

VALUED FIELDS, METASTABLE GROUPS

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ABSTRACT. We introduce a class of theories called *metastable*, including the theory of algebraically closed valued fields (ACVF) as a motivating example. The key local notion is that of definable types dominated by their stable part. A theory is metastable (over a sort Γ) if every type over a sufficiently rich base structure can be viewed as part of a Γ -parametrized family of stably dominated types. We initiate a study of definable groups in metastable theories of finite rank. Groups with a stably dominated generic type are shown to have a canonical stable quotient. Abelian groups are shown to be decomposable into a part coming from Γ , and a definable direct limit system of groups with stably dominated generic. In the case of ACVF, among definable subgroups of affine algebraic groups, we characterize the groups with stably dominated generics in terms of group schemes over the valuation ring. Finally, we classify all fields definable in ACVF.

1. INTRODUCTION

Let V be a variety over a valued field (F, val) . By a *val-constructible set* we mean a finite Boolean combination of sets of the form $\{x \in U : \text{val}(f(x)) \leq \text{val}(g(x))\}$ where U is an open affine, and f, g are regular functions on U . Two such sets can be identified if they have the same points in any valued field extension of F , or equivalently, by a theorem of Robinson, in any fixed algebraically closed valued field extension K of F .

In many ways these are analogous to constructible sets in the sense of the Zariski topology, and (more closely) to semi-algebraic sets over real fields. However while the latter two categories are closed under quotients by equivalence relations, the valuative constructible sets are not. For instance, the valuation ring $\mathcal{O} = \{x : \text{val}(x) \geq 0\}$, is constructible, and has constructible ideals $\alpha\mathcal{O} = \{x : \text{val}(x) \geq \alpha\}$ and $\alpha\mathfrak{M} = \{x : \text{val}(x) > \alpha\}$. For any group scheme G over \mathcal{O} , one obtains corresponding congruence subgroups; but the quotients $\mathcal{O}/\alpha\mathcal{O}$ and $G(\mathcal{O}/\alpha\mathcal{O})$ are not constructible. We thus enlarge the category by formally adding quotients, referred to as the *imaginary sorts*; the objects of the larger category will be called the *definable sets*. It is explained in [HHM06] that instead of this abstract procedure, it suffices to add sorts for the homogeneous spaces $\text{GL}_n(K)/\text{GL}_n(\mathcal{O})$, for all n , and also $\text{GL}_n(K)/I$ for a certain subgroup I ; we will not require detailed knowledge of this here. One such sort that will be explicitly referred to is the value group $\Gamma := \text{GL}_1(K)/\text{GL}_1(\mathcal{O})$; this is a divisible ordered Abelian group, with no additional induced structure. We will refer to *stable sorts* also. These include first of all the residue field $k = \mathcal{O}/\mathfrak{M}$, but also vector spaces over k of the form $L/\mathfrak{M}L$, where $L \subseteq K^n$ is a lattice.

The paper [HHM08], continuing earlier work, studied the category of quantifier-free definable sets over valued fields, especially with respect to imaginaries. As usual, the direct study of a concrete structure of any depth is all but impossible, if it is not aided by a general theory. We first tried to find a generalization of stability (or simplicity) in a similar format, capable of dealing with valued fields as stability does with differential fields, or simplicity with difference fields. To this we encountered resistance; what we found instead was not a new analogue of stability, but a new method of utilizing classical stability in certain unstable structures.

Even a very small stable part can have a decisive effect on the behavior of a quite “large,” unstable type. This is sometimes analogous to the way that the (infinitesimal) linear approximation to a variety can explain much about the variety; and indeed, in some cases, casts tangent spaces and Lie algebras in an unexpected model theoretic role. Two main principles encapsulate the understanding gained:

- (1) Certain types are dominated by their stable parts. They behave “generically” as stable types do.
- (2) Uniformly definable families of types make an appearance; they are indexed by the linear ordering Γ of the value group, or by other, piecewise-linear structures definable in Γ . An arbitrary type can be viewed as a definable limit of stably dominated types (from principle (1)).

A general study of stably dominated types was initiated in [HHM08]; it is summarized in §2. Principle (2) was only implicit in the proofs there. We state a precise version of the principle, and call a theory satisfying (2) *metastable*. We concentrate here on finite rank metastability.

Principle (1) is given a general group-theoretic rendering in Proposition 4.6. Stably dominated groups are defined, and it is shown that a group homomorphism into a stable group controls them generically. Theorem 5.15 clarifies the second principle in the context of Abelian groups. A metastable Abelian group of finite rank is shown to contain nontrivial stably dominated groups T_α , unless it is internal to Γ . Moreover, the groups T_α are shown to form a definable direct limit system, so that the group is described by three ingredients: Γ -groups, stably dominated groups, and definable direct limits of groups.

We then apply the theory to groups definable in algebraically closed valued fields. Already the case of Abelian varieties is of considerable interest; all three ingredients above occur, and the description beginning with a definable map into a piecewise linear definable group L takes a different aspect than the classical one. The points of L over a non-Archimedean local field will form a finite group, related to the group of connected components in the Néron model. On the other hand over $\mathbb{R}(t)$ the same formulas will give tori over \mathbb{R} .

Non-commutative groups definable in algebraically closed valued fields include examples such as $\mathrm{GL}_n(\mathcal{O})$, $\mathrm{GL}_n(\mathcal{O}/a\mathcal{O})$ and “congruence subgroups” such as the kernel of $\mathrm{GL}_n(\mathcal{O}) \rightarrow \mathrm{GL}_n(\mathcal{O}/a\mathcal{O})$, which are all stably dominated. In Corollary 6.4 and Theorem 6.11 we relate stably dominated groups to group schemes over \mathcal{O} . Previous work in this direction used other theories, inspired by topology; see [Pil89], [HP94] regarding the p -adics.

We now describe the results in more detail. Our main notion will be that of a stably dominated group. In a metastable theory, we will try to analyze arbitrary groups, and to some extent types, using them. In the case of valued fields, this notion is related to but distinct from compactness (over those fields where topological notions make sense, i.e. local fields). For groups defined over local fields, stable domination implies compactness of the group of points over every finite extension. Abelian varieties with bad reduction show that the converse fails; this failure is explained by another aspect of the theory, definable homomorphic quotients defined over Γ .

Let T be a first order theory. It is convenient to view a projective system of definable sets D_i as a single object, a pro-definable set; similarly for a compatible system of definable maps $\alpha_i : P \rightarrow D_i$. Precise definitions of the terminology and notations used in the following paragraphs can be found in §2.2 and §2.3.

Definition 1.1 (Stable domination). A type p over C is *stably dominated* if there exists, over C , a pro-definable map $\alpha : p \rightarrow D$, D stable and stably embedded, such that for any $a \neq p$ and

tuple b , $\alpha(a) \downarrow_C D \cap \text{dcl}(b)$ implies:

$$\text{tp}(b/C\alpha(a)) \equiv \text{tp}(b/Ca).$$

Let Γ be a stably embedded definable set which is orthogonal to the stable part. In this paper we will mostly be focusing on the case where Γ is o -minimal, as is the case in algebraically closed valued fields. Many of the results presented here should remain true in Henselian valued fields with algebraically closed residue field and different value groups, such as \mathbb{Z} ; where every definable subset of Γ is still a Boolean combination of \emptyset -definable sets and intervals.

Definition 1.2 (Metastability). T is *metastable* (over Γ) if any set of parameters C_0 is included in a set C , called a metastability basis, such that for any a , there exists a pro-definable map (over C), $\gamma_c : p \rightarrow \Gamma$ with $\text{tp}(a/\gamma_c(a))$ stably dominated. In addition, we assume:

- (E) Every type over an algebraically closed subset of a model of T^{eq} has an automorphism-invariant extension to the model.

The main reason we require property (E) is because of descent (Proposition 2.11.(2)) which is not known to hold without the the additional hypothesis that $\text{tp}(B/C)$ has a global C -invariant extension.

- Question 1.3.**
- (1) *Can descent be proved without the additional hypothesis that $\text{tp}(B/C)$ has a global C -invariant extension?*
 - (2) *If we assume Γ to be o -minimal, does the invariant extension property follow from the existence of metastability bases?*

- Remark 1.4.**
- (1) Instead of considering all C -definable stable sets, it is possible, in the definition of stable domination, to take a proper subfamily \mathcal{S}_C with reasonable closure properties. The notion of \mathcal{S} -domination is meaningful even for stable theories.
 - (2) In the definition of metastability, one can also replace the single sort Γ with a family of sorts Γ_i , or with a family of parametrized families of definable sets \mathcal{G}_A , with no loss for the results of the present paper.

As in stability theory, a range of finiteness assumptions is possible. Let T be metastable over Γ .

Definition 1.5 (cf. Definition 2.17). The theory T has (FD) if:

- (1) Γ is o -minimal.
- (2) Morley dimension is uniformly finite and definable in families.
- (3) Let D be a definable set. The Morley dimension of $f(D)$, where f ranges over all definable functions with parameters such that $f(D)$ is stable, takes a maximum value.
- (4) Similarly, the o -minimal dimension of $g(D)$, where g ranges over all definable functions with parameters such that $g(D)$ is Γ -internal, takes a maximum value.

Let $A \leq M \models T$. We define St_A to be the family of all stable, stably embedded A -definable sets. Some statements will be simpler if we also assume:

Definition 1.6 (cf Definition 2.21). The theory T has (FD_ω) if, in addition to (FD), any set is contained in a metastability basis M which is also a model. Moreover, for any acl-finitely generated $G \subseteq \Gamma$ and $S \subseteq St_{M \cup G}$ over M , isolated types over $M \cup S$ are dense.

- Remark 1.7.**
- (1) (FD) and (FD_ω) both hold in ACVF, with all imaginary sorts included (cf Proposition 2.31). (FD), at least, is valid for all C -minimal expansions of ACVF, in particular the rigid analytic expansions (cf. [HM94, Lip93, LR98]).

- (2) Existentially closed valued fields with a contractive derivative (cf. [Rid]) and separably closed valued fields (cf. [HKR]) are also metastable, but of infinite rank. The results presented here will only hold in finite rank definable groups.
- (3) In practice, the main structural results will use finite weight hypotheses; this is a weaker consequence of (FD). Recall that a definable type p in a stable theory is said to have *weight* $\leq n$ if for any M and independent b_1, \dots, b_{n+1} over M , and any $a \models p|_M$, for some i , $a \models p|_M b_i$. In a definable set of Morley rank n , every definable type has weight $\leq n$.

A group is *stably dominated* if it has a generic type that is stably dominated (See Definitions 3.1 and 4.1). In this case, the stable domination is witnessed by a group homomorphism (cf. Proposition 4.6). One cannot expect every group to be stably dominated. But one can hope to shed light on any definable group by studying the stably dominated groups inside it. We formulate the notion of a *limit stably dominated* group: it is a direct limit of connected metastable groups by a pro-definable direct limit system (cf. Definition 5.6).

Theorem 1.8 (cf. Theorem 5.15). *Let T be a metastable theory with (FD_ω) . Let A be a definable Abelian group. Then there exists a definable group $\Lambda \subset \Gamma^n$, and a definable homomorphism $\lambda : A \rightarrow \Lambda$, with $K := \ker(\lambda)$ limit stably dominated.*

In fact, under these assumptions, K is the union of a definable directed family of definable groups, each of which is stably dominated. Assuming only bounded weight (in place of (FD_ω)), we obtain a similar result but with K ∞ -definable.

In the non-Abelian case the question remains open. The optimal conjecture would be a positive answer to:

Problem 1.9. *(FD_ω) Does any definable group G have a limit stably dominated definable subgroup H with $H \backslash G / H$ internal to Γ ?*

Another goal of this paper is to relate definable groups in ACVF to group schemes over \mathcal{O} . We recall the analogous results for the algebraic and real semi-algebraic cases. Consider a field K of characteristic zero. Then the natural functor from the category of algebraic groups to the category of constructible groups is an equivalence of categories. This follows locally from Weil's group chunk theorem; nevertheless some additional technique is needed to complete the theorem. It was conjectured by Poizat and proved in [vdD90] using definable topological manifolds, and in [Hru90] using stability theoretic notions like definable types and germs. This methods will be explained in §3. Let us only remark here that an irreducible algebraic variety has a unique generic behavior, in that any definable subset has lower dimension or a complement of lower dimension; this is typical of stable theories.

In ACVF, we certainly cannot hope every definable group to be a subgroup of an algebraic group or even of the definable homomorphic image of an algebraic group. We can, however, hope for all definable groups to be towers of such groups:

Problem 1.10. *Let G be a definable group in $K \models \text{ACVF}$. Does there exist definable normal subgroups $(1) = G_0 \leq \dots \leq G_n = G$ of G and definable homomorphisms f_i , with kernel G_i , from G_{i+1} into the definable homomorphic image of an algebraic group over K ?*

Call a definable set D *boundedly imaginary* if there exists no definable map with parameters from D onto an unbounded subset of Γ . If D is defined over a local field L , then D is boundedly imaginary if and only if $D(L')$ is finite for every finite extension L' of L . We prove:

Proposition 1.11 (cf. Proposition 6.1). *Let H be a stably dominated connected group definable in ACVF. Then there exists an algebraic group G over K and a definable homomorphism $f : H \rightarrow G(K)$ with boundedly imaginary kernel.*

Corollary 1.12. *Let H be a stably dominated group definable in ACVF with parameters from a local field L . Then there exists a definable homomorphism $f : H(L) \rightarrow G(L)$ with G an algebraic group over L , with finite kernel.*

Proposition 1.11 reduces the study of a stably dominated group definable in the field sort of ACVF to that of stably dominated subgroups of algebraic groups G . We proceed then to describe these.

There exists an exact sequence $1 \rightarrow A \rightarrow G \rightarrow_f L \rightarrow 1$, with A an Abelian variety and L an affine algebraic group. We show (cf. Lemma 4.3 and Corollary 4.5) that a definable subgroup H of G is stably dominated if and only if $H \cap A$ and $f(H)$ are. For linear groups, we have:

Theorem 1.13 (cf. Theorem 6.11). *Let G be an affine algebraic group and let H be a stably dominated definable subgroup of G . Then H is isomorphic to $\mathbb{H}(\mathcal{O})$, \mathbb{H} an algebraic group scheme over \mathcal{O} .*

If H is Zariski dense in G , \mathbb{H} can be taken to be K -isomorphic to G .

Finally, we show that enough of the structure of definable groups in ACVF is known to be able to classify all the fields: as expected, they are all isomorphic either to the valued field itself or to its residue field (cf. Theorem 6.21).

Let us conclude this introduction with a series of examples that illustrate some of the structure results proved in this paper:

Example 1.14 (Definable generic types of algebraic groups). Let K be an algebraically closed valued field.

- (1) $\mathrm{SL}_n(\mathcal{O})$ is stably dominated, with a unique definable generic.
- (2) $\mathbb{G}_m(K)$ has two definable generics: elements of small valuation and elements of large valuation (i.e. infinitesimals). Both of these definable generics are attributable to Γ . The group $\mathbb{G}_m(\mathcal{O})$ is its largest stably dominated subgroup and $\mathbb{G}_m(K)/\mathbb{G}_m(\mathcal{O}) \simeq \Gamma$.
- (3) $\mathbb{G}_a(K)$ has a unique definable generic: elements of small valuation. This is a limit stably dominated example. Here the definable generic types correspond to cofinal definable types in the (partially) ordered set $(\Gamma, >)$ indexing the stably dominated subgroups $\alpha\mathcal{O}$. The fact that the poset has a unique cofinal definable type is however a phenomenon of dimension one.
- (4) $\mathbb{G}_a^2(K)$ has a large family of definable generics. For any definable curve E in Γ^2 , cofinal in the sense that for any (a, b) there exists $(d, e) \in E$ with $d < a, e < b$, there exists a definable generic type of $\mathbb{G}_a^2(K)$ whose projection to Γ^2 concentrates on E . This is also a limit stably dominated group.
- (5) Let G be the solvable group of upper diagonal matrices. Then G has left, right as well as two sided definable generics. One of the latter is given as follows. Let $(x_{i,j})$ be the matrix coefficients of an element of $x \in \mathrm{GL}_n(K)$. A two-sided generic of G is determined by: $\mathrm{val}(x_{i,j}) \ll \mathrm{val}(x_{i',j'})$ when $(i, j) < (i', j')$ lexicographically (and $i < j$). The valuation of the diagonal coefficients is a group homomorphism into Γ^n with stably dominated kernel.
- (6) For $n > 1$, $\mathrm{SL}_n(K)$ (or $\mathrm{GL}_n(K)$) has no definable left generic type. For suppose p is a definable type. Then $(d_p x)(\mathrm{val}(x_{11}) \leq \mathrm{val}(x_{21}))$ or $(d_p x)(\mathrm{val}(x_{11}) > \mathrm{val}(x_{21}))$. Multiplying on the left by the elementary matrix with 1's on the diagonal, and t (with $\mathrm{val}(t) < 0$) in the $(2, 1)$ -entry (respectively the $(1, 2)$ -entry), this situation becomes reversed. Thus p is not invariant under left multiplication. Since neither SL_n nor $\mathrm{GL}_n(K)$ have proper subgroups of finite index (definable or not), p cannot be a left generic.

Also $\mathrm{SL}_n(\mathcal{O})$ (respectively $\mathrm{GL}_n(\mathcal{O})$) is a maximal stably dominated subgroup of $\mathrm{SL}_n(K)$ (respectively $\mathrm{GL}_n(K)$), as are all of its conjugates. The double coset space is a disjoint union of $n!$ copies of Γ^{n-1} (respectively Γ^n).

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2. PRELIMINARIES

We recall some material from stability, stable domination, \mathcal{o} -minimality and the model theory of algebraically closed valued fields, in a form suitable for our purposes.

We fix a theory T that eliminates imaginaries and a universal model \mathbb{U} (a sufficiently saturated and homogeneous model). Any model of T that we consider will be an elementary submodel of \mathbb{U} . We write $\models \phi$ as a shorthand for $\mathbb{U} \models \phi$. By a *definable* set or function we mean one defined in T_A for some $A \subseteq \mathbb{U}$. If we wish to specify the base of definition, we say A -definable.

A pro-definable set X is a (small) projective filtered system $(X_i)_{i \in I}$ of definable sets and definable maps. We think of X as $\varprojlim_i X_i$. Pro-definable sets can equivalently be presented as a collection of formulas in potentially infinitely many variables. If all the maps in the system describing X are injective, we say that X is ∞ -definable and we can identify X with a subset of any of the X_i . If all the maps in the system describing X are surjective, we say that X is strict pro-definable; equivalently, the projection of X to any X_i is definable. A relatively definable subset of X is a definable subset of one of the X_i ; we identify it as a subset of X by pulling it back to X . Equivalently, if $x = (x_i)$ is a tuple of variables where each x_i ranges over X_i , a relatively definable subset of X is the set of realizations in X of some formula $\phi(x)$.

A pro-definable map $f : \varprojlim_i X_i \rightarrow Y$ where Y is definable is a definable map from some X_i to Y . A pro-definable map $f : \varprojlim_i X_i \rightarrow \varprojlim_j Y_j$ is a compatible collection of maps $f_j : \varprojlim_i X_i \rightarrow Y_j$. A pro-definable subset of a pro-definable set $X = \varprojlim_i X_i$ is a pro-definable set Y with an injective pro-definable map $f : Y \rightarrow X$; it can be identified with a compatible collection of ∞ -definable subsets of the X_i , or with an intersection of relatively definable subsets of X .

More generally, By a *piecewise pro-definable* set, we mean a family $(X_i)_{i \in I}$ of pro-definable over a directed order I , with a compatible system of injective pro-definable maps $X_i \rightarrow X_j$, for all $i \leq j$, viewed as inclusion maps. The direct limit is thus identified with the union, and denoted $X = \bigcup_i X_i$. A pro-definable subset of X is a pro-definable subset of one of the X_i .

