that, in DCF_0, the prolongation space of a definable set is definable in ACF, whereas in $\mathrm{VDF}_{\mathcal{EC}}$ we only obtain a partial type in ACVF.

Finally, in Section IV.5, we study groups definable in $\mathrm{VDF}_{\mathcal{EC}}$ and their relationship to groups definable in ACVF. Almost all of this section consists in showing (following [Hrub]), that certain tools developed in stable theories to study and build groups generalise to the unstable

context as long as the group has a definable generic. In particular, we show the equivalent of Hrushovski's result [Hru90] that a $\star$-definable group in a stable theory is a pro-limit of definable groups. We also show how to build groups out of "group chunks". The notion of "group chunks" that we use here is somewhat more general than the one usually considered in model theory and allows us to consider non connected groups.
We deduce from these "neostable" considerations that the proof that a group definable in DCF_0 can be definably embedded in a group definable in ACF (and hence in an algebraic group), can be reproduced in an abstract setting. It follows that certain definable groups in $\mathrm{VDF}_{\mathcal{EC}}$ can be definably embedded in groups definable in ACVF.

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