2.1. Definable types and filters. A type will mean a complete type in possibly infinitely many variables (equivalently an ultrafilter on the Boolean algebra of relatively definable sets). We will refer to possibly incomplete types as filters.

A filter $\pi(x)$ over \mathbb{U} in the (possibly infinite) tuple of variables x is said to be C -definable if for all formulas $\phi(x, y)$, there exists a formula $\theta(y) =: (d_\pi x)\phi(x, y)$ over C , such that $\pi = \{\phi(x, m) : \mathbb{U} \models (d_\pi x)\phi(x, y)\}$. Note that π is completely determined by the map $\phi(x, y) \mapsto (d_\pi x)\phi(x, y)$, its *definition scheme*. If π is a type over \mathbb{U} , then this map is a Boolean homomorphism.

If C is a set of parameters, we define $\pi|C := \{\phi(x, m) : \mathbb{U} \models (d_\pi x)\phi(x, y) \text{ and } m \in C\}$. Note that if C is a model, $\pi|C$ completely determines π .

We say that π *concentrates* on a pro-definable set D if $\pi(x) \vdash x \in D$. By a *pro-definable function on π* , we mean a pro-definable map $f : D \rightarrow X$ such that π concentrates on D .

Definition 2.1. If f is a pro-definable function on a C -definable filter π , we define the push-forward $f_*\pi$ by:

$$(d_{f_*\pi} u)\phi(u, v) = (d_\pi x)\phi(f(x), v).$$

So that if $a \models \pi|C$ then $f(a) \models f_*\pi|C$.

Definition 2.2. Let $\pi(x), \mu(y)$ be C -definable filters. Define $\rho(x, y) := \pi(x) \otimes \mu(y)$ by:

$$(d_{\rho xy})\phi(x, y, z) = (d_\pi x)(d_\mu y)\phi(x, y, z).$$

Then, if $a \models \pi|C$ and $b \models \mu|Ca$, $ab \models (\pi \otimes \mu)|C$. If π and μ are types, then $f_*\pi$ and $\pi \otimes \mu$ are types.

We will occasionally use a more general construction. Assume $p(x)$ is a C -definable type. Let $a \models p$ and let $q_a(y)$ be a Ca -definable type of the theory T_{Ca} .

Lemma 2.3. *There exists a unique definable type $r(xy)$ such that for any $B \supseteq C$, if $ab \models r|B$ then $a \models p|B$ and $b \models q_a|Ba$.*

Proof. Given a formula $\phi(xy, z)$, let $\psi(x, z)$ be a formula such that $\psi(a, z) = (d_{q_a} y)\phi(a, y, z)$. The formula ψ is not uniquely defined, but if ψ and ψ' are two possibilities then $(d_p x)(\psi \leftrightarrow \psi')$. Therefore we can define:

$$(d_r xy)\phi(xy, z) = (d_p x)\psi(x, y, z).$$

It is easy to check that this definition scheme works. □

In fact, over $C = \text{acl}(C)$, it suffices that q_a be definable over $\text{acl}(Ca)$. This follows from:

Lemma 2.4. *Let M be a model and let $C = \text{acl}(C) \subseteq M$. Let $\text{tp}(a/M)$ be C -definable. Let $c \in \text{acl}(Ca)$. Then $\text{tp}(ac/M)$ is C -definable. Indeed, $\text{tp}(a/M) \cup \text{tp}(ac/C) \vdash \text{tp}(ac/M)$.*

Proof. Let $\phi(x, y)$ be a formula over C such that $\phi(a, c)$ holds, and such that $\phi(a, y)$ has k solutions. Let $\psi(x, y, z)$ be any formula. Then the equivalence relation $z_1 E z_2$ defined by $(d_p x)(\forall y)\phi(x, y) \implies (\psi(x, y, z_1) \leftrightarrow \psi(x, y, z_2))$ where $p := \text{tp}(a/M)$ is C -definable and has at most 2^k classes. It follows that each of these classes, denote Z_i , are C -definable. Finally, for all m , $\models \psi(a, c, m)$ if and only if $m \in Z_i$ for some i such that $(\forall z \in Z_i)\psi(x, y, z) \in \text{tp}(ac/C)$. □

See Proposition 2.10.(4) for a stronger statement in the stably dominated case.

Definition 2.5 (Germs of definable functions). Let p be a definable type. Two definable functions $f(x, b)$ and $g(x, b')$ are said to have the same p -germ if

$$\models (d_p x)f(x, b) = g(x, b').$$

We say that *the p -germ of $f(x, b)$ is defined over C* if whenever $\text{tp}(b/C) = \text{tp}(b'/C)$, $f(x, b)$ and $f(x, b')$ have the same p -germ. Note that the equivalence relation $b \sim b'$ defined by “ $f(x, b), f(x, b')$ have the same p -germ” is definable; the p -germ of $f(x, b)$ is defined over C if and only if $b/\sim \in \text{dcl}(C)$.

2.2. Stable domination.

Definition 2.6. Let D be a C -definable set.

- (1) D is said to be *stably embedded* if for any definable $X \subseteq D^n$, for some n , is $C \cup D$ -definable.
- (2) D is said to be *stable* if every formula $\phi(x; y)$ with parameters in C , implying $y \in D^n$ for some n , is a stable formula.

Since being a stable implies being stably embedded, the later is often referred to as being *stable*, *stably embedded* in the literature. See e.g. [Pil83] for a treatment of basic stability and [CH99, Appendix] for the properties of stably embedded and stable sets.

For all C , let $(D_i)_i$ enumerate all C -definable stable sets and $\text{St}_C := \prod_i D_i$. It is a strict pro-definable set. We often consider St_C as a (stable) structure, whose sorts are the D_i , with the full induced structure. By a pro-definable map into St_C , we mean (somewhat abusively) a collection of maps $f = (f_j)_j$ where the range of each f_j is stable. For all b , let $\text{St}_C(b)$ denote $\text{dcl}(Cb) \cap \text{St}_C$.

Notation 2.7. For all a, b, C , we write $a \downarrow_C b$ if there exists an $\text{acl}(C)$ -definable type p such that $a \models p|_{\text{acl}(Cb)}$.

Note that if $a, b, C \in \text{St}_D$ for some $D \subseteq C$ then $a \downarrow_C b$ if and only if they are (forking) independent in St_D over C . Let us now recall the definition of stable domination:

Definition 2.8. Let $p = \text{tp}(a/C)$ and $\alpha : p \rightarrow \text{St}_C$ be a pro- C -definable map. The type p is said to be *stably dominated* via $\alpha = (\alpha_i)_i$ if for any tuple b , if $\text{St}_C(b) \downarrow_C \alpha(a)$, then $\text{tp}(b/C\alpha(a)) \vdash \text{tp}(b/Ca)$.

A type p over C is said to be stably dominated if it is stably dominated via some C -definable map $\alpha : p \rightarrow \text{St}_C$. For $a \models p$, let $\theta_C(a)$ enumerate $\text{St}_C(a)$, then p is stably dominated if and only if it is stably dominated via θ_C .

Let us now recall some of the result from [HHM08] regarding stable domination.

Proposition 2.9 ([HHM08, Corollary 3.31.(iii) and Proposition 3.13]). *For all a and C ,*

- (1) $\text{tp}(a/C)$ is stably dominated if and only if $\text{tp}(a/\text{acl}(C))$ is.
- (2) If $C = \text{acl}(C)$ and $\text{tp}(a/C)$ is stably dominated via f , then $\text{tp}(a/C)$ has a unique C -definable extension p . Moreover, for all $B \supseteq C$, $a \models p|_B$ if and only if $\text{St}_C(B) \downarrow_C f(a)$.

Proposition 2.10. *Assume $\text{tp}(a/C)$ is stably dominated.*

- (1) Let q be a global $\text{acl}(C)$ -definable type and $b \models q|_{\text{acl}(C)}$, $a \downarrow_C b$ implies $b \downarrow_C a$. In particular, if $\text{tp}(b/C)$ is stably dominated, $a \downarrow_C b$ if and only if $b \downarrow_C a$.
- (2) $a \downarrow_C bd$ if and only if $a \downarrow_C b$ and $a \downarrow_{Cb} d$.
- (3) If $\text{tp}(b/Ca)$ is stably dominated, then so is $\text{tp}(ab/C)$.
- (4) If $b \in \text{acl}(Ca)$, then $\text{tp}(b/C)$ is stably dominated.

Proof. (1) and (2) are easy to check. (3) is [HHM08, Proposition 6.11] and (4) is [HHM08, Corollary 6.12]. \square

Let p be a global C -invariant type. We say that p is stably dominated over C if $p|_C$ is stably dominated.

Proposition 2.11 ([HHM08, Proposition 4.1 and Theorem 4.9]). *Let $B \supseteq C$ and p be a global C -invariant type.*

- (1) Let f be a pro- C -definable function. If p is stably dominated over C via f , then it is also stably dominated over B via f .

- (2) If p is stably dominated over B and $\text{tp}(B/C)$ has a global C -invariant extension, then p is stably dominated over C .

Proposition 2.12 (Strong germs, [HHM08, Theorem 6.3]). *Let $C = \text{acl}(C)$, p be a global C -invariant type stably dominated over C and f be a definable function defined at p .*

- (1) *The p -germ of f is strong; i.e. there exists a $[f]_p C$ -definable function g with the same p -germ as f .*
- (2) *If $f(a) \in \text{St}_C$ for $a \models p$, then $[f]_p \in \text{St}_C$.*

Let us now recall the definition of metastability. Let Γ be an \emptyset -definable stably embedded set. We will also assume that Γ is orthogonal to the stable part: no infinite definable subset of Γ^{eq} is stable. For any C and a , let $\Gamma_C(a)$ denote $(C \cup \Gamma^{\text{eq}}) \cap \text{dcl}(Ca)$.

Definition 2.13. The theory T is metastable (over Γ) if, for any C :

- (1) (Metastability bases) There exists $D \supseteq C$ such that for any tuple a , $\text{tp}(a/\Gamma_D(a))$ is stably dominated.
- (2) (Invariant extension property) If $C = \text{acl}(C)$, then, for all tuple a , there exists a global C -invariant type p such that $a \models p|C$.

A D such as in (1) is called a metastability basis.

Remark 2.14. Since Γ is orthogonal to the stable part, if $\text{tp}(a/C)$ is stably dominated, so is $\text{tp}(a/C\gamma)$ for any tuple $\gamma \in \Gamma$. It follows that if C is a metastability basis, so is $C\gamma$.

Proposition 2.15 (Orthogonality to Γ , [HHM08, Corollary 10.8]). *Assume that the theory T is metastable over Γ . A global type C -invariant type p is stably dominated if and only if for any definable map $g : p \rightarrow \Gamma^{\text{eq}}$, the p -germ of g is constant (equivalently, g_*p is a realized type).*

Write $g(p)$ for the constant value of the p -germ. The property of p in the proposition above is referred to as *orthogonality of p to Γ* . Note that this is strictly weaker than orthogonality of D to Γ for some definable $D \in p$.

2.3. Ranks and weights. Let us now recall our “finite rank” assumptions. But, first let us recall the definition of internality.

Definition 2.16. A definable set X is Γ -*internal* if $X \subseteq \text{dcl}(F\Gamma)$ for some finite set F ; equivalently for any $M < M' \models T$, $X(M') \subseteq \text{dcl}(M\Gamma(M'))$.

The same condition with acl replacing dcl is called *almost internality*.

Thus a definable X is almost Γ -internal if over some finite set F of parameters and for some finite m , X admits a definable m -to-one function $f : X \rightarrow Y/E$ for some definable $Y \subseteq \text{dcl}(\Gamma^m)$ and definable equivalence relation E . When Γ eliminates imaginaries, E does not need to be mentioned. Note that X is Γ -internal whenever we can choose $m = 1$.

A definable set D is called *o -minimal* if there exists a definable linear ordering on D such that every definable subset of D (with parameters) is a finite union of intervals and points. There is a natural notion of dimension for definable subsets of D^m , such that D is dimension one (see [vdD98]).

Definition 2.17. T has (FD) if:

- (1) Γ is o -minimal.
- (2) Morley rank (in the stable part, denoted MR) is uniformly finite and definable in families: if $(D_t)_t$ is a definable family of definable sets, then, when D_t is stable, the Morley rank of D_t is finite and bounded uniformly in t and for all $m \in \mathbb{Z}_{>0}$, $\{t : \text{MR}(D_t) = m\}$ is definable.

- (3) For all definable D , the Morley rank of $f(D)$, where f ranges over all functions definable with parameters whose range is stable, takes a maximum value $\dim_{\text{st}}(D) \in \mathbb{Z}_{>0}$.
- (4) For all definable D , the o -minimal dimension of $f(D)$, where f ranges over all functions definable with parameters whose range is Γ -internal, takes a maximum value $\dim_o(D) \in \mathbb{Z}_{>0}$.

Note that, in that case, Γ is an o -minimal group, so it eliminates imaginaries. So, in (3), we could equally well ask that $f(D) \subseteq \Gamma^n$ for some n .

Definition 2.18. If $X = \varprojlim_i X_i$ is pro-definable, we define $\dim_{\text{st}}(X) = \max\{\dim_{\text{st}}(X_i) : i\}$ and $\dim_o(X) = \max\{\dim_o(X_i) : i\}$.

We define $\dim_{\text{st}}(a/B) := \min\{\dim_{\text{st}}(D) : a \in D \text{ and } D \text{ is } B\text{-definable}\}$ and $\dim_o(a/B) := \min\{\dim_o(D) : a \in D \text{ and } D \text{ is } B\text{-definable}\}$.

Note that if D is a stable definable set then $\dim_{\text{st}}(D) = \text{MR}(D)$ and, similarly, for all tuples $a \in \text{St}_C$, $\dim_{\text{st}}(a/C) = \text{MR}(a/C)$. Note that, in general, we may have $\dim_{\text{st}}(d/B) > \dim_{\text{st}}(\text{St}_B(d)/B)$. Similar statements hold for \dim_o .

A set A is acl-finitely generated over $B \subseteq A$ if $A \subseteq \text{acl}(Ba)$ for some finite tuple a from A . If $c \in \text{acl}(Ba)$ is a finite tuple, then $\dim_{\text{st}}(ca/B) = \dim_{\text{st}}(a/B)$. In particular, it makes sense to speak of the stable dimension (respectively the o -minimal dimension) over B of any set finitely acl-generated over B .

Lemma 2.19. (FD) Let C be a set of parameters and a be a finite tuple. Then $\Gamma_C(a)$ and $\text{St}_C(a)$ are acl-finitely generated over C .

Proof. Any tuple $d \in \text{St}_C(a)$ can be written $d = h(a)$ for some C -definable function h . By the dimension bound in (FD), $\dim_{\text{st}}(d/C) \leq \dim_{\text{st}}(a/C)$. It follows that if $d_i \in \text{St}_C(a)$ then for sufficiently large n , $d_n \in \text{acl}(Cd_1, \dots, d_{n-1})$. So $\text{St}_C(a)$ is finitely acl-generated. The proof for $\Gamma_C(a)$ is the same. \square

Lemma 2.20. (FD) Let $f : P \rightarrow Q$ be a definable map between definable sets P and Q . Let $P_a = f^{-1}(a)$. Then there exists m such that if P_a is finite, then $|P_a| \leq m$.

Proof. Say f, P, Q are \emptyset -definable. By compactness, it suffices to show that if P_a is infinite then there exists a \emptyset -definable set Q' with $a \in Q'$ and P_b infinite for all $b \in Q'$.

Claim 2.20.1. If P_a is infinite then either $\dim_{\text{st}}(P_a) > 0$ or $\dim_o(P_a) > 0$.

Proof. Let M be a good base, with $a \in M$. Let $c \in P_a \setminus M$. If $\Gamma_M(c) \neq \Gamma(M)$ then $\dim_o(P_a) > 0$. Otherwise, by metastability, $\text{tp}(c/M)$ is stably dominated, say via f . If $f(c) \in M$, then $\text{tp}(b/M) \vdash \text{tp}(b/Mc)$ for all b , and taking $b = c$ it follows that $c \in M$. Thus $f(c) \in \text{St}_M \setminus M$. It follows that $\dim_{\text{st}}(P_a) > 0$. \square

If $\dim_{\text{st}}(P_a) > 0$, then there exists a definable family of stable definable sets D_t with $\dim_{\text{st}}(D_t) = k > 0$, and a definable function $f(x, u)$ such that for some b and t , $f(P_a, b) = D_t$. Then the formula $(\exists u)(\exists t)f(P_y, u) = D_t$ is true of $y = a$, and implies that P_y is infinite. The case $\dim_o(P_a) > 0$ is similar. \square

Definition 2.21. T has (FD_ω) if, in addition to (FD), any C is contained in a metastability basis M such that M is a model and such that for any finitely acl-generated $G \subseteq \Gamma$ and $S \subseteq \text{St}_{M \cup G}$ over M , isolated types over $M \cup S$ are dense.

Say $\text{tp}(a/C)$ is *strongly stably dominated* if there exists $\phi(x, c) \in \text{tp}(a/\text{St}_C(a))$ such that for any tuple b with $\text{St}_C(b) \downarrow_C \text{St}_C(a)$, $\text{tp}(a/Cb)$ is isolated via ϕ . In particular, $\text{tp}(a/\text{St}_C(a))$

is isolated via ϕ . Moreover, $\text{tp}(a/C)$ is stably dominated, since $\text{tp}(a/\text{St}_C(a))$ implies ϕ which implies $\text{tp}(a/\text{St}_C(a)b)$ for any b as above.

When $\text{tp}(a/C)$ is strongly stably dominated, the parameters c of ϕ may be written as $c_0h(a)$ for some C -definable function h into St_C and tuple $c_0 \in C$. In this situation we say that $\text{tp}(a/C)$ is strongly stably dominated via h . Conversely, if $\text{tp}(a/C)$ is stably dominated via h and $\text{tp}(a/Ch(a))$ is isolated, then $\text{tp}(a/C)$ is strongly stably dominated via h .

Lemma 2.22. (*FD $_\omega$*) *Let D be C_0 -definable and $h : D \rightarrow S$ a definable map to a stable definable set of maximal possible dimension $\dim_{\text{st}}(S) = \dim_{\text{st}}(D)$. Then there exists M containing C_0 and $a \in D$ such that, with $C := \Gamma_M(a)$, we have $\dim_{\text{st}}(D) = \dim_{\text{st}}(h(a)/C) = \dim_{\text{st}}(\text{St}_C(a)/C)$, and $\text{tp}(a/C)$ is strongly stably dominated via h .*

Proof. We may assume that h is defined over C_0 . Let $a_0 \in D$ be such that $\dim_{\text{st}}(D) = \dim_{\text{st}}(h(a_0)/C_0)$. Extend C_0 to a metastability basis M as in (*FD $_\omega$*), with $h(a_0)$ independent from M over C_0 . Then $\dim_{\text{st}}(D) = \dim_{\text{st}}(h(a_0)/M) = \dim_{\text{st}}(\text{St}_M(a_0)/M)$. Choose a_0 such that $\dim_{\text{o}}(\Gamma_M(a_0)/M)$ is as large as possible—given the other constraints. Let $C := \Gamma_M(a_0)$ and $B := \text{St}_C(a_0)$; then, by Lemma 2.19, C and B are finitely generated over M . Moreover, $\dim_{\text{st}}(h(a_0)/C) = \dim_{\text{st}}(D) \geq \dim_{\text{st}}(B/C)$ so $B \subseteq \text{acl}(Ch(a_0))$. Let D' be B -definable set contained in D , containing a , and such that for any $a \in D'$, $B \subseteq \text{acl}(Ch(a))$. By (*FD $_\omega$*), there exists $a \in D'$ such that $\text{tp}(a/B)$ is isolated. By choice of D' we have $B \subseteq \text{acl}(Ch(a))$. So $\text{tp}(B/Ch(a))$ is atomic and, therefore, $\text{tp}(a/Ch(a))$ is isolated.

Similarly, by maximality of $\dim_{\text{o}}(\Gamma_M(a_0)/M)$ and the fact that $\Gamma_M(a_0) \subseteq B \cap \Gamma \subseteq \text{acl}(Ca) \cap \Gamma \subseteq \Gamma_M(a)$, we have $\Gamma_M(a_0) = C$. By metastability, $\text{tp}(a/C)$ is stably dominated. But $\dim_{\text{st}}(\text{St}_C(a)/C) \leq \dim_{\text{st}}(D) = \dim_{\text{st}}(B/C)$ and $B \subseteq \text{acl}(Ch(a)) \subseteq \text{acl}(\text{St}_C(a))$, so $\text{St}_C(a) \subseteq \text{acl}(B) \subseteq \text{acl}(Ch(a))$ and $\text{tp}(a/C)$ is stably dominated via h . It follows that $\text{tp}(a/C)$ is strongly stably dominated via h . \square

Finally, let us introduce weight.

Definition 2.23. Let p be a definable type and X be a pro-definable set. We say that X has p -weight smaller or equal to n if whenever $b \in X$, $(a_1, \dots, a_n) \models p^{\otimes n}$, we have $a_i \models p|b$ for some i . The set X has bounded weight if for some n , for every stably dominated type p concentrating on X , X has p -weight smaller or equal to n .

For all definable D and stably dominated type p concentrating on D , the p -weight of D is bounded by $\dim_{\text{st}}(D)$. It follows that if (*FD*) holds, every definable set has bounded weight. In ACVF, for example, the weight of any definable subset of a variety V is bounded by the dimension of V .

2.4. Some o -minimal lemmas. This subsection contains some lemmas on o -minimal partial orders and groups. The former will yield a generic type of limit stably dominated groups. The latter will be used to improve some statements from “almost internal” to “internal”.

Let (P, \leq) be a definable partial ordering, and p a definable type concentrating on P . We say p is cofinal if for any $c \in P$, $\models (d_p x)x \geq c$. Equivalently, for every non-cofinal definable $Q \subseteq P$, $\models \neg(d_p x)Q(x)$. The following result can also be found in [HL16, Lemma 4.2.18].

Lemma 2.24. *Let P be a definable directed partial ordering in an o -minimal structure Γ . Then there exists a definable type p cofinal in P .*

Proof. We assume P is \emptyset -definable, and work with \emptyset -definable sets; we will find an \emptyset -definable type with this property.

Note first that we may replace P with any \emptyset -definable cofinal subset. Also if Q_1 and Q_2 are non-cofinal subsets of P , there exist a_1 and a_2 such that no element of Q_i lies above a_i ; but by

directedness there exists $a \geq a_1, a_2$; so no element of $Q_1 \cup Q_2$ lies above a , i.e. $Q_1 \cup Q_2$ is not cofinal. In particular if $P = P' \cup P''$, at least one of P', P'' is cofinal in P (hence also directed).

If $\dim_o(P) = 0$ then P is finite, so according to the above remarks we may assume it has one point; in which case the lemma is trivial. We use here the fact that in an o -minimal theory, any point of a finite \emptyset -definable set is definable.

If $\dim_o(P) = n > 0$, we can divide P into finitely many \emptyset -definable sets P_i , each admitting a map $f_i : P_i \rightarrow \Gamma$ with fibers of dimension strictly smaller than n . We may thus assume that there exists an \emptyset -definable $f : P \rightarrow \Gamma$ with fibers of dimension strictly smaller than n . Let $P(\gamma) = f^{-1}(\gamma)$, and $P(a, b) = f^{-1}(a, b)$.

Claim 2.24.1. *One of the following holds:*

- (1) *For any $a \in \Gamma$, $P(a, \infty)$ is cofinal in P .*
- (2) *For some \emptyset -definable $a \in \Gamma$, for all $b > a$, $P(a, b)$ is cofinal.*
- (3) *For some \emptyset -definable $a \in \Gamma$, $P(a)$ is cofinal.*
- (4) *For some \emptyset -definable $a \in \Gamma$, for all $b < a$, $P(b, a)$ is cofinal.*
- (5) *For any $a \in \Gamma$, $P(-\infty, a)$ is cofinal.*

Proof. Suppose (1) and (5) fail. Then $P(a, \infty)$ is not cofinal in P for some a ; so $P(-\infty, b)$ must be cofinal, for any $b > a$. Since (5) fails, $\{b : P(-\infty, b) \text{ is cofinal}\}$ is a non-empty proper definable subset of Γ , closed upwards, hence of the form $[A, \infty)$ or (A, ∞) for some \emptyset -definable $A \in \Gamma$. In the former case, $P(-\infty, A)$ is cofinal, but $P(-\infty, b)$ is not cofinal for $b < A$, so $P(b, A)$ is cofinal for any $b < A$; thus (4) holds.

In the latter case, $(-\infty, b)$ is cofinal for any $b > A$, while $(-\infty, A)$ is not; so $P([A, b))$ is cofinal for any $b > A$. Thus either (2) or (3) hold. \square

Let p_1 be an \emptyset -definable type of Γ , concentrating on sets X with $f^{-1}(X)$ cofinal; for instance in case (1), p_1 concentrates on intervals (a, ∞) .

Claim 2.24.2. *For any $c \in P$, if $a \models p_1|c$ then there exists $d \in P(a)$ with $d \geq c$.*

Proof. Let $Y(c) = \{x : (\exists y \in P(x))y \geq c\}$. Then $P^{-1}(\Gamma \setminus Y(c))$ is not cofinal in P , so it cannot be in the definable type p_1 . Hence $Y(c) \in p_1|c$. \square

Now let $M \models T$. Let $a \models p_1|M$. By induction, let q_a be an a -definable type, cofinal in $P(a)$, and let $b \models q_a|Ma$. Then $\text{tp}(ab/M)$, and hence $\text{tp}(b/M)$ is definable (see Lemma 2.3). If $c \in M$ then by Claim 2.24.2, there exists $d \in P(a)$ with $d \geq c$. So $\{y \in P(a) : \neg(y \geq c)\}$ is not cofinal in $P(a)$. Therefore this set is not in q_a . Since $b \models q_a|Ma$, we have $b \geq c$. This shows that $\text{tp}(b/M)$ is cofinal in P . \square

Discussion. If there exists a definable weakly order preserving map $j : \Gamma \rightarrow P$ with cofinal image, then we can use the definable type at ∞ of Γ , r_∞ , to obtain a cofinal definable type of P , namely j_*r_∞ .

When Γ admits a field structure, perhaps such a map j always exists. In general, it is not always possible to find a one-dimensional cofinal subset of P . For instance, when Γ is a divisible ordered Abelian group, consider the product of two closed intervals of incommensurable sizes; or the subdiagonal part of a square.

Lemma 2.25. *Let G be a definable group. Assume G is almost internal to a stably embedded definable set Γ . Then there exists a finite normal subgroup N of G with G/N internal to Γ .*

Proof. The assumption implies the existence of a definable finite-to-one function $f : G \rightarrow Y$, where $Y \subseteq \text{dcl}(\Gamma)$. Given a definable $Y' \subseteq Y$, let $m(Y')$ be the least integer m such that (possibly over some more parameters), there exists a definable m -to-one map $f^{-1}(Y') \rightarrow Z$, for

some definable $Z \subseteq \text{dcl}(\Gamma)$. We may assume that f is $m(Y)$ -to-one. Let I be the family of all definable subsets Y' of Y with $(Y' = \emptyset \text{ or } m(Y') < m(Y))$. This is clearly an ideal (closed under finite unions, and definable subsets). Let $F := \{Y \setminus Y' : Y' \in I\}$ be the dual filter. For $g \in G$, let $D(g)$ be the set of $y \in Y$ such that for some (necessarily unique) y' , $g \cdot f^{-1}(y) = f^{-1}(y')$; and define $g_*(y) = y'$. The function $x \mapsto (f(x), f(g \cdot x))$ shows that $\{y : |f(g \cdot f^{-1}(y))| > 1\} \in I$. Since $X := \{y : |f^{-1}(y)| = m\} \in F$ and $(Y \setminus I) \cap X \subseteq D(g)$, it follows that $D(g) \in F$.

For any definable $Z \subseteq Y$, let $F|Z = \{W \cap Z : W \in F\}$. Let G_0 be the set of bijections $\phi : Y' \rightarrow Y''$ with $Y', Y'' \in F$, carrying the filter $F|Y'$ to $F|Y''$. Note that $g_* : D(g) \rightarrow D(g^{-1})$ lies in G_0 . Write $\phi \sim \phi'$ if ϕ and ϕ' agree on some common subset of their domains, lying in F ; and let $G' = G_0 / \sim$. Composition induces a group structure on G' and we obtain a homomorphism $G \rightarrow G'$, $g \mapsto g_* / \sim$. Let N be the kernel of this homomorphism. Let us prove that $|N| \leq m(Y)$. For suppose n_0, \dots, n_m are elements of N . Then for some $y \in \cap D(n_i)$ we have $(n_i)_*(y) = y$. It follows that $n_i(f^{-1}(y)) = f^{-1}(y)$. Fix any $g \in f^{-1}(y)$ and since $|f^{-1}(y)| = m$, for some $i \neq j$ we have $n_i \cdot g = n_j \cdot g$ and hence $n_i = n_j$.

Let σ be an automorphism fixing Γ , then for any $g \in G$, since g_* is Γ -definable, $\sigma(g_*) = g_*$ and hence $g \cdot N = N$. It follows that G/N is Γ -internal. \square

Lemma 2.26. *Let G be a definable group. Assume G is almost internal to an o-minimal definable set Γ . Then G is Γ -internal.*

Proof. By Lemma 2.25, there exists a definable surjective homomorphism $f : G \rightarrow B$ with B a group definable over Γ , and $N = \ker(f)$ a group of finite size n . Let B^0 be the connected component of the identity in B ; then B/B^0 is finite, and it suffices to prove the lemma for $f^{-1}(B^0)$. Assume therefore that B is connected.

If G has a proper definable subgroup G_1 of finite index, then $f(G_1) = B$ by connectedness of B . It follows that N is not contained in G_1 , so $N_1 = N \cap G_1$ has smaller size than N . Hence using induction on the size of the kernel, G_1 is Γ -internal; hence so is G . Thus we may assume G has no proper definable subgroups of finite index. Since the action of G on N by conjugation has kernel of finite index, N must be central.

Let $Y = \{g^n : g \in B\}$. By [Edm03, Theorem 7.2], there exists a definable function $\alpha : Y \rightarrow B$ with $\alpha(b)^n = b$. Define $\beta : Y \rightarrow G$ by $\beta(b) = a^n$ where $f(a) = \alpha(b)$; this does not depend on the choice of a , and we have $f(\beta(b)) = f(a)^n = \alpha(b)^n = b$. It follows that $f^{-1}(Y) = N\beta(Y) \subseteq \text{dcl}(N, \Gamma)$ is Γ -internal.

Similarly, let $[B, B] = \{[g, h] : g, h \in B\}$. As above, there exists a definable $\alpha_1 : [B, B] \rightarrow B$ such that $(\exists y)[\alpha_1(b), y] = b$, and $\alpha_2 : [B, B] \rightarrow B$ such that $[\alpha_1(b), \alpha_2(b)] = b$. Define $\beta' : [B, B] \rightarrow G$ by $\beta'(b) = [a_1, a_2]$ where $f(a_i) = \alpha_i(b)$. Again β' is definable and well-defined, and shows that $f^{-1}([B, B])$ is Γ -internal.

Hence for any k , letting $X^{(k)} = \{x_1 \dots x_k : x_1, \dots, x_k \in X\}$, $(f^{-1}(Y \cup [B, B]))^{(k)} = f^{-1}((Y \cup [B, B])^{(k)})$ is Γ -internal. So we are done once we show:

Claim 2.26.1. *Let B be any definably connected group definable in an o-minimal structure. Let $Y = Y_n(B) = \{g^n : g \in B\}$. Then for some $k \in \mathbb{Z}_{\geq 0}$, $(Y \cup [B, B])^{(k)} = B$.*

Proof. If the Claim holds for a normal subgroup H of B with bound k' , and also for B/H (with bound k''), then it is easily seen to hold for B (with bound $k' + k''$).

We use induction on $\dim_o(B)$. If B has a non-trivial proper connected definable normal subgroup H , then the statement holds for H and for B/H . We may thus assume B has no such subgroups H . Thus any definable normal subgroup of B is finite and by connectedness, central.

If B is centerless, it is definably simple. By [PPS00], B is elementarily equivalent to a simple Lie group. In this case it is easy to see that every element is the product of a bounded number of commutators, by considering root subgroups.

Since the center Z of B is finite, then B/Z is definably simple and so the claim holds (say with $k_{B/Z}$). Let $W_l = (Y \cup [B, B])^{(l)}$, and let $\pi : B \rightarrow B/Z$ be the quotient homomorphism. Then $\pi(W_l) = B/Z$ for $l \geq k_{B/Z}$. For such l , $W_{5l} \cap Z = W_l \cap Z$ implies that $W_{4l} = W_{2l}$. Thus W_{2k} is a subgroup of B mapping onto B/Z , for some $k \leq k_{B/Z} 5^{|Z|}$. By connectedness $W_{2k} = B$. \square

This concludes the proof. \square

We conclude this section with a quick discussion of definable compactness. First, one can define the limit of a function along a definable type, generalizing the limit along a one-dimensional curve. Let X be a definable space over an o -minimal structure Γ .

Definition 2.27. Given a definable set D , a definable type p on D , and a definable function $g : D \rightarrow X$, we can define $\lim_p g = x \in X$ if for any definable neighborhood U of x , $g_{*}p$ concentrates on U .

When X is $\Gamma \cup \{-\infty, +\infty\}$, the limit always exists. Consider the definable type $g_{*}p$. It must of the form $r_{\infty}, r_{-\infty}, r_a := (x = a), r_a^+$ the type of elements infinitesimally bigger than a , or r_a^- . By definition $\lim_p g$ is $\infty, -\infty, a, a, a$ in the respective cases.

The following definition is equivalent to the one in [PS99].

Definition 2.28. The definable set X is *definably compact* if for any definable type p on Γ and any definable function $f : \Gamma \rightarrow X$, $\lim_p f$ exists.

Let A be an Abelian group. A definable set is called *generic* if finitely many translates of that set cover the group. Say A has the property (NG) if the non-generic definable sets form an ideal. It is shown in [PP07, Corollary 3.9] that in any definably compact Abelian group A in an o -minimal theory, (NG) holds; moreover any definable subsemigroup of A is a group (cf. [PP07, Theorem 5.1]). We include the deduction of the latter fact in a case we will need.

Lemma 2.29. *Let A be an Abelian group with (NG). Let Y be a definable semi-group of A , such that $Y - Y = A$. Then $Y = A$.*

Proof. Note first that $A \setminus Y$ is not generic. Otherwise, some finite intersection $\bigcap_{i=1}^n (c_i + Y) = \emptyset$. By assumption, there exist $b_i \in Y$ with $b_i + c_i \in Y$. If $b := \sum b_i$ then $b + c_i \in Y$; so by translating we may assume each $c_i \in Y$. But then $\sum c_i \in c_i + Y$ for each i , a contradiction.

As A is generic, Y cannot be in the non-generic ideal, so Y is generic and $\bigcup_{i=1}^n d_i + Y = G$ for some $d_i \in G$. Again we find $e \in Y$ with $e + d_i \in Y$. We have $\bigcup_{i=1}^n (e + d_i) + Y = G$. But $e + d_i + Y \subseteq Y$. So $Y = G$. \square

2.5. Valued fields: imaginaries and resolution. Let K be an algebraically closed valued field, with valuation ring \mathcal{O} , maximal ideal \mathfrak{M} and value group Γ . The geometric language for valued fields has a sort for the valued field itself, and certain other sorts. In particular, there is a sort S_n such that $S_n(K)$ is the space of free \mathcal{O} -modules in K^n , equivalently $S_n(K) = \text{GL}_n(K)/\text{GL}_n(\mathcal{O})$. In that section, we work in a model M of the theory ACVF of algebraically closed fields in the geometric language.

By a *substructure*, we mean a subset of M , closed under definable functions. For any geometric sort S and substructure $A \subseteq M$, $S(A)$ denotes $S \cap A$; in particular $K(A)$ is the set of field points of A . A substructure $A \subseteq M$ is called *resolved* if $A \subseteq \text{dcl}(K(A))$. When $\Gamma(A) \neq 0$ and A is algebraically closed this just amounts to saying that $A < M$. Recall that, in any first order theory, if A is a substructure of a model M , M is *prime* over A if any elementary map $A \rightarrow N$ into another model, extends to an elementary map $M \rightarrow N$; and M is *minimal* over A if there is no $M' < M$ with $A \subset M'$. If a minimal model over A and a prime one exist, then any two minimal or prime models over A are isomorphic.

Proposition 2.30. *Let A be a substructure of M , finitely generated over a subfield L of K , and assume $\Gamma(A) \neq 0$. Then there exists a minimal prime model \tilde{A} over A which enjoys the following properties.*

- (1) \tilde{A} is a minimal resolution of A . Moreover it is the unique minimal resolution, up to isomorphism over A . It is atomic over A .
- (2) $St_L(\tilde{A}) = St_L(A)$.
- (3) Let $A \leq A'$, with A' finitely generated over A . Then \tilde{A} embeds into \tilde{A}' over A . If $A \leq A' \leq \tilde{A}$, then \tilde{A} is the prime resolution of A' .
- (4) For a valued field extension L' of L , let $L'(A)$ be the structure generated by $L' \cup A$. Then $L'(\tilde{A})^{\text{alg}}$ is a prime resolution of $L'(A)$.
- (5) If $\text{tp}(A/L)$ is stably dominated, then $\text{tp}(\tilde{A}/L)$ is stably dominated.
- (6) If $\text{tp}(A'/A)$ is stably dominated, and $A' \downarrow_A \tilde{A}$, then there exists a prime resolution \tilde{A}' of A' such that $\text{tp}(\tilde{A}'/\tilde{A})$ is stably dominated.

Proof. The existence, uniqueness and minimality of \tilde{A} are shown in [HHM08, Theorem 11.14]. It is also shown there that $k(\tilde{A}) = k(\text{acl}(A))$ and $\Gamma(\tilde{A}) = \Gamma(\text{acl}(A))$, where k is the residue field; and that \tilde{A}/A is atomic, i.e. $\text{tp}(c/A)$ is isolated for any tuple c from A .

- (2) Since L is a field, for any $B = \text{acl}(B)$ with $L \subset B$, $St_L(B) = \text{dcl}(B, k(B))$.
- (3) This is immediate from the definition of prime resolution: since \tilde{A}' is a resolution of A , \tilde{A} embeds into \tilde{A}' . If $A \leq A' \leq \tilde{A}$, then \tilde{A} is clearly a minimal resolution of A' ; hence by (1) it is the prime resolution.
- (4) Let B be the prime resolution of $L'(A)$. Then \tilde{A} embeds into B . Within B , $L'(\tilde{A})^{\text{alg}}$ is a resolution of $L'(A)$; by minimality of B , $B = L'(\tilde{A})^{\text{alg}}$.
- (5) We may assume $L = L^{\text{alg}}$. Let \bar{L} be a maximal immediate extension of L . Choose it in such a way that $\text{tp}(\tilde{A}/\bar{L})$ is a sequentially independent extension of $\text{tp}(\tilde{A}/L)$. Since A/L is stably dominated, $\Gamma(\bar{L}(A)^{\text{alg}}) = \Gamma(\bar{L})$. According to (4), $\bar{L}(\tilde{A})^{\text{alg}}$ is the prime resolution of $\bar{L}(A)^{\text{alg}}$, and so $\Gamma(\bar{L}(\tilde{A})^{\text{alg}}) = \Gamma(\bar{L})$. Thus, by [HHM08, Theorem 10.12], $\text{tp}(\tilde{A}/\bar{L})$ is stably dominated. Using descent (cf. Proposition 2.11.(2)), $\text{tp}(\tilde{A}/L)$ is stably dominated.
- (6) Since $A' \downarrow_A \tilde{A}$ and $\text{tp}(A'/A)$ is stably dominated, $\text{tp}(A'/\tilde{A})$ is stably dominated. By (5), $\text{tp}(B/\tilde{A})$ is stably dominated, where B is a resolution of $\tilde{A}(A')$. But B contains a resolution of A' whose type over \tilde{A} is therefore stably dominated. \square

Proposition 2.31. *The theory ACVF is metastable, with (FD) and (FD $_{\omega}$).*

Proof. The fact that, in ACVF, maximally complete submodels are metastability bases is proved in [HHM08, Theorem 12.18.(ii)] and the existence of invariant extensions is shown in [HHM08, Corollary 8.16]. The fact that (FD) holds follows from the fact that the value group Γ is a divisible ordered Abelian group, that the stable part, being internal to an algebraically closed field, has finite Morley rank and that, for any definable subset D of the field, $\dim_{\text{st}}(D)$ and $\dim_{\circ}(D)$ are bounded by the dimension of the Zariski closure of D .

To see that (FD $_{\omega}$) holds, let $M \models \text{ACVF}$ and C be finitely generated over M — C might contain imaginaries. By Proposition 2.30, the resolution N of C is a model of ACVF which is atomic over M . Hence isolated types over C are dense. \square

Definition 2.32. (1) The (imaginary) element $e \in M$ is said to be *purely imaginary* over $C \subseteq M$ if $\text{acl}(Ce)$ contains no field elements other than those in $\text{acl}(C)$.
(2) An ∞ -definable set D is *purely imaginary* if there exists no definable map (with parameters) from D onto an infinite subset of the field K .

Note that D is purely imaginary if and only if, for all C such that D is C -definable, any $e \in D$ is purely imaginary over C . For every $\alpha \in \Gamma$, let $\alpha\mathcal{O} := \{x \in K : \text{val}(x) \geq \alpha\}$, and $\alpha\mathfrak{M} := \{x \in K : \text{val}(x) > \alpha\}$.

Lemma 2.33. *The following are equivalent:*

- (1) e is purely imaginary
- (2) $\text{dcl}(Ce) \cap K \subseteq \text{acl}(C)$
- (3) For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma_C(e)$, $e \in \text{dcl}(C, \beta_0, \beta_1, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$.

Proof. Note that (1) implies (2) trivially. Let us now prove that (2) implies (1). Let $d \in \text{acl}(Ce)$ be a field element. The finite set of conjugates of d over Ce is coded by a tuple d' of field elements and $d' \in \text{dcl}(Ce)$. By (2), $d' \in \text{acl}(C)$. Since $d \in \text{acl}(d')$ we have $d \in \text{acl}(C)$.

A non-empty ball in K cannot be a definable image of $(\beta\mathcal{O}/\alpha\mathfrak{M})^n$, since in some models of ACVF, $\mathcal{O}/\alpha\mathfrak{M}$ is countable while every ball is uncountable. Hence any such definable image is finite. Thus (3) implies (2).

Finally, let us prove that (2) implies (3): By [HHM06, Theorem 1.0.2], and using (2), there exists an e -definable \mathcal{O} -submodule Λ of K^m (for some m), such that e is a canonical parameter for Λ , over $\text{acl}(C)$. The K -vector space $V = K \otimes_{\mathcal{O}} \Lambda$ is coded by an element w of some Grassmanian $G_{m,l}$; by (2) we have $w \in K_0 := K \cap \text{acl}(C)$; since K_0 is an algebraically closed field, V is K_0 -isomorphic to K^l , so we may assume $V = K^l$. Dually, let $V' = \{v \in V : K \cdot v \subseteq \Lambda\}$. Then V' is a $K \cap \text{acl}(Ce) = K_0$ -definable K -vector subspace of V . Replacing Λ by the image in V/V' , we may assume $V' = (0)$. It follows, by [HHM06, Lemma 2.2.4], that $V' \subseteq \beta_0\mathcal{O}^l$ for some $\beta_0 < 0$ that we may choose in $\Gamma_C(e)$.

Let v_1, \dots, v_l be a standard basis for $V = K^l$. Since $v_i \in K \otimes_{\mathcal{O}} \Lambda$, we have $c_i v_i \in \Lambda$ for some $c_i \in \mathcal{O}$. So $\sum r_i c_i v_i \in \Lambda$ for all $r_1, \dots, r_l \in \mathcal{O}$. Let $\Lambda'(\beta) = \{\sum r_i v_i : \text{val}(r_i) > \beta\}$. Then $\Lambda'(\beta) \subseteq \Lambda$ for sufficiently large β ; namely for $\beta \geq \max_i \text{val}(c_i)$. The set $\{\beta : \Lambda'(\beta) \subseteq \Lambda\}$ is definable, hence it contains $\{\beta : \beta > \beta_1\}$ for some Ce -definable β_1 . It follows that Λ is determined by its image D in $(\beta_0\mathcal{O}/\beta_1\mathfrak{M})^l$. Pick $c \in \beta_0\mathcal{O}/\beta_1\mathfrak{M}$; then $r \mapsto r \cdot c$ is an isomorphism $\mathcal{O}/(\beta_1 - \beta_0)\mathfrak{M} \rightarrow b_0\mathcal{O}/b_1\mathfrak{M}$. By [HT06], $\mathcal{O}/(\beta_1 - \beta_0)\mathfrak{M}$ is stably embedded; hence so is $\beta_0\mathcal{O}/\beta_1\mathfrak{M}$. Thus D is $\beta_0\mathcal{O}/\beta_1\mathfrak{M}$ -definable and e is in $\text{dcl}(C, \beta_0, \beta_1, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$. \square

Remark 2.34. As we can see from the proof above, every geometric sort other than K is purely imaginary.

- Definition 2.35.**
- (1) An element $e \in M$ is said to be *boundedly imaginary* over $C \subseteq M$ if for any $\gamma \in \Gamma_C(e)$, $\text{tp}(\gamma/C)$ is bounded, i.e. it neither the type at $+\infty$ nor the one at $-\infty$.
 - (2) An ∞ -definable set D is *boundedly imaginary* if there exists no definable map (with parameters) from D onto an unbounded subset of Γ .

Note that D is boundedly imaginary if and only if any $e \in D$ is boundedly imaginary over any C over which D is defined.

Lemma 2.36. *Any definable function $f : (\alpha\mathcal{O}/\beta\mathfrak{M})^n \rightarrow \Gamma$ is bounded.*

Proof. Suppose first $n = 1$. Since parameters are allowed, we may assume $\alpha = 0$, and consider $f : \mathcal{O}/\beta\mathfrak{M} \rightarrow \Gamma$, defined over C . Let q be the type of elements of Γ greater than any element of $\Gamma(C)$. For $\gamma = q$, let $X(\gamma)$ be the pullback to \mathcal{O} of $f^{-1}(\gamma)$. This is a finite Boolean combination of balls of valuative radii $\delta_1(\gamma), \dots, \delta_m(\gamma)$, with $0 \leq \delta_i(\gamma) \leq \beta$. But any C -definable function into a bounded interval in Γ is constant on q . Thus $X(\gamma)$ is a finite Boolean combination of balls of constant valuative radii $\delta_1, \dots, \delta_m$. However, it is shown in [HHM06, Proposition 2.4.4] that any definable function on a finite cover of Γ into balls of constant radius has finite image. Hence

$X(\gamma)$ is constant on q . But if $X(\gamma) = X(\gamma') \neq \emptyset$ then for any $x \in X(\gamma)$, $\gamma = f(x + \beta\mathfrak{M}) = \gamma'$. It follows that $X(\gamma) = \emptyset$ for $\gamma \neq q$, i.e. f is bounded. Now, given $f : (\alpha\mathcal{O}/\beta\mathfrak{M})^2 \rightarrow \Gamma$, let $F(x) = \sup\{f(x, y) : y \in \alpha\mathcal{O}/\beta\mathfrak{M}\}$. Then F is bounded, so f is bounded. This shows the case $n = 2$, and the general case is similar. \square

Corollary 2.37. *Let D be a C -definable set in ACVF. Then the following are equivalent:*

- (1) D is boundedly imaginary.
- (2) There exists a definable surjective map $g : (\mathcal{O}/\beta\mathfrak{M})^n \rightarrow D$.
- (3) For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$, for any $e \in D$, $e \in \text{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$.

Note that infinite definable subsets of K contain balls which are not boundedly imaginary. It follows that boundedly imaginary definable sets are purely imaginary.

Proof. by Lemma 2.36, (1) follows from (2) and (3) implies (2) easily. To prove that (1) implies (3), by compactness it suffices to fix $e \in D$ and show that $e \in \text{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$ for some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$. Since e is purely imaginary over C , by Lemma 2.33.(3), there exists such $\beta_0, \beta_1 \in \Gamma_C(e)$. But $\beta_i = f_i(e)$ for some C -definable functions f_i ; by (1), f_i is bounded on D , and the upper and lower bounds are C -definable. \square

It follows, by compactness, that any ∞ -definable boundedly imaginary D is contained in a definable boundedly imaginary D' .

Let us conclude that section with a characterization of independence of stably dominated types of field elements in ACVF in terms of a *maximum modulus* principle.

Proposition 2.38 (Maximum Modulus, [HHM08, Theorem 14.12]). *Let p be a stably dominated C -definable type concentrating on an affine variety V defined over $K \cap C$. Let $P = p|_C$ and F be a regular function on V over $L \supseteq C$. Then $\text{val}(F)$ has an infimum $\gamma_{\min}^F \in \Gamma(L)$ on P . Moreover for $a \models P$, $a \models p|_L$ if and only if $\text{val}(F(a)) = \gamma_{\min}^F$ for all such F .*

Corollary 2.39 ([HHM08, Theorem 14.13]). *Let U and V be varieties over the algebraically closed valued field C . Let p and q be a stably dominated types over C of elements of U and V respectively. Let F be a regular function on $U \times V$. Then there exists $\gamma_F \in \Gamma$ such that:*

- (1) If $(a, b) \models p \otimes q$ then $\text{val}(F(a, b)) = \gamma_F$.
- (2) For any $a \models p$, $b \models q$, we have $\text{val}(F(a, b)) \geq \gamma_F$.
- (3) Assume U and V are affine. If $a \models p$ and $b \models q$ and $\text{val}(F(a, b)) = \gamma_F$ for all regular F on $U \times V$, then $(a, b) \models p \otimes q$.

Proof. Since V admits a finite cover by open affines, and q concentrates on one of these affines, we may assume V is affine. Let $a \models p$. Then the statement follows from Proposition 2.38 applied to q over Ca . As p is stably dominated, the value of γ_F does not depend on a . \square

3. GROUPS WITH DEFINABLE GENERICS

A pro-definable group is a pro-definable set with a pro-definable group law. We will, in a few cases, need to consider a larger class of groups. A *piecewise pro-definable group* is a piecewise pro-definable set $G = \varprojlim_i G_i$ together with pro-definable maps $m_i : G_i \times G_i \rightarrow G_j$, for some $j \geq i$, compatible with the inclusions, and inducing a group structure $m : G \times G \rightarrow G$. By a pro-definable subgroup H of G we mean a pro-definable subset $H \subset G_i$ for some definable piece G_i of G , such that $m(H^2) \subset H$ and (H, m) is a subgroup of (G, m) .

Let us now fix G a pro-definable group.

3.1. Definable generics. For all filters π concentrating on G and $g \in G$, let ${}^g\pi := \{\phi(x, a) : \pi(x) \vdash \phi(g \cdot x, a)\}$. Similarly, we define $\pi^g := \{\phi(x, a) : \pi(x) \vdash \phi(x \cdot g, a)\}$. Note that if π is C -definable, then ${}^g\pi$ and π^g are Cg -definable.

Definition 3.1. Let π be a filter concentrating on G . We say that π is right generic in G over C if, for all $g \in G$ ${}^g\pi$ is C -definable. We say that it is left generic over C if, for all $g \in G$, π^g is C -definable.

We say that G admits a generic filter if there exists a filter π concentrating on G which is left or right generic in G (over some small set of parameters).

Lemma 3.2. *Let π be a definable filter concentrating on G then π is right generic if and only if π has boundedly many left translations by G . Moreover if π is right generic then for every formula ϕ , the set $\{({}^g\pi)|_\phi : g \in G\}$ is finite.*

Proof. If π is right generic, then its orbit must be bounded since there are only boundedly many C -definable filters. Conversely, if the orbit of π is bounded we can find a small C such that every translate is C -definable.

Let us now assume that π is right generic and ϕ be some formula. Then the equivalence relation $g_1 \sim g_2$ if $({}^{g_1}\pi)|_\phi = ({}^{g_2}\pi)|_\phi$ is definable by $(\forall y)(d_\pi x)\phi(g_1 \cdot x, y) \leftrightarrow (d_\pi x)\phi(g_2 \cdot x, y)$. Since this equivalence relation has bounded many class, it must have finitely many. \square

Remark 3.3. Let π be a right generic filter of G . Then it follows easily for the above that $\mu := \bigcap_{g \in G} {}^g\pi$ is a definable filter concentrating on G . It is obviously invariant under left translation by G .

Proposition 3.4. *Assume G is pro- C -definable and admits a generic filter over C , then G is pro- C -definably isomorphic to a pro-limit of C -definable groups. In particular, if G is ∞ - C -definable, G is C -definably isomorphic to $\bigcap_i G_i$ where the G_i are all C -definable subgroups of some C -definable group H .*

Proof. By remark 3.3, there is a C -definable G -invariant filter π concentrating on G . Let us first assume that $G = \bigcap_i X_i$ is ∞ -definable. By compactness, there exists $i_0 \in I$ such that if x, y and $z \in X_{i_0}$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot 1 = 1 \cdot x = x$. Because I is filtered, we may assume that i_0 the smallest element of I . Let $Y_i = \{a \in X_{i_0} : \pi(x) \vdash a \cdot x \in X_i\}$, which is C -definable.

Claim 3.4.1. $G = \bigcap_{i \in I} Y_i$.

Proof. Let $a \in G$ and $c \vDash \pi$. Then $a \cdot c \in G \subseteq X_i$ and $a \in Y_i$. Conversely, let $a \in \bigcap_i Y_i$. Then $\pi(x) \vdash a \cdot x \in \bigcap_i X_i = G$. Let $c \vDash \pi$, we have $b := a \cdot c \in G$ and hence $a = bc^{-1} \in G$. \square

Let i_1 be such that $Y_{i_1} \cdot Y_{i_1} \subseteq X_{i_0}$ and $S_i := \{a \in Y_i : Y_i \cdot a \subseteq Y_i\}$, which is C -definable.

Claim 3.4.2. *For all $i \geq i_1$, (S_i, \cdot) is a monoid containing G .*

Proof. Pick any a and $c \in S_i$. For all $x \in Y_i$, $x \cdot a \in Y_i$ and $x \cdot c \in Y_i$, hence $x \cdot a \cdot c \in Y_i$ and, since $a \in Y_i$, $a \cdot c \in Y_i$. Since we obviously have $1 \in S_i$, S_i is a monoid. Now, let $a \in G \subseteq Y_{i_1}$ and $c \in Y_i \subseteq Y_{i_1}$. We have $c \cdot a \in X_{i_0}$. Moreover, $\pi(x) \vdash c \cdot x \in X_i$ and, therefore, ${}^a\pi = \pi \vdash c \cdot a \cdot x \in X_i$. It follows that $c \cdot a \in Y_i$ and $a \in S_i$. \square

Then $G = \bigcap_{i \geq i_1} H_i$, where H_i is the group of invertible elements in S_i .

Let us now assume that $G = \varprojlim_{i \in I} X_i$. For all $i \in I$, let $G_i = \{a \in G : \pi^{\otimes 2}(x, y) \vdash \rho_i(x^{-1} \cdot a \cdot y) = \rho_i(x^{-1} \cdot y) = \rho_i(x^{-1} \cdot a^{-1} \cdot y)\}$, where $\rho_i : G \rightarrow X_i$ is the projection.

Claim 3.4.3. $G_i \trianglelefteq G$.

Proof. By definition, if $a \in G_i$, then $a^{-1} \in G_i$. Let now $a, c \in G_i$. For all $(x, y) \models \pi^{\otimes 2} | Cac$, $c \cdot y \models {}^c \pi | Cacx = \pi | Cacx$. It follows that $(x^{-1} \cdot a \cdot (c \cdot y))_i = (x^{-1} \cdot (c \cdot y))_i = (x \cdot y)_i$. Similarly $(x^{-1} \cdot (a \cdot c)^{-1} \cdot y)_i = ((c \cdot x)^{-1} \cdot a^{-1} \cdot y)_i = (x \cdot y)_i$. Finally, if $a \in G_i, c \in G$ and $(x, y) \models \pi^{\otimes 2} | Cac$, then $(c \cdot x, c \cdot y) \models ({}^c \pi)^{\otimes 2} | Cac = \pi^{\otimes 2} | Cac$. It follows that $(x^{-1} \cdot c^{-1} \cdot a \cdot c \cdot y)_i = ((c \cdot x)^{-1} \cdot a \cdot (c \cdot y))_i = ((c \cdot x)^{-1} \cdot (c \cdot y))_i = (x \cdot y)_i$. \square

Since G_i is relatively C -definable, $H_i := G/G_i$ is an ∞ - C -definable group.

Claim 3.4.4. *The natural map $G \rightarrow H := \varprojlim H_i$ is a group isomorphism.*

Proof. Let $a \in \bigcap_i G_i$ and $(x, y) \models \pi^{\otimes 2} | Ca$. Then for all $i \in I$, $(x^{-1} \cdot a \cdot y)_i = (x^{-1} \cdot y)_i$ and thus $x^{-1} \cdot a \cdot y = x^{-1} \cdot y$. It follows that $a = 1$. Surjectivity follows, by compactness, from the fact that each map $G \rightarrow H_i$ is surjective. \square

The image of π in H_i is generic, so we can conclude by the ∞ -definable case. \square

Similar results hold for rings:

Proposition 3.5. *Let R be an pro- C -definable ring. Assume that $(R, +)$ admits a generic filter over C and that there exists a C -definable filter π concentrating on R^* such that the stabilizer of π under (left) multiplication generates R . Then R is pro- C -definably isomorphic to a pro-limit of C -definable rings. In particular, if R is ∞ - C -definable, R is C -definably isomorphic to an intersection of C -definable subrings of some C -definable ring.*

The above hypothesis on R hold in particular if both the additive group and the units of R admit a generic filter over C and that the units generate R .

Proof. Let us first assume $R = \bigcap_{i \in I} P_i$ is ∞ -definable. By Proposition 3.4, we may assume that the P_i are subgroups of some group $(P, +)$ and by compactness we may assume that multiplication is an associative bilinear map on P that 1 is the identity. For all $i \in I$, let $Q_i = \{a \in P : \pi(x) \vdash a \cdot x \in P_i\}$ which is C -definable. It is easily checked that Q_i is a subgroup of P . Moreover, the multiplicative stabilizer of π is a subset of Q_i which therefore contains R . Conversely, if $a \in \bigcap_i Q_i$, then for any $c \models \pi | Ca$, $a \cdot c \in \bigcap_i P_i = R$ and hence $a \in Rc^{-1} = R$. By compactness, there exists i_0 such that $Q_{i_0} \cdot Q_{i_0} \subseteq P$. Let $R_i = \{a \in Q_i : Q_i \cdot a \subseteq Q_i\}$. One can check that, for all $i \geq i_0$, S_i is a ring, and proceeding as in Claim 3.4.2, we can show that if a stabilizes π multiplicatively, and $c \in Q_i$, then $c \cdot a \in Q_i$. So S_i contains R and $R = \bigcap_{i \geq i_0} S_i$.

Now, let $R = \varprojlim_{i \in I} P_i$. By, Proposition 3.4, we may assume that each P_i is an additive group. For all $i \in I$, let $J_i := \{a \in R : \pi^{\otimes 2}(x, y) \vdash (x^{-1} \cdot a \cdot y)_i = 0_i\}$, which is relatively C -definable. It is clear that J_i is an additive subgroup. Now pick any $a \in J_i$ and let $S_a := \{c \in R : c \cdot a \in J_i\}$. Then S_a is additive subgroup. Moreover, if c stabilizes π , then so does c^{-1} and hence for any $(x, y) \models \pi^{\otimes 2}$, $(c^{-1} \cdot x, y) \models \pi^{\otimes 2}$. Since $a \in J_i$, it follows that $(x^{-1} \cdot c \cdot a \cdot y)_i = ((c^{-1} \cdot x)^{-1} \cdot a \cdot y)_i = 0_i$, so $c \in S_a$ and hence $S_a = R$. Similarly, we show that J_i is a two sided ideal. Bijectivity of the ring homomorphism $R \rightarrow \varprojlim_i R/J_i$ follows as in the group case and, since R/J_i is an ∞ - C -definable ring, we conclude by the ∞ -definable case. \square

3.2. Stabilizers. The stabilizer of a definable filter can be viewed as an adjoint notion for the generic of a group. In this section, we will assume that G is a pro- \emptyset -definable group.

Let $\Delta(x)$ be a set of formulas $\phi(x, y)$ where x ranges over G , which is preserved by G (on the left): for all $\phi(x, m)$ instance of Δ and $g \in G$, $\phi(g \cdot x, m)$ is equivalent to an instance of Δ .

Definition 3.6. Let π be a definable filter concentrating on G . We define

$$\text{Stab}_\Delta(\pi) := \{g \in G : {}^g \pi|_\Delta = \pi|_\Delta\}.$$

If Δ is the set of all formulas, we write $\text{Stab}(\pi)$ for $\text{Stab}_\Delta(\pi)$.

Note that since Δ is preserved by G , G acts on restrictions of filters to instance of Δ . It follows that $\text{Stab}_\Delta(\pi)$ is a pro-definable subgroup of G .

Remark 3.7. Assume G is a pro-limit of \emptyset -definable groups. If $\phi(x, y)$ is any formula where x ranges over G , then $\phi'(x, y, t) = \phi(t \cdot x, y)$ where t also ranges over G is preserved by G . It follows that any finite set of formulas is contained in a finite set Δ preserved by G . For such a Δ , $\text{Stab}_\Delta(\pi)$ is a relatively definable subgroup of G .

For infinite Δ , it follows that $\text{Stab}_\Delta(\pi) = \bigcap_{\phi \in \Delta} \text{Stab}_{\phi'}(\pi)$ is an intersection of relatively definable subgroups of G , i.e. a pro-limit of definable groups.

Remark 3.8. If π concentrates on $\text{Stab}(\pi)$, then π is generic in $\text{Stab}(\pi)$.

Remark 3.9. If p is a generic type, let $G_p^0 := \bigcap_{g \in G} \text{Stab}(^g p)$. Since they are boundedly many translates, G_p^0 has bounded index in G . So p concentrates on a coset of G_p^0 and p is a translate of a generic type of G_p^0 —cf. Lemma 3.11, in the case p is symmetric.

If π and μ are definable filters, we also define $\text{Stab}(\pi, \mu) := \{g \in G : ^g \pi = \mu\}$. It might be empty, but when it is non-empty, it is a left torsor of $\text{Stab}(\mu)$ and a right torsor of $\text{Stab}(\pi)$. In particular, these two groups are then conjugate (by any element in $\text{Stab}(\pi, \mu)$).

We will also consider the following more general variant of the stabilizer.

Definition 3.10. Let π be a definable filter. Let $\mathfrak{F}(\pi)$ be the piecewise pro-definable semigroup under composition of \mathbb{U} -definable functions h such that $h_* \pi = \pi$. This semigroup has an quotient consisting of the π -germs of elements of $\mathfrak{F}(\pi)$. The invertible germs form an piecewise pro-definable group, denoted $\mathfrak{G}(p)$.

If π is a filter concentrating on a pro-definable group G , then the stabilizer $\text{Stab}(\pi)$ embeds naturally into $\mathfrak{G}(p)$: c maps to the germ of left translation by c .

3.3. Symmetric generics. Let p be a definable type. We say that p is symmetric if for all definable type q and C such that p and q are defined over C , $a \models p|_{\text{acl}(C)}$ and $b \models q|_{\text{acl}(Ca)}$, then $a \models p|_{\text{acl}(Cb)}$. Proposition 2.10.(1) exactly states that stably dominated types are symmetric.

Lemma 3.11. *Assume G admits a symmetric right generic type, then left and right generics coincide, they are all symmetric and there is a unique left (respectively right) orbit of generics. In particular, there are only boundedly many generics in G .*

Moreover, there is a smallest pro-definable subgroup $G^0 \leq G$ of bounded index and G/G^0 is pro-finite. In fact, G^0 is the left and right stabilizer of any generic of G .

Proof. Let p be a symmetric right generic. Then p^{-1} is a symmetric left generic. Let q be any left generic and C be a model over which p and q are defined, $a \models p|_C$ and $b \models q|_{\text{acl}(Ca)}$. By definition $ba \models q^a|_{\text{acl}(Ca)}$. Since p is symmetric, $a \models p|_{\text{acl}(Cb)}$ and hence $ba \models {}^b p|_{\text{acl}(Cb)}$. So q^a and ${}^b p$ agree on C but, since both are C -definable and C is a model, $q^a = {}^b p$. Since right and left translation commute, it follows that p is left generic. Moreover if q_1 and q_2 are both left generic, then, by the previous argument, we find b_1 and b_2 and a such that $q_1^a = {}^{b_1} p$ and $q_2^a = {}^{b_2} p$. It follows that ${}^{b_2 b_1^{-1}} q_1 = q_2$ and all left generic are left translates of one another. Since p is a symmetric left generic, all left generic are symmetric.

By a similar argument proof starting with any (symmetric) left generic, we get that all left generic are right generic, that all right generic are right translates of each other and that all right generic are symmetric (and hence left generic). We have proved that left and right genericity coincide and that they are all left and right translates of each other.

The group $G^0 := \text{Stab}(p) = \text{Stab}(p^g)$ does not depend on the choice of generic p . Moreover G^0 has bounded index in G since p has boundedly many left translates. Since G^0 has bounded

index, it follows that p concentrates on a coset of G^0 and, translating, we may assume that p concentrates on G^0 . Since G^0 stabilizes p , it cannot contain any pro-definable subgroup of bounded index. Furthermore, for all formulas ϕ preserved by G —which exist since G is a pro-limit of definable groups, G_ϕ^0 has finite index in G and $G/G^0 = \varprojlim_{\phi} G/G_\phi^0$ is pro-finite. \square

The unique generic type concentrating on G^0 is called the principal generic. If $G = G^0$, we say that G is connected.

Remark 3.12. Assume G admits a symmetric generic type p . Then the following are equivalent:

- (1) p is the unique generic type of G .
- (2) For all $g \in G$, ${}^g p = p$.
- (3) G is connected.

Proof. Assume (1). By definition of genericity, for any $g \in G$, ${}^g p$ is right generic. Hence by uniqueness, ${}^g p = p$. Conversely given (2), let q be generic. Then by Lemma 3.11, $q = {}^g p$ for some g . By (2), $p = q$. The equivalence with (3) is immediate, given that G^0 is the left stabilizer of any generic. \square

Lemma 3.13. *Assume G is pro- C -definable and admits a symmetric generic type. Then G^0 is pro- C_0 -definable for some $C_0 \subseteq C$ of size at most $|\mathcal{L}| + |x|$ where x ranges over G .*

Proof. Let p be the principal generic of G . Since p is fixed by all automorphism fixing G globally, it is C -definable. Moreover, it is determined by the function $\phi(x, y) \mapsto (d_p x)\phi(x, y)$ and there are at most $|\mathcal{L}| + |x|$ formulas ϕ to consider. So p is C_0 -definable for some $C_0 \subseteq C$ of size at most $|\mathcal{L}| + |x|$ and hence so is $G^0 = \text{Stab}(p)$. \square

3.4. Group chunks. This idea, in the context of algebraic groups, is due to Weil. Let G be a definable group, and π a left G -invariant definable filter concentrating on G . The $\pi \otimes \pi$ -germ of multiplication is called the *group chunk* corresponding to (G, π) .

Definition 3.14. An abstract group chunk over C is a C -definable filter π and a pro- C -definable map F defined on $\pi^{\otimes 2}$ (or $\pi \otimes \pi$ -germ of such a function) such that:

- (1) For all $a \vDash \pi|C$, $(F_a)_* \pi = \pi$, where $F_a(x) = F(a, x)$.
- (2) For all $a, b \vDash \pi^{\otimes 2}|C$, $a \in \text{dcl}(C, b, F(a, b))$ and $b \in \text{dcl}(C, a, F(a, b))$.
- (3) $\pi^{\otimes 3}(x, y, z) \vdash F(x, F(y, z)) = F(F(x, y), z)$.

Proposition 3.15. *Let (π, F) be an abstract group chunk over C . Then (π, F) is pro- C -definably isomorphic to the group chunk of a pro- C -definable group G : there exists an injective pro- C -definable map $f : \pi \rightarrow G$ such that $\pi^{\otimes 2}(x, y) \vdash f(F(x, y)) = f(x) \cdot f(y)$ and $G = \text{Stab}(f_* \pi)$.*

Proof. Let $P = \pi|C$. For any $a \in P$, by (2), the π -germ $f(a)$ of $F(a, x)$ is invertible. It is therefore an element of the group $\mathfrak{G}(\pi)$ (cf. Definition 3.10). By (2) also, the map f is injective. Let G be the subgroup of $\mathfrak{G}(\pi)$ generated by the elements $(f(a))_{a \in P}$ and their inverses. Let I enumerate all finite sets of the variables of π . For all $i \in I$, and $g \in \mathfrak{G}(\pi)$, let g_i denote the π -germ of $\rho_i \circ h$, where h is any map whose π -germ is g and ρ_i is the projection to the variables in i .

To show that G can be identified with a pro-definable group, it suffices to check that:

- Any element of G is a product $f(a) \cdot f(b)^{-1}$ for some $a, b \in P$.
- The set $\{(a_1, \dots, a_6) \in P^6 : (f(a_1) \cdot f(a_2)^{-1} \cdot f(a_3) \cdot f(a_4)^{-1})_i = (f(a_5) \cdot f(a_6)^{-1})_i\}$ is relatively definable.

The second point is immediate from the definability of π . As for the first point, it suffices to show that a product $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1}$ has the required form. Note that, by (1), when $a \in P$ and $b \vDash \pi|Da$ for any $D \supseteq C$, $F(a, b) \vDash \pi|Da$ and there exists $c \vDash \pi|Da$ such that $F(a, c) = b$ and, by (3), $f(a)^{-1} \cdot f(b) = f(c)$. Thus given any $a, b, c, d \in P$, let $e_0 \vDash \pi|Cabed$; then $f(d)^{-1} \cdot f(e_0) = f(e_1)$ for some $e_1 \vDash \pi|Cabed$; so $f(c) \cdot f(e_1) = f(e_2)$ for some $e_2 \vDash \pi|Cabed$; continuing this way, we obtain that $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1} \cdot f(e_0) = f(e_4)$, for some $e_4 \vDash \pi|Cabed$. Therefore, $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1} = f(e_4) \cdot f(e_0)^{-1}$ as required.

Finally, if $a \in P$ and $b \vDash \pi|Da$ for some $D \supseteq C$, $f(a) \cdot f(b) = f(F(a, b))$ where $F(a, b) \vDash \pi|Da$ so $f^{(a)} f_* \pi = f_* \pi$ and we do have $G = \text{Stab}(f_* \pi)$. This concludes the construction of G and the proof of its various properties. \square

The uniqueness of G is Proposition 3.15 is guaranteed by the following:

Proposition 3.16. *Let G_1 and G_2 be pro- C -definable groups, π be left G_1 -invariant C -definable filter concentrating on G_1 and $f : \pi \rightarrow G_2$ be a pro- C -definable map such that $\pi^{\otimes 2}(x, y) \vdash f(x \cdot y) = f(x) \cdot f(y)$. Then there exists a unique pro- C -definable homomorphism $g : G_1 \rightarrow G_2$ such that $\pi(x) \vdash f(x) = g(x)$.*

Proof. Uniqueness of g is clear, since $\pi|C$ generates G : if $g \in G$, $a \vDash \pi|Cg$, then $g \cdot a \vDash \pi|Cg$ and $g = (g \cdot a) \cdot a^{-1}$.

Existence is also clear, provided we show that, for all $a, b, c, d \vDash \pi|C$ such that $a \cdot b^{-1} = c \cdot d^{-1}$, $f(a) \cdot f(b)^{-1} = f(c) \cdot f(d)^{-1}$. It suffices to show that $f(a) \cdot f(b)^{-1} \cdot f(e) = f(c) \cdot f(d)^{-1} \cdot f(e)$ for any $e \vDash \pi|Cabed$. But it follows from our hypothesis on f that $f(b)^{-1} \cdot f(e) = f(b^{-1} \cdot e)$ and $f(d)^{-1} \cdot f(e) = f(d^{-1} \cdot e)$. Moreover, $b^{-1} \cdot e \vDash \pi|Cabed$, so $f(a) \cdot f(b^{-1} \cdot e) = f(a \cdot b^{-1} \cdot e)$. Similarly, $f(c) \cdot f(d^{-1} \cdot e) = f(c \cdot d^{-1} \cdot e)$. Since $a \cdot b^{-1} \cdot e = c \cdot d^{-1} \cdot e$, the equality holds.

Since $g(a) = b$ if and only if $(d_p x) f(a \cdot x) = b \cdot f(x)$, g is pro- C -definable. \square

Analogous statements for group actions exist.

3.5. Products of types in groups. Let G be a piecewise pro-definable group, let p_1, \dots, p_n be definable types of elements of G , and let w be an element of the free group F on generators $\{1, \dots, n\}$. The words of the free group will be denoted by expressions such as $1\bar{2}3$; $\bar{2}$ is the inverse of the generator 2.

We construct a definable type $p_w = w_*(p_1, \dots, p_n)$. Let $a_i \vDash p_i|acl(Ca_1 \dots a_{i-1})$. Let $a_w = w(a_1, \dots, a_n)$ be the image of w under the homomorphism $F \rightarrow H$ with $i \mapsto a_i$. Let $p_w|C = \text{tp}(a_w/C)$.

If w is the product of the generators 1, 2, we write $p_1 \star p_2$ for p_w . We denote $p_{123\dots n}$ as $\otimes_{i=1}^n p_i$. Note that if G is Abelian and p is symmetric, then the order of enumeration does not actually matter. If a single type p is given, rather than a sequence p_w will refer to the sequence (p, p, \dots, p) . Let p^{*n} denote $p_{123\dots n}$, $p^{\pm 2n} = p_{1\bar{2}3\bar{4}\dots(2n)}$ and $p^{\pm 2n+1} = p_{1\bar{2}3\bar{4}\dots(2n)(2n+1)}$.

4. STABLY DOMINATED GROUPS

In this section, T is assumed to be metastable.

Definition 4.1. A pro-definable group G is *stably dominated* if G has a stably dominated generic type.

Since stably dominated types are symmetric, it follows from Lemma 3.11 that all generics of G are both left and right generics and they are all stably dominated.

Remark 4.2. The class of stably dominated pro-definable groups is closed under Cartesian products and image under a definable group homomorphism. If G and H are stably dominated

pro-definable groups and p (respectively q) is a stably dominated generic of G (respectively H), then $p \otimes q$ is a stably dominated generic of $G \times H$. If $f : G \rightarrow H$ is a pro-definable surjective group homomorphism, then f_*p is a stably dominated generic of H .

Lemma 4.3. *Let G be a pro-definable group and $N \leq G$ be a stably dominated pro-definable subgroup. Assume that there exists a stably dominated type concentrating on G/N whose orbit under G -translations is bounded. Then G is stably dominated.*

In particular, if $N \trianglelefteq G$ and $N, G/N$ are stably dominated, then so is G .

Proof. Let p be the generic type of N^0 and q be a stably dominated type of G/N , with bounded orbit under G -translations. Let C be such that G, N are defined over C and p, q are stably dominated over C .

Assume first that $N = N^0$. For all $n \in N$, ${}^n p = p$. Thus, if $c = dn$ for some $n \in N$, then ${}^c p = {}^{dn} p = {}^d p$. So the type ${}^c p$ depends only on the coset $s = cN$. We denote it p_s . Let us now define the type r as follows: a realization of r over $B \supseteq C$ is a realization of p_s over Bs where $s \models q|B$.

Since p is stably dominated over C , by descent (Proposition 2.11.(2)), p_s is stably dominated over Cs . It follows, by transitivity (Proposition 2.10.(3)), that r is stably dominated over C . By construction, r has a bounded orbit under G -translations. That concludes the proof in the case $N = N^0$.

In general, the map $\pi : G/N^0 \rightarrow G/N$ has pro-finite fibers and hence for any $s \in G/N^0$, $s \in \text{acl}(C\pi(s))$. It follows, by Proposition 2.10.(4), that any q_0 such that $\pi_* q_0 = q$ is stably dominated over C . Fix such a q_0 , then q_0 has bounded orbit under G -translations and, by the above, G is stably dominated. \square

Lemma 4.4. *Let G be a stably dominated pro-definable group and $N \leq G$ be a pro-definable subgroup. Let $\eta : G \rightarrow G/N$ be the map $\eta(g) = g \cdot N$. Assume that there exists a pro-definable $Y \subset G$ such that $\eta|_Y$ is surjective, bounded to one. Then N is stably dominated.*

Proof. Let p be the principal generic of G , C be such that G, N, Y, p are C -definable. Let Q be the set of types $\text{tp}(h^{-1} \cdot g/C)$ where $g \models p|C$, $h \in Y$ and $h \cdot N = g \cdot N$. Because there are boundedly many choices for h , the set Q is bounded. Moreover, for all $g \models p|C$, $h^{-1} \cdot g \in \text{acl}(Cg)$ and hence, by Proposition 2.10.(4), the types in Q are all stably dominated types concentrating on N .

Let us now show that Q is $N \cap G^0$ -invariant. Pick $n \in N$, $g \models p|Cn$, $h \in Y$ such that $h \cdot N = g \cdot N$ and $q := \text{tp}(h^{-1} \cdot g/C) \in Q$. We have that $h^{-1} \cdot g \cdot n \models q^n|Cn$, but since $g \cdot yn \models p^n = p$ and $g \cdot n \cdot N = g \cdot N = h \cdot N$, $q^n|C = \text{tp}(h^{-1} \cdot g \cdot n/C) \in Q$. Since $N/N \cap G^0$ can be embedded in G/G^0 , which is pro-finite, it follows that the types in Q all have a bounded orbit under the action of N by translation. So they are all generic. \square

Corollary 4.5. *Let G be an algebraic group, $N \leq G$ an algebraic subgroup. Let $H \leq G$ be definable in ACVF and stably dominated. Then $H \cap N$ is stably dominated.*

Proof. By elimination of imaginaries in algebraically closed fields, the coset space $X := G/N$ can be identified with a subset of some Cartesian power of the field sort. Let η be the projection $G \rightarrow X$. Let $M_0 \models \text{ACVF}$ be such that G, N and H are defined over M_0 . For all $h \in H$, $M = M_0(\eta(h))^{\text{alg}} \models \text{ACVF}$, so there exists $y \in H(M) \cap h \cdot N$. By compactness, we find a definable subset Z of the graph of $\eta|_H$ with finite projection to X . Let $Y \subseteq H$ be the projection on H . Then, applying Lemma 4.4 to $H, H \cap N$ and Y , we get that $H \cap N$ is stably dominated. \square

4.1. Domination via a group homomorphism.

Proposition 4.6. *Let G be a stably dominated pro-definable group. There exists a pro-definable stable group \mathfrak{g} , and a pro-definable homomorphism $g : G \rightarrow \mathfrak{g}$, such that the generics of G are stably dominated via g .*

If (FD) holds and G is definable then \mathfrak{g} can be taken to be definable.

We say that such a G is stably dominated via g .

Proof. Let p be generic in G , C be such that G and p are C -definable. Let $\theta(a)$ enumerate $\text{St}_C(a)$, and, for all $a, b \in G$, let $f_a(b) = \theta(a \cdot b)$. Fix $g \in G$. Since ${}^g p$ is stably dominated, by Proposition 2.12, the ${}^g p$ -germ of f_a is strong and is in $\text{St}_C(a) = \theta(a)$. It follows that for all $b \models {}^g p|Ca$, $f_a(b) = h_{\theta(a)}(b)$ for some C -definable map h . Since St_C is stably embedded, h factors through $\theta(b)$. Indeed, let $c = f_a(b) = h_{\theta(a)}(b)$. Since $c \in \text{St}_C$, $\text{tp}(\theta(a)c/C\theta(b)) \vdash \text{tp}(\theta(a)c/Cb)$. Thus $c \in \text{dcl}(\theta(a)\theta(b))$. We have proved that, for all $g_1, g_2 \in G$, there exists a C -definable map F_{g_1, g_2} such that, for all $a \models {}^{g_1} p|C$, $b \models {}^{g_2} p|Ca$, $\theta(ab) = F_{g_1, g_2}(\theta(a), \theta(b))$. By compactness, we may assume that $F = F_{g_1, g_2}$ does not depend on g_1 or g_2 .

Recall that for all $g \in G$, ${}^g p$ is stably dominated over C . It follows that, for all $g_1, g_2 \in G$, $a \models {}^{g_1} p|C$, $b \models {}^{g_2} p|Ca$ and $c = a \cdot b$, $\text{tp}(\theta(b)\theta(c)/C\theta(a)) \vdash \text{tp}(bc/Ca)$ and hence $\theta(b) \in \text{dcl}(\theta(a), \theta(c)) = \text{dcl}(\theta(a), F(\theta(a), \theta(b)))$. Symmetrically, we have $\theta(a) \in \text{dcl}(\theta(b), F(\theta(a), \theta(b)))$.

Thus, F is a group chunk on $\pi := \bigcap_{g \in G} \theta_* {}^g p$ and, by Proposition 3.15, F is the restriction of the multiplication map of some pro-definable group \mathfrak{g} in St_C whose unique generic filter is $\theta_* \pi$. By Proposition 3.16, θ extends to a pro-definable group homomorphism $g : G \rightarrow \mathfrak{g}$.

Let us now assume (FD). By Proposition 3.4, $\mathfrak{g} := \varprojlim_{i \in I} \mathfrak{g}_i$ where the \mathfrak{g}_i form a (filtered) projective system of definable groups. Let $g_i : G \rightarrow \mathfrak{g}_i$ be the canonical projection. Since there are no descending chains of definable subgroups of \mathfrak{g}_i , every ∞ -definable subgroup is definable, in particular $g_i(G)$ is definable. Then $\mathfrak{g} = \varprojlim_{i \in I} g_i(G)$ and we may assume that every g_i is surjective.

Let a realize the principal generic of G over C . By Lemma 2.19, $\text{St}_C(a)$ is acl-finitely generated over C ; say $\text{St}_C(a) \subseteq \text{acl}(Cd)$ for some tuple $d \in \text{St}_C$. Since $\dim_{\text{st}}(d/C(g_i(a)))$ decreases as i increases, it stabilizes at some i_0 (Note that, if C is sufficiently large, i_0 does not depend on C). But all $g_i(a)$ are in $\text{acl}(Cd)$, and hence $g_i(a) \in \text{acl}(C(g_{i_0}(a)))$ for all $i \geq i_0$. It follows $\text{tp}(a/C)$ is dominated via g_{i_0} and so are all of its left translates. \square

Remark 4.7. If (FD) holds, we have also proved the following. If $g : H \rightarrow \mathfrak{g} = \varprojlim_i \mathfrak{g}_i$ is a pro-definable surjective group homomorphism and that $\dim_{\text{st}}(\mathfrak{g}) = n < \infty$, then there exists i_0 such that $\dim_{\text{st}}(\mathfrak{g}_{i_0}) = n$. Moreover, since there are no descending chains of definable subgroups of \mathfrak{g}_{i_0} , $g_{i_0}(G)$ is definable. So we can actually find a pro-definable surjective group homomorphism onto a *definable* stable group of Morley rank n .

In such a situation, we say that H has stable homomorphic image of Morley rank n .

The homomorphism g of Proposition 4.6 is not uniquely determined by G ; even if G is stable, there may be a non-trivial homomorphism of this kind. We do however have a maximal pro- C -definable homomorphism into stable pro-definable groups.

Remark 4.8. There exists a pro- C -definable stable group \mathfrak{g} , and a pro- C -definable homomorphism $g : G \rightarrow \mathfrak{g}$, maximal in the sense that any pro- C -definable homomorphism $g' : G \rightarrow \mathfrak{g}'$ into a pro- C -definable stable group factors through g .

The kernel of this maximal g is uniquely determined. If G is stably dominated it will be stably dominated via this maximal homomorphism.

Lemma 4.9. *Let G be a pro- C -definable group stably dominated via some pro- C -definable group homomorphism $g : G \rightarrow \mathfrak{g}$. Then $\text{tp}(a/C)$ is generic in G (i.e. $a \models r|\text{acl}(C)$ for some generic r of G) if and only if $\text{tp}(g(a)/C)$ is generic in \mathfrak{g} .*

Proof. Assume $\text{tp}(g(a)/C)$ is generic in \mathfrak{g} , then so is $\text{tp}(g(a)/\text{acl}(C))$. So we may assume $C = \text{acl}(C)$. Let p be a generic type of G , stably dominated via $g : G \rightarrow \mathfrak{g}$. Then $q = {}^a p$ is a generic in G and hence stably dominated via g . Let $b \models p|Ca$. Note that $a \cdot b \models q|Ca$. Since $g(a)$ is generic in \mathfrak{g} , we have $g(a \cdot b) = g(a) \cdot g(b) \downarrow_C g(b)$. Since $p|C$ is stably dominated via g , $b \models p|Cg(a \cdot b)$ and hence $\text{St}_C(b) \downarrow_C g(a \cdot b)$. Since $q|C$ is stably dominated via g , $a \cdot b \models q|Cb$. So $a \models q^{b^{-1}}|Cb$, and in particular $a \models q^{b^{-1}}|C$, where $q^{b^{-1}}$ is generic in G .

The converse is obvious since for any generic r of G , $g^{(x)}g_*r = g_*({}^x r)$. \square

Corollary 4.10. *Let G be a pro-definable group stably dominated via some pro-definable group homomorphism $g : G \rightarrow \mathfrak{g}$. Let $H \leq G$ be a relatively definable. If $g(H) = \mathfrak{g}$, then H has finite index in G .*

Proof. Work over some C over which G and H are definable. Let $b \in H$ be such that $g(b)$ is generic in \mathfrak{g} over C . By Lemma 4.9, $\text{tp}(b/C)$ is generic in G . Thus a generic of G lies in H , so H has finite index in G . \square

Corollary 4.11. *(FD_ω) Let G be a C -definable group stably dominated via some C -definable group homomorphism $g : G \rightarrow \mathfrak{g}$. Then G^0 is definable, i.e. $[G : G^0]$ is finite.*

Proof. Note that $g^{-1}(\mathfrak{g}^0)$ is a definable subgroup of G with finite index, so we may assume that \mathfrak{g} is connected. Since $[G : G^0]$ is bounded, we can find C such that every coset of G^0 in G contains a point from C . We can also assume that C is a metastability basis as in the definition of (FD_ω) . Let a_0 be such that $g(a_0)$ is generic in \mathfrak{g} over C . By (FD_ω) , there exists $a \in G$ such that $g(a) = g(a_0)$ and $\text{tp}(a/Cg(a))$ is isolated. Since $g(a)$ is generic in \mathfrak{g} over C , by Lemma 4.9, $\text{tp}(a/C)$ is generic in G . Let p be the principal generic of G . Then there exists $c \in G(C)$ such that $b := c \cdot a \models p|C$. It follows that $\text{tp}(c \cdot a/Cg(a)) = \text{tp}(b/Cg(b))$ is isolated.

On the other hand, let $G^0 = \bigcap_i G_i$ where $\{(G_i)_{i \in I}\}$ is a bounded family of definable subgroups of finite index, closed under intersections. Note that by Lemma 4.9, $\{x \in G_i : i \in I\} \cup \{g(x) = g(b)\}$ generates $\text{tp}(b/Cg(b))$. This type being isolated, there exists $i_0 \in I$ such that it is generated by $g(x) = g(b)$ and $x \in G_{i_0}$. Since \mathfrak{g} is connected, any generic of G_i has a realization d such that $g(d) = g(b)$ and hence is equal to p . So $G_i = G^0$, and G^0 is definable. \square

Here is a characterization of generics that does not explicitly mention g and \mathfrak{g} .

Corollary 4.12. *Let G be a stably dominated pro- C -definable group, $\dim_{\text{st}}(G) < \infty$. Then the generic types of G over C are precisely the types $\text{tp}(c/C)$ such that for h generic in G over Cc , $\dim_{\text{st}}(\text{St}_C(h \cdot c)/Ch) = \dim_{\text{st}}(G)$.*

Proof. Let g and \mathfrak{g} be as in Proposition 4.6. Since $h \cdot c$ is generic in G over C , $\text{acl}(\text{St}_C(h \cdot c)) = \text{acl}(Cg(h \cdot c)) = \text{acl}(Cg(h)g(c))$. So $\dim_{\text{st}}(G) = \dim_{\text{st}}(\text{St}_C(h \cdot c)/Ch) = \dim_{\text{st}}(g(h) \cdot g(c)/Ch) = \dim_{\text{st}}(g(c)/Ch) \leq \dim(g(c)/C)$. It follows that $g(c)$ is generic in \mathfrak{g} over C and, by Lemma 4.9, c is generic in G over C .

Conversely, if $c \models p|\text{acl}(C)$ for some stably dominated generic p , by symmetry, $c \models p|\text{acl}(Ch)$ and hence $h \cdot c$ is generic in G over Ch . Thus $g(h \cdot c)$ is generic in \mathfrak{g} and $\dim_{\text{st}}(\text{St}_C(h \cdot c)/Ch) \geq \dim_{\text{st}}(\mathfrak{g}) = \dim_{\text{st}}(G)$. \square

Note that the proof uses, rather than proves, the existence of a generic.

Remark 4.13. If G is piecewise-definable, in a superstable theory, and p is a type of maximal rank in G , then p is a translate of a generic of $\text{Stab}(p)$, a definable subgroup of G .

For stably dominated types, the analog need not hold. In ACVF, consider $G = \mathbb{G}_a(\mathcal{O})^2$, fix γ a positive element of the value group and let p be the stably dominated type of elements such that x is generic in $\mathbb{G}_a(\mathcal{O})$ and y is generic (over x) in the closed ball of radius γ centered at x^2 . Then $\dim_{\text{st}}(G) = 2$ and the stable part of p has dimension 2. But, although p has maximal rank, it is not a generic type. However, we do have that p^{*2} is a stably dominated generic of G .

Let us now prove a certain converse to Proposition 4.6:

Proposition 4.14. *Let G be a pro-definable group. Assume G admits a surjective pro-definable homomorphism $g : G \rightarrow \mathfrak{g}$, with \mathfrak{g} stable and with $\dim_{\text{st}}(\mathfrak{g}) = \dim_{\text{st}}(G) = n < \infty$. Then, there exists an pro-definable normal subgroup T of G with:*

- (1) T connected stably dominated;
- (2) G/T almost internal to Γ ;
- (3) $g(T) = \mathfrak{g}^0$;

and T is uniquely determined by (1,2) or by (1,3). Moreover, T is stably dominated via $g|_T$.

Proof. Let Z be the collection of types q of elements of G , definable over some set of parameters, with g_*q generic in \mathfrak{g} , and q stably dominated.

Claim 4.14.1. $Z \neq \emptyset$.

Proof. Let $C = \text{acl}(C)$ be a metastability basis over which G and g are pro-definable. Let $a \in G$ be such that $g(a)$ generic in \mathfrak{g} over $\text{acl}(C)$. Let $\gamma := \Gamma_C(a)$. Any relatively $\text{acl}(C\gamma)$ -definable subset of \mathfrak{g} is already relatively C -definable, so $g(a)$ is generic in \mathfrak{g} over $\text{acl}(C\gamma)$. By metastability, $\text{tp}(a/\text{acl}(C\gamma))$ is stably dominated and extends to a unique $\text{acl}(C\gamma)$ -definable type which is in Z . \square

Claim 4.14.2. *Any two elements of Z are left and right translates of each other.*

Proof. Pick any $q, r \in Z$; say both are B -definable for some $B = \text{acl}(B)$ over which G and g are also defined. Let $a \models q|B$, $b \models r|Ba$ and let $c = a \cdot b^{-1}$. Then $g(a)$ and $g(b)$ are B -independent generic elements of \mathfrak{g} . Hence so are $g(a)$ and $g(a) \cdot g(b)^{-1} = g(c)$. Since $\dim_{\text{st}}(g(a)/B) = n = \dim_{\text{st}}(G) \geq \dim_{\text{st}}(\text{St}_B(a)/B) \geq \dim_{\text{st}}(g(a)/B)$, we have $\text{St}_B(a) \subseteq \text{acl}(Bg(a))$. Similarly $\text{St}_B(c) \subseteq \text{acl}(Bg(c))$. Thus $\text{St}_B(c) \perp_B \text{St}_B(a)$. By stable domination, $a \models q|Bc$. Similarly $b \models r|Bc$. It follows that $q = {}^c r$. By the symmetric argument, $q = r^d$ where $d = b^{-1}a$. \square

Since Z is closed under right translates, it follows that $T := \text{Stab}(q)$ is a pro-definable normal subgroup of G that does not depend on the choice of $q \in Z$. Moreover if a and b are independent realizations of q over B and $c = a \cdot b^{-1}$, then the proof of Claim 4.14.2 shows that ${}^c q = q$ and hence $r := q^{b^{-1}}$ is the unique generic type of T . Thus T is connected stably dominated. Since $g(T)$ contains a generic of \mathfrak{g} and is connected, we have $g(T) = \mathfrak{g}^0$. Also, as in the above claim, we have that $\text{St}_C(c) \subseteq \text{acl}(g(c))$. It follows that $\text{tp}(c/C)$, and hence r , is stably dominated via $g|_T$.

Since $T = \text{Stab}(q)$, it is the intersection of a family of relatively definable subgroups $(T_i)_{i \in I}$ of G .

Claim 4.14.3. *For all $i \in I$, G/T_i is almost Γ -internal.*

Proof. Pick any $a \in G$ and let C be a metastability basis over which G , T and g are defined. If $c \models r|Ca$, then $a \cdot c \in a \cdot T$ and $g(a \cdot c) = g(a) \cdot g(c)$ is generic in \mathfrak{g} over $\text{acl}(C)$. So we may assume that $g(a)$ is generic in \mathfrak{g} over $\text{acl}(C)$ without changing the coset $a \cdot T$. By the proof of Claim

4.14.1, there exists a generic q , $\text{acl}(\Gamma_C(a))$ -definable, such that $a \models q|_{\Gamma_C(a)}$. By the proof of Claim 4.14.2, q concentrates on $a \cdot T$. It follows that for all $i \in I$, $a \cdot T_i \in \text{acl}(\Gamma_C(a))$ and hence G/T_i is almost Γ -internal. \square

Let us now prove the uniqueness of T . Let T' be a connected stably dominated pro-definable subgroup of G . Then $T'/(T \cap T') \simeq T \cdot T'/T \leq G/T$ is both almost Γ -internal and connected stably dominated, hence trivial. So $T' \leq T$. If G/T' is almost Γ -internal, the symmetric proof shows that $T \leq T'$. So T is indeed characterized by (1,2). Now assume that $g(T') = \mathfrak{g}^0$ and let $T' = \bigcap_j T'_j$ where the T'_j are relatively definable. By Corollary 4.10, $[T : T'_j \cap T]$ is finite and therefore $T \leq T'_j$. It follows that $T \leq T'$ and T is also characterized by (1,3). \square

4.2. Stably dominated subgroups of maximal rank. While not all stably dominated subgroups are definable, we will show that subgroups whose residual rank is $\dim_{\text{st}}(G)$ are.

Lemma 4.15. *Let T be an ∞ - C -definable connected and stably dominated normal subgroup of some definable group G with G/T almost internal to Γ .*

(FD) *There exists a C -definable subgroup stably dominated S of G with $S^0 = T$.*

(FD $_{\omega}$) *T itself is definable.*

Proof. Let assume (FD). Let p be the principal generic of T , then $T = \text{Stab}(p)$ is an intersection of C -definable subgroups S_i of G ; and G/S_i is almost Γ -internal. By Proposition 4.6, there exists a definable homomorphism $g : T \rightarrow \mathfrak{g}$, with \mathfrak{g} stable, and T stably dominated via g . By compactness, g extends to a definable homomorphism on some S_i ; replacing G by this S_i , we may assume $g : G \rightarrow \mathfrak{g}$.

By Lemma 2.25, some quotient of G/S_i by a finite normal subgroup is Γ -internal. Define $\dim_o(G/S_i)$ to be the o -minimal dimension of any such quotient. By (FD), for any $a \in G$ and any B , $\dim_o(\Gamma_B(a)/B)$ is finite, so $\dim_o(G/S_i)$ is bounded independently of i . Thus for some i , for all $j \geq i$, $\dim_o(G/S_j) = \dim_o(G/S_i)$. It follows that the natural map $G/S_j \rightarrow G/S_i$ has zero-dimensional fibers, hence it has finite kernel. This shows that for some i , for all $j \geq i$, S_i/S_j is finite. Thus $[S_i : T]$ is bounded, S_i is stably dominated, $S_i^0 = T$, and (1) holds.

If we assume (FD $_{\omega}$), definability of T follows from Corollary 4.11. \square

Corollary 4.16. *Let G be a definable group, H be a connected ∞ - C -definable stably dominated subgroup of G , with a stable homomorphic image of Morley rank $n := \dim_{\text{st}}(G)$.*

(FD) *There exists a C -definable subgroup stably dominated S of G with $S^0 = H$.*

(FD $_{\omega}$) *H itself is definable.*

Note that the group S is stably dominated and has a stable homomorphic image of Morley rank $n := \dim_{\text{st}}(G)$.

Proof. Assume (FD). Let $g : H \rightarrow \mathfrak{g}$ be a definable surjective homomorphism with \mathfrak{g} definable stable and $\dim_{\text{st}}(\mathfrak{g}) = n$. Since H is the stabilizer of its principal generic, $H = \bigcap_i H_i$ for some definable subgroups H_i of G . By compactness, g extends to a definable surjective group homomorphism $H_{i_0} \rightarrow \mathfrak{g}$. Note that $n = \dim_{\text{st}}(\mathfrak{g}) \leq \dim_{\text{st}}(H_{i_0}) \leq \dim_{\text{st}}(G) = n$ so we may assume that g extends to G . Note also that $g(H) = \mathfrak{g} = \text{Stab}(g \cdot p) = \mathfrak{g}^0$. By the uniqueness in Proposition 4.14, H is normal in G and G/H is almost Γ -internal. We can now conclude using Lemma 4.15. \square

5. ABELIAN GROUPS

We will use the notation of §3.2 and §3.5.

Lemma 5.1. *Let G be a piecewise pro-definable Abelian group and p a symmetric definable type concentrating on G . Assume that G has p -weight strictly smaller than $2n$. Then there exists a pro-definable connected $H \leq G$ with generic $p^{\pm 2n}$ such that p concentrates on a coset of H .*

Proof. Let $(a_1, a_2, \dots, a_{2n}) \models p^{\otimes 2n}|C$, where C is a model such that G is pro- C -definable and p is C -definable. Let $b = a_1^{-1} \cdot a_2 \cdot \dots \cdot a_{2n} \models p^{\pm 2n}|C$. By the weight assumption, $a_i \models p|Cb$ for some i . Say i is odd. Since p is symmetric and G is commutative, $\text{tp}(a_1, a_2, \dots, a_{2n}/b)$ is $\text{Sym}(n)$ -invariant (for the action by permutation on the pairs a_{2i-1}, a_{2i}), so $a_1 \models p|Cb$ and hence $b \models p^{\mp 2n}|Ca_1$. Since $a_1 \cdot b \models a_1 p^{\pm 2n}$ and $a_1 b = a_2 \cdot \dots \cdot a_{2n} \models p^{\pm 2n-1}|C$, it follows that $a_1 \in X := \text{Stab}(p^{\pm 2n}, p^{\pm 2n-1})$. If i is even, then $a_{2n} \models p|Cb^{-1}$ and hence $a_{2n} \in X$.

In both cases, p concentrates on X , a coset of $H := \text{Stab}(p^{\pm 2n})$. It follows that $p^{\pm 2}$ and therefore $p^{\pm 2n}$, concentrate on $H = \text{Stab}(p^{\pm 2n})$. So $p^{\pm 2n}$ is the generic of H . \square

Remark 5.2. If p is stably dominated, so is $p^{\pm 2n}$ and H is connected stably dominated.

For the rest of this section, we assume T metastable.

Proposition 5.3. *Let A be a pro-definable Abelian group of bounded weight. Let $(B_i)_{i \in I}$ be connected stably dominated pro-definable subgroups of A . Then there exists a stably dominated connected pro-definable subgroup B containing all the B_i .*

Proof. Let C be a metastability basis such that A and all the B_i are defined over C . Let \mathfrak{F} be the collection of all C -definable functions on A into Γ , seen as a pro-definable function. Then, for any $c \in A$: $\text{tp}(c/C, \mathfrak{F}(c))$ is stably dominated.

Let p_i be the principal generic of B_i . Consider the partial type:

$$q_0 = \{(d_{p_i} y) f(x) = f(y \cdot x) : i \in I, f \in \mathfrak{F}\}$$

Claim 5.3.1. q_0 is consistent.

Proof. It suffices to show that any finite number of formulas, concerning the types $(p_i)_{i \in I_0}$ for some finite $I_0 \subseteq I$, can be satisfied. Note that the sum of two stably dominated groups B_1 and B_2 is again stably dominated, by Remark 4.2: $B_1 + B_2$ is a homomorphic image of $B_1 \times B_2$. Let $B = \sum_{i \in I_0} B_i$ and p be the principal generic of B .

If $(a, c) \models p_i \otimes p|C$, then by genericity of p , $a \cdot c \models p|Ca$. By symmetry, $a \models p_i|Cc$. Now, for any $f \in \mathfrak{F}$, as p is stably dominated, and thus orthogonal to Γ , there exists γ_f such that for any $c' \models p|C$, $f(c') = \gamma_f$. In particular, $f(c) = \gamma_f = f(a \cdot c)$. Thus $\models (d_{p_i} y) f(c) = f(y \cdot c)$. Since this is true for each $i \in I_0$ and $f \in \mathfrak{F}$, q_0 is consistent. \square

Let $c \models q_0$, $C' = C \cup \mathfrak{F}(c)$ and $C'' = \text{acl}(C)$. Recall that $\text{tp}(c/C')$ is stably dominated. Let p be the unique C'' -definable extension of $\text{tp}(c/C')$.

Claim 5.3.2. For all $a \models p_i$, ${}^a p|C' = p|C'$.

Proof. If $a \models p_i| \text{acl}(Cc)$, by definition of q_0 , $\text{tp}(a \cdot c/C) = \text{tp}(c/C)$ —consider characteristic functions of definable sets. Thus $\text{tp}(a \cdot c, \mathfrak{F}(a \cdot c)/C) = \text{tp}(c, \mathfrak{F}(c)/C)$. But, for all $f \in \mathfrak{F}$, $f(a \cdot c) = f(c)$, so $\text{tp}(a \cdot c, \mathfrak{F}(c)/C) = \text{tp}(c, \mathfrak{F}(c)/C)$. Since, by symmetry, $c \models p| \text{acl}(C'a)$, this shows that $p|C' = \text{tp}(c/C') = \text{tp}(a \cdot c/C') = {}^a p|C'$. \square

Note also that since p and p_i are invariant over C'' , so is ${}^a p$. It is therefore one of the boundedly many C'' -definable extensions of $\text{tp}(c/C')$.

Moreover, by Lemma 5.1, there exist a stably dominated pro-definable group B with generic $p^{\pm 2n}$, for some n , such that p concentrates on a coset of B . Since the orbit of $p^{\pm 2n}$ under left translation by $B_i B$ is bounded, it follows that $B_i/(B \cap B_i) \simeq (B_i B)/B$ is bounded and, as B_i is connected, $B_i \leq B$. \square

In the previous proof, the subgroup B we constructed is, *a priori*, pro- C'' -definable, but we can assume it is pro- C' -definable:

Lemma 5.4. *Let A be a pro-limit of C -definable Abelian groups and $B \leq A$ be a stably dominated connected pro- $\text{acl}(C)$ -definable group. Then, there exists a stably dominated pro- C -definable $B' \leq A$ containing B .*

Proof. Let p be the principal generic of B and $(p_i)_{i \in I}$ be its conjugates over C . For all finite sets Δ of formulas (with parameters in C) preserved by left translation by A , let I_Δ be a finite set of representatives for $\{p_i|_\Delta : i \in I\}$ and let $q_\Delta = \bigotimes_{i \in I_\Delta} p_i$. Note that, since A is a pro-limit of definable groups, any formula is contained in such a Δ .

Claim 5.4.1. *For any $i_0 \in I$, p_{i_0} concentrates on $\text{Stab}_\Delta(q_\Delta)$.*

Proof. Let $c \models p_{i_0}|_M$ for some model M containing C'' and $(a_i)_i \models (\bigotimes_{i \in I_\Delta} p_i)|_M c$. Also, pick $i_1 \in I_\Delta$ such that $p_{i_0}|_\Delta = p_{i_1}|_\Delta$. For every instance $\phi(x, y) \in \Delta$ and $m \in M$, we have that $\phi(x, m) \in q_\Delta$ if and only if $\models \phi(\sum_i a_i, m)$, which is equivalent to $\phi(\sum_{i \neq i_1} a_i + x, m) \in p_{i_1}$, by symmetry of stably dominated types. That last statement is equivalent to $\phi(\sum_{i \neq i_1} a_i + x, m) \in p_{i_0}$. Since p_{i_0} concentrates on $\text{Stab}_\Delta(p_{i_0})$, this is equivalent to $\phi(\sum_{i \neq i_1} a_i + c + x, m) \in p_{i_0}$ and hence to $\models \phi(c + \sum_i a_i, m)$, i.e. $\phi(x, m) \in {}^c q_\Delta$. \square

It immediately follows that q_Δ concentrates on $\text{Stab}_\Delta(q_\Delta)$ and that, for any $J \supseteq I_\Delta$, $(\bigotimes_{i \in J} p_i)|_\Delta = q_\Delta|_\Delta$. Let q be the unique type such that for all Δ , $q|_\Delta = q_\Delta|_\Delta$. For all $i \in I$, p_i concentrates on $\text{Stab}_\Delta(q) = \text{Stab}_\Delta(q_\Delta)$ and hence so does q_Δ .

Note that for all finite set Δ , $\text{Stab}_\Delta(q)$ is relatively C -definable and thus $B' = \text{Stab}(q)$ is pro- C -definable. There remains to show that q concentrates on B' . For all Δ , let $\Delta' \supseteq \Delta$ be a finite set of formulas preserved by left translation by A such that $\text{Stab}_\Delta(q)$ is defined by an instance of Δ' . Since $q_{\Delta'}$ concentrates on $\text{Stab}_{\Delta'}(q_{\Delta'}) \leq \text{Stab}_\Delta(q_{\Delta'}) = \text{Stab}_\Delta(q)$, so does q . \square

Lemma 5.5. (FD) *Let C be a metastability basis and A be a pro-limit of C -definable Abelian groups with $\dim_{\text{st}}(A) = n < \infty$. Then A contains a stably dominated pro- CT -definable subgroup S with stable homomorphic image of dimension n .*

Proof. Let $g \in A$ with $\dim_{\text{st}}(\text{St}_C(g)/C) = n$, $C' = \Gamma_C(g)$, $C'' = \text{acl}(C')$ and $p = \text{tp}(g/C'')$. So p is stably dominated. Moreover, for all $B \supseteq C''$ and $a \models p|_B$, $\dim_{\text{st}}(\text{St}_B(a)/B) = n$.

Claim 5.5.1. *For all C , a, b , $\dim_{\text{st}}(\text{St}_C(a, b)/C) \leq \dim_{\text{st}}(\text{St}_{C_a}(b)/C_a) + \dim_{\text{st}}(\text{St}_C(a)/C)$.*

Proof. We have that $\dim_{\text{st}}(\text{St}_C(a, b)/C) = \dim_{\text{st}}(\text{St}_C(a, b)/\text{St}_C(a)) + \dim_{\text{st}}(\text{St}_C(a)/C)$ and that $\dim_{\text{st}}(\text{St}_C(a, b)/\text{St}_C(a)) = \dim_{\text{st}}(\text{St}_C(a, b)/C_a) \leq \dim_{\text{st}}(\text{St}_{C_a}(b)/C_a)$. \square

Claim 5.5.2. *If p and q are stably dominated over C concentrating on A , $a \models p|_C$, $b \models q|_C a$, $c = a \cdot b$ and $\dim_{\text{st}}(\text{St}_C(a)/C) = \dim_{\text{st}}(\text{St}_C(b)/C) = n$, then $\dim_{\text{st}}(\text{St}_C(c)/C) = n$.*

Proof. Since $\text{tp}(ab/C)$ is stably dominated, so is $\text{tp}(c/C)$. Since $\text{St}_C(a) \downarrow_C \text{St}_C(b)$,

$$\dim_{\text{st}}(\text{St}_C(a, b)/C) \geq \dim_{\text{st}}(\text{St}_C(a)\text{St}_C(b)/C) = 2n.$$

Moreover, $\dim_{\text{st}}(\text{St}_C(a, b)/C) = \dim_{\text{st}}(\text{St}_C(b, c)/C) \leq \dim_{\text{st}}(\text{St}_{C_c}(b)/C_c) + \dim_{\text{st}}(\text{St}_C(c)/C) = n + \dim_{\text{st}}(\text{St}_C(c)/C)$. So $\dim_{\text{st}}(\text{St}_C(c)/C) \geq n$. But $\dim_{\text{st}}(A) = n$, so $\dim_{\text{st}}(\text{St}_C(c)/C) = n$. \square

By Lemma 5.1, A contains a pro- C'' -definable stably dominated group R , with generic type $p^{\pm 2m}$ for some m . By Claim 5.5.2, for any $a \models p^{\pm 2m}|_{C''}$, $\dim_{\text{st}}(\text{St}_{C''}(a)/C'') = n$. By Lemma 5.4, we find $S \geq R$ pro- C' -definable with stably dominated type q —we can check that, by construction and Claim 5.5.2, for all $a \models q|_{C'}$, $\dim_{\text{st}}(\text{St}_{C'}(a)/C') = n$. We can now conclude with Proposition 4.6. \square

5.1. Limit stably dominated groups.

Definition 5.6. Let G be a pro- C -definable group and q be a (potentially infinitary) type of Γ over C . For all $t \models q$, let S_t be pro- Ct -definable subgroup of G (uniformly in t). We call $(S_t)_{t \models q}$ a limit stably dominated family for G if:

- (1) S_t is a connected stably dominated subgroup of G .
- (2) If $W \leq G$ is connected and stably dominated, pro- C -definable, then $W \leq S_t$ for some $t \models q$.
- (3) The family $(S_t)_{t \models q}$ is directed: any small set of realizations of q has an upper bound in the order defined by $t_1 \leq t_2$ if $S_{t_1} \leq S_{t_2}$.

The group $H = \bigcup_{t \models q} S_t$ is called the limit stably dominated subgroup of G . If $G = H$, we say that G is limit stably dominated.

We view H as the limit of a Γ -internal directed limit of stably dominated groups. It is clearly independent of the particular choice of limit stably dominated family $(S_t)_{t \models q}$.

Lemma 5.7. *Let G be a pro-definable group and $H \leq G$ be its limit stably dominated subgroup. Then H is pro-definable.*

Proof. Since the S_t are pro-definable, there exists a directed family of relatively definable S_t^i such that $S_t = \bigcap_i S_t^i$. By compactness, we have that $a \in H$ if and only if, for all i and all finite $q_0 \subseteq q$, there exists t such that $t \models q_0$ and $a \in S_t^i$. So H is an intersection of relatively definable subsets of G , i.e. a pro-definable subset. \square

Lemma 5.8. *Let G be a definable group and $(S_t)_{t \models q}$ a limit stably dominated family for G . Assume that S_t is the connected component of some definable group R_t , then the order $t_1 \leq t_2$ is relatively definable.*

Proof. If $S_{t_1} \leq R_{t_2}$, then $S_{t_1} \leq S_{t_2}$. It follows that $t_1 \leq t_2$ if and only if $S_{t_1} \leq R_{t_2}$, equivalently $(d_{p_{t_1}} x) x \in R_{t_2}$ where, p_t denotes the principal generic of S_t . \square

Two different behaviors are possible, according to whether or not the direct limit system has a maximal element. The latter is equivalent to the existence of a maximal connected stably dominated subgroup of G .

Proposition 5.9. *Let A be a pro-limit of definable Abelian groups. Assume A has bounded weight. Then, the limit stably dominated subgroup of A exists.*

Proof. Let C_1 be an \aleph_1 -saturated model over which A is defined and let $C \supseteq C_1$ be a metastability basis. Let \mathfrak{F} be the family of all C -definable functions into Γ . Let $(A_i)_{i \in I}$ be the family of all connected stably dominated pro- C -definable subgroups of A . Let S be as in Proposition 5.3. Note that, by Lemma 5.4, we may assume $S = S_t$ to be pro- Ct -definable for some tuple t in Γ . By Lemma 3.13, we can take t to be a small tuple and we can find a small $C_0 \subset C$ such that S_t is pro- C_0t -definable. Let $q = \text{tp}(t/C_0)$.

If $W \leq A$ is a connected stably dominated group, we must show that $W \leq S_{t'}$, for some $t' \models q$. For this purpose we can replace W by a conjugate, under the group of automorphisms of the universal domain over C_0 . Thus we may assume W is defined over C . In this case, $W \leq S_t$. This proves (2) of the definition of a limit stably dominated family. Directness of the family $(S_t)_{t \models q}$ follows from Proposition 5.3, together with (2). \square

Lemma 5.10. *Let C be a metastability basis. Let A be a C -definable Abelian group and let H be a connected stably dominated ∞ - C -definable subgroup of A .*

(FD) *H is contained in a $C\Gamma$ -definable stably dominated subgroup.*

(FD $_{\omega}$) H is contained in a CT-definable connected stably dominated subgroup.

Proof. Assume (FD). Since H is the stabilizer of its principal generic, $H = \bigcap_i H_i$ for some C -definable subgroups H_i of A . Then $\dim_{\text{st}}(A/H_i)$ is non-decreasing with i , and eventually stabilizes; we may assume it is constant, $\dim_{\text{st}}(A/H_i) = n$. Clearly $\dim_{\text{st}}(A/H) = n$. By Lemma 5.5, A/H contains a stably dominated pro-CT-definable subgroup S with stable homomorphic image of dimension n . Let S_i be the image of S in A/H_i . Then S_i is stably dominated and $S = \varprojlim_i S_i$. The stable homomorphic image of dimension n of S factorizes through S_i for large enough i , so we may assume that, for all i , S_i has stable homomorphic image of dimension $n = \dim_{\text{st}}(A/H_i)$. By Corollary 4.16, there exists a CT-definable group $R_i \leq A$ such that $S_i = (R_i/H_i)^0$. Note that for any $j \geq i$, R_i/H_j has a stable homomorphic image of dimension $n = \dim_{\text{st}}(A/H_j) = \dim_{\text{st}}(R_i/H_j)$. So replacing A by any of the R_i , we may assume that for all i , A/H_i has a stable homomorphic image of dimension $n = \dim_{\text{st}}(A/H_i)$.

By Proposition 4.14 there exists a unique ∞ -definable group T_i with $H_i \leq T_i \leq A$, A/T_i is almost Γ -internal and T_i/H_i is connected stably dominated. Note that, since it is unique, T_i is ∞ -CT-definable. By Lemma 4.15, there exists $W_i \leq A$, CT-definable, such that $T_i/H_i = (W_i/H_i)^0$.

If $j \geq i$, then $H_j \leq H_i$. Since T_j/H_j is connected stably dominated and A/T_i is almost Γ -internal, we have $T_j/H_j \leq T_i/H_j$, i.e. $T_j \leq T_i$. Since A/T_j is (almost) Γ -internal, the argument of Lemma 4.15 shows that for large enough i_0 , for all $i \geq i_0$, $[T_{i_0} : T_i]$ is bounded. Let $T = \bigcap_i T_i$. Then $[T_{i_0} : T]$ is bounded. Moreover, $T/T \cap H_i$ is (isomorphic to) a bounded index subgroup of T_i/H_i , it is therefore stably dominated. It follows that $T/H = \varprojlim_i T/T \cap H_i$ is also stably dominated. By Lemma 4.3, so is T and hence so are the T_i . Since $[W_i : T_i]$ is bounded, any of the W_i is CT-definable and stably dominated.

If (FD $_{\omega}$) holds, since $H \leq T^0$, the second statement follows from Corollary 4.11. \square

A definable family H_t of subgroups of some definable group G is said to be *certifiably* stably dominated if for all t , H_t is stably dominated and there exists t' such that $H_t \leq H_{t'}$ and $H_{t'}$ is connected.

Proposition 5.11. (FD $_{\omega}$) *Let A be a definable Abelian group. Then for some C , there exist C -definable families H^v of definable subgroups H_t^v of A , where t is a Γ -tuple, such that:*

- (1) *Any H_t^v is stably dominated.*
- (2) *Any connected stably dominated ∞ -definable subgroup of A (over any set of parameters) is contained in some H_t^v .*

Proof. Let $(S_t)_{t \in q}$ be a limit stably dominated family for A ; it exists by Proposition 5.9. Let C be a metastability basis such that A and the family S_t are defined over C . To prove (2), it suffices to consider the groups S_t , in particular, it suffices to consider ∞ -CT-definable connected stably dominated subgroups of A . By Lemma 5.10 (and Remark 2.14), it suffices to consider CT-definable connected stably dominated subgroups of A .

For a CT-definable Abelian group A , define invariants n and k as follows: $n = \dim_{\text{st}}(A)$. Let $Z(A)$ be the collection of CT-definable subgroups $S \leq B$ with stable homomorphic images of dimension n ; by Lemma 5.5 and Corollary 4.16, $Z(B) \neq \emptyset$. Let $Z_2(B) = \{(S, T) : S \in Z(B), T \leq S \text{ CT-definable, } S/T \text{ } \Gamma\text{-internal}\}$. Let $k = \max\{\dim_o(T/K) : (T, K) \in Z_2(B)\}$; by (FD), such a maximum exists. The pairs (n, k) are ordered lexicographically.

Pick any definable connected stably dominated CT-definable $B \leq A$. If $(S/B, T/B)$ attains the maximum for A/B , then the pullbacks to A show that $(n, k)(A) \geq (n, k)(A/B)$. Thus increasing B has the effect of decreasing $(n, k)(A/B)$. Let B_0 be such that $(n, k)(A/B_0)$ is minimal. For any stably dominated $H \leq A$, $H + B_0$ is also stably dominated and if S_t is a family

of stably dominated subsets of A/B_0 , its lifting to A is a family of stably dominated subsets of A ; so it suffices to find a families $\{S_t^\nu\}$ for A/B_0 . Thus, we may assume $(n, k)(A/B) = (n, k)(A)$ for any connected stably dominated $C\Gamma$ -definable $B \leq A$. Let $(n, k) = (n, k)(A)$.

Claim 5.11.1. *Let S be a $C\Gamma$ -definable subgroup of A admitting a $C\Gamma$ -definable surjective homomorphism $h : S \rightarrow \mathfrak{g}$ to a $C\Gamma$ -definable stable group of Morley rank n and a $C\Gamma$ -definable surjective homomorphism $g : S \rightarrow W$ to a $C\Gamma$ -definable Γ -internal group of o -minimal dimension k . Then there exists a C -definable family H_t of stably dominated subgroups of A , with t a Γ -tuple, such that $\ker(g) = H_t$ for some t .*

Proof. Clearly S, h, \mathfrak{g}, g, W lie in a family $(S_t, h_t, \mathfrak{g}_t, g_t, W_t)$ such that \mathfrak{g}_t is stable, $\dim_{\text{st}}(\mathfrak{g}_t) = n$, W_t is a Γ -internal, $\dim_o(W_t) = k$, $h_t : S_t \rightarrow \mathfrak{g}$ is a surjective homomorphism, $g_t : S_t \rightarrow W_t$ is a surjective homomorphism. Let $H_t = \ker(g_t)$. Let $T_t \leq S_t$ be as in Proposition 4.14. By Lemma 4.15, T_t is C_t -definable. By Lemma 2.26, S_t/T_t is Γ -internal. The group $T_t/H_t \cap T_t$ is Γ -internal and stably dominated, it is therefore finite and hence trivial, since T_t is connected. By maximality of k , $H_t/T_t \cap H_t$ is Γ -internal of o -minimal dimension 0 so it is finite. It follows that H_t is stably dominated. \square

Let B be any $C\Gamma$ -definable stably dominated connected subgroup of A . Let $(S, T) \in Z_2(A/B)$ be such that S has a stable homomorphic image of dimension n and S/T is Γ -internal of o -minimal dimension k . Let (S', T') be the pullbacks of (S, T) to A . Then $B \leq T'$. Clearly S' admits a stable homomorphic image of dimension n and S'/T' is Γ -internal of o -minimal dimension k . By the Claim 5.11.1, there exists a C -definable family H_t of stably dominated groups, with t a Γ -tuple, such that $T' = H_t$ for some t . \square

Corollary 5.12. *(FD_ω) Let A be a definable Abelian group. There exists a definable certifiably stably dominated family H_t , where t is a Γ -tuple, such that any connected stably dominated ∞ -definable subgroup of A is contained in some H_t .*

Proof. Let C and H_t^ν be as in Proposition 5.11.

Claim 5.12.1. *There exists a type p over C and a ν such that any ∞ -definable connected stably dominated subgroup of A is contained in some H_t^ν , with $t \models p$.*

Proof. Assume not, then for any ν and p , there exists a ∞ -definable stably dominated $B_{p,\nu} \leq A$ which is not contained in $\bigcup_{t \models p} H_t^\nu$. By Proposition 5.3, there exists an ∞ -definable stably dominated $B \leq A$ containing all the $B_{p,\nu}$. This B is contained in some H_t^ν for some t and ν . Let $p = \text{tp}(t/C)$. Then $B_{p,\nu} \leq H_t^\nu$, a contradiction. \square

Fix p and ν as in the Claim and $\gamma \models p$. Then $(H_\gamma^\nu)^0$ is definable and has finite index, say l , in H_γ^ν . Let L_t be a C -definable family of subgroups of H_t such that $[H_t : L_t] \leq l$ and whenever $t \models p$, $L_t = H_t^{\nu,0}$. Then every L_t is stably dominated and any ∞ -definable connected stably dominated subgroup of A is contained in some L_t with $t \models p$, which is connected by construction. \square

Before we can state the next theorem, we need to give an explicit definition of Γ -internality for certain hyperimaginaries.

Definition 5.13. Let D be a pro-definable set and E be a pro-definable equivalence relation of D^2 . We say that D/E is Γ -internal if there exists a pro-definable map $f : D \rightarrow X$ such that:

- (1) X is a pro-definable set in Γ^{eq} .
- (2) Each fiber of f intersects only boundedly many classes of E .

Remark 5.14. Note that if E is the intersection of relatively definable equivalence relations on D , then D/E is a pro-definable set and the above definition is equivalent to the usual notion of almost Γ -internality.

In particular, if G is a pro-definable group and H is a pro-definable subgroup admitting a generic and G/H is almost Γ -internal, then G/H is a direct limit of almost Γ -internal groups. If, moreover, Γ is o -minimal, then, by Lemma 2.26, G/H is (pro-definably isomorphic to) a group pro-definable in Γ .

Theorem 5.15. *Let A be a pro-limit of definable Abelian group. Assume A has bounded weight, then the limit stably dominated subgroup H exists and A/H is almost internal to Γ .*

If (FD_ω) holds and A is definable, then H is definable, admits a unique generic type p and A/H is Γ -internal.

Remark 5.16. Instead of (FD_ω) , it suffices to assume that there exists a certifiably stably dominated family H_t , where t is a Γ -tuple, such that any ∞ -definable connected stably dominated subgroup is contained in some H_t .

Proof. The existence of H is proved in Proposition 5.9. Let C be a metastability basis over which A and H are defined. Let f be a pro-definable function enumerating all C -definable maps from A into Γ .

Claim 5.16.1. *For all $c, d \in A$, if $\text{tp}(c/\text{acl}(Cf(c))) = \text{tp}(d/\text{acl}(Cf(c)))$, then $c \cdot d^{-1} \in H$.*

Proof. By definition of a metastability basis, $\text{tp}(c/Cf(c))$ is stably dominated. Let p be the unique $\text{acl}(Cf(c))$ -definable extension of $\text{tp}(c/\text{acl}(Cf(c)))$. By Lemma 5.1, there exists a stably dominated pro- $\text{acl}(Cf(c))$ -definable connected stably dominated subgroup $S \leq A$ such that p concentrates on a coset of S . Since p is $\text{acl}(Cf(c))$ -definable, it follows that $p|_{\text{acl}(Cf(c))}$ concentrates on a coset of S ; i.e. $c \cdot d^{-1} \in S \leq H$. \square

It immediately follows that each fiber of f intersects boundedly many cosets of H and, since the image of f is a pro-definable set in Γ , A/H is almost Γ -internal.

Let us now assume (FD_ω) . Let $(S_t)_{t=q}$ be a limit stably dominated family for H . By Corollary 5.12 we can also find a certifiably stably dominated family $(H_{t'}^i)_{t' \in Q}$ where $Q \leq \Gamma^n$ for some n , such that any ∞ -definable connected stably dominated subgroup of A is contained in some $H_{t'}$. It is easy to see that $\bigcup_{t=q} S_t = H = \bigcup_{t' \in Q} H_{t'}$ and hence that H is definable.

Define a partial order on D by $t'_1 \leq t'_2$ if $H_{t'_1} \leq H_{t'_2}$. Note that this order is directed. By Lemma 2.24, Q has a cofinal definable type q' . For any $t' \models q'$, let $p_{t'}$ be the principal generic of $H_{t'}^0$. Let r be the type such that $a \models r|C$ if and only if $a \models p_{t'}|C$ for some $t' \models q'|Ct'$. By Lemma 2.3, r is definable. Let C be such that A, H and r are C -definable and pick any $g \in H$. Then there exists $t \models q$ and $t' \models q'|Cg$ such that $g \in S_t \leq H_{t'}^0$. Let $a \models p_{t'}|Cgt'$, then $g \cdot a \models p_{t'}|Cgt'$ and hence $g \cdot a \models r|Cg$. It follows that ${}^g p = p$ and p is the unique generic of H .

The fact that A/H is Γ -internal now follows from the fact that it is almost Γ -internal and Lemma 2.26 (cf. Remark 5.14). \square

Corollary 5.17. *(FD_ω) Let A be a definable Abelian group. There exists a universal pair (f, B) with B a Γ -internal definable group, and $f : A \rightarrow B$ a definable homomorphism. In other words for any (f', B') of this kind there exists a unique definable homomorphism $h : B \rightarrow B'$ with $f' = h \circ f$.*

Proof. Let H be the limit stably dominated subgroup of A . Then H is definable. For any pair (f', B') as above, f vanishes on any definable connected stably dominated group. Since H is a union of such groups, f vanishes on H . But A/H is Γ -internal, so the canonical homomorphism $A \rightarrow A/H$ clearly solves the universal problem. \square

Corollary 5.18. (FD_ω) *Let A be a definable Abelian group. There exists a smallest definable subgroup A^0 of A of finite index.*

Proof. As in Corollary 5.17, any subgroup A' of finite index must contain each connected stably dominated group, and hence the limit stably dominated group H . The question thus reduces to the o -minimal group A/H , where it is known (cf. [Pil88, Proposition 2.12]). \square

Question 5.19. *In ACVF, is G/G^0 always finite, where G^0 denotes the intersection of all definable finite index subgroups of G ?*

The answer is presumably yes (cf. Problem 1.9 and the proof of Corollary 5.18) and it is already known to be true of stably dominated groups (cf. Corollary 4.11). A positive answer to that question would imply that Corollaries 5.17 and 5.20 hold for non-Abelian groups too.

Corollary 5.20. (FD_ω) *Let A be a definable Abelian group. There exists a universal pair (f, B) with B a stable definable group, and $f : A \rightarrow B$ a definable homomorphism. Equivalently, there exists a smallest definable subgroup H with A/H stable.*

Proof. Clearly if A/H and A/H' are stable, so is $A/(H \cap H')$. It suffices to show that there are no strictly descending chains $A \supset H_1 \supset H_2 \dots$ of such subgroups. By (FD), $\dim_{\text{st}}(A/H_n)$ is bounded; so we may assume it is constant, $\dim_{\text{st}}(A/H_n) = d$. Then H_n/H_{n+1} is finite. By Corollary 5.18, the chain stabilizes. \square

Remark 5.21. Since a pro-definable stable group always has a generic and hence is a prolimit of groups, any ∞ -definable normal subgroup of a definable group A such that A/H is stable is the intersection of definable subgroups H_i such that A/H_i is stable. It follows that if A is Abelian, the smallest ∞ -definable subgroup H with A/H stable exists and is definable.

It is then straightforward to prove that any pro-limit A of Abelian groups—in particular, any pro-definable Abelian group with a generic—has a smallest pro-definable subgroup H such that A/H is stable.

Corollary 5.22. (FD_ω) *Let A be a connected definable Abelian group. Then for almost all primes p , $p \cdot A = A$.*

Proof. Let H be the limit stably dominated group of A . We have $H = \cup_{t \neq q} S_t$ where q is a complete type over some C and S_t is a definable connected stably dominated. Let A_t be the maximal stable quotient of S_t and $m = \dim_{\text{st}}(A_t)$ rank of A_t . Let $A_t(p) = \{x \in A_t : p \cdot x = 0\}$. Then $A_t(p)$ can be infinite for at most m values of p . For any other p , $A_t(p)$ is finite and hence $p \cdot A_t = A_t$. It follows that $p \cdot S_t$ has finite index in S_t (Corollary 4.10); so $p \cdot S_t = S_t$. Thus $pH = H$. On the other hand $p(A/H) = A/H$ since A/H is a connected o -minimally definable group. So $p \cdot A = A$. \square

5.2. Metastable fields. By a rng we mean an Abelian group with a commutative, associative, distributive multiplication (but possibly without a multiplicative unit element.) If a rng has no zero-divisors, the usual construction of a field of fractions makes sense.

Proposition 5.23. *Let F be a definable field with $0 < \dim_{\text{st}}(F) = n < \infty$. Then there exists an ∞ -definable subrng D of F , with $(D, +)$ connected stably dominated. Moreover F is the field of fractions of D .*

Proof. Using Lemma 5.5, we find a connected ∞ -definable subgroup M of F^* with a stably dominated generic type p of stable dimension n .

Let D be a subgroup of $(F, +)$ with generic type $p^{\pm 2n}$, generated additively by $p^{\pm 2}$. Then p stabilizes D multiplicatively. Since D is generated by $p^{\pm 2n}$, so does D ; so D is a subrng of F .

Let $F' \subseteq F$ be the fraction field of D . If $a \in F \setminus F'$, then the map $(x, y) \mapsto x + a \cdot y$ takes $D^2 \rightarrow F$ injectively—if $x + a \cdot y = x' + a \cdot y'$ and $(x, y) \neq (x', y')$ then $y \neq y'$ and $a = (x' - x)/(y - y')$. Thus $D + a \cdot D \subseteq F$ is definably isomorphic to D^2 . But $\dim_{\text{st}}(D^2) \geq 2n > n = \dim_{\text{st}}(F)$, a contradiction. We have proved that $F = F'$ is the field of fractions of D . \square

Remark 5.24. Assuming (FD_ω) , in the previous proposition, if D is a ring, i.e. $1 \in D$, then the generic type of $(D, +)$ is also generic in (D^*, \cdot) .

Proof. Note first that by construction, the generic type q of $(D, +)$ has stable dimension $n = \dim_{\text{st}}(F)$, so, by Corollary 4.16, D is definable. Let N be the smallest definable subgroup of $(D, +)$ with D/N stable (cf. Corollary 5.20). Clearly N is multiplicatively invariant under the units of D . Thus N is an ideal of D and D/N is a definable stable ring. Then D/N has only finitely many maximal ideals which are all definable (cf. [CR76, Theorem 2.6]). It follows that the generic of D/N avoids all these maximal ideals and therefore concentrates on $(D/N)^*$.

Let $a \models q$. The image in D/N of the ideal $a \cdot D$ contains the generic $a + N$ which is invertible. So that image equals D/N . By Corollary 4.10, $a \cdot D = D$ —recall that D is connected—and a is invertible. So the additive generic of D concentrates on D^* and since, for any $c \in D^*$, the map $x \mapsto c \cdot x$ is a definable isomorphism, it must preserve q . \square

Remark 5.25. Let F be a definable field of finite weight then the limit stably dominated subgroup H of $(F, +)$ is an ideal so it is either trivial—in which case F is Γ -internal—or coincides with F . If $(F, +)$ is stably dominated then its minimal definable subgroup N with F/N stable is a proper ideal and hence is trivial—in which case F is stable.

So, unless F is stable or Γ -internal, $(F, +)$ is properly limit stably dominated.

Proposition 5.26. (FD_ω) *Let F be an infinite definable field which is neither stable nor internal to Γ . Then F^* has a definable homomorphic image that is Γ -internal and not definably compact.*

Proof. Let H be a stably dominated ∞ -definable subgroup of F^* with generic p . As in the proof of Proposition 5.23, we find a subrng R , additively generated by $p^{\pm 2}$, invariant under multiplication by H , whose additive group is connected stably dominated and such that p concentrates on an additive coset of R .

If $H_1 \leq H_2$ are two ∞ -definable subgroup of F^* , let p_i be their generics and R_i the corresponding subrng. Let C be such that everything is defined over C . If $a, b \in H_1$ and $e \models p_2|Cab$, then both $a \cdot e$ and $b \cdot e$ realize $p_2|Cab$. It follows that $a - b = e^{-1}(a \cdot e - b \cdot e) \in R_2$; so $p_1^{\pm 2}$ concentrates on R_1 and $R_1 \subseteq R_2$.

Let $M = \bigcup_{t \models q} M_t$ be the limit stably dominated subgroup of F^* and R_t the corresponding subrng. Note that, since M_t can be—and is—chosen definable and that $M_t - M_t$ generates R_t , R_t is definable. Note that if $\dim_{\text{st}}(F) = 0$, F , working over a metastability basis C , for any $a \in F$, $\text{tp}(a/\Gamma_C(a))$ is realized and F is internal to Γ . So $\dim_{\text{st}}(F) > 0$. Let R (and M) be as in Proposition 5.23. By the previous paragraph, for some $t \models q$, $R \subseteq R_t$. It follows that for any $t \models q$, F is the fraction field of R_t . Let N_t be the minimal quotient of $(R_t, +)$ such that R_t/N_t is stable (cf. Corollary 5.20). Since N_t is characteristic, it is invariant under multiplication by elements of M_t . It is therefore an ideal of R_t . By Proposition 4.6, the principal generic of R_t is dominated by a group morphism, so if $N_t = R_t$, $R_t = \{0\}$. On the other hand, if $N_t = (0)$, R_t , and hence F is stable. It follows that N_t is a proper non-trivial ideal. If $t \leq s$, $R_t/N_s \cap R_t$ is stable, so $N_t \subseteq N_s$. If $a \in N_t$ is not 0, then $a^{-1} \notin R_s$ for any $s > t$, it follows that $R := \bigcup_t R_t \neq F$.

Let $V := F^*/M$. It is Γ -internal. Let $V^+ := (R_t \setminus \{0\})/M$ for some choice of t . It is a definable subsemigroup of V . Since R is stabilized by M , $MR_t \subseteq R \neq F$, so $V^+ \neq V$. Also, since F is the

fraction field of R_t , $V = V^+ - V^+$. By Lemma 2.29, V does not have (NG), so it is not definably compact. \square

6. VALUED FIELDS: STABLY DOMINATED GROUPS AND ALGEBRAIC GROUPS

In this section, we work in ACVF. Let K be an algebraically closed valued field; \mathcal{O} denotes the valuation ring. Occasionally we will assume K to be saturated.

Proposition 6.1. *Let $C = \text{acl}(C)$ and H be a connected ∞ - C -definable group with a (right) generic. Then there exists an algebraic group G over $C \cap K$ and a C -definable homomorphism $f : H \rightarrow G$, with purely imaginary kernel.*

Proof. Let $F = C \cap K$. Let p be the principal generic type of H ; let $(a_1, a_2) \models p^{\otimes 2}|_C$ and $a_3 = a_1 \cdot a_2$. Let $\tau = \{a_1, a_2, a_3, a_{12} = a_1 \cdot a_2, a_{23} = a_2 \cdot a_3, a_{123} = a_1 \cdot a_2 \cdot a_3\}$. We view this six-element set as a *matroid*, given by specifying the collection of algebraically dependent subsets of τ . This data is called a group configuration.

For $c \in \tau$, let $A(c) = \text{acl}(Cc) \cap K$; this is the set of field elements in the algebraic closure of c over C . Pick a tuple $\alpha(c) \in A(c)$ such that $A(c) = F(\alpha(c))^{\text{alg}}$. We view $c \mapsto \alpha(c)$ as a map on τ into the matroid of algebraic dependence in the algebraically closed field K over F . Then α preserves both dependence and independence. For independence this is clear. Preservation of dependence follows from:

Claim 6.1.1. *Let E_1 and E_2 be two algebraically closed substructures of a model of ACVF, all sorts allowed. Let L_i be the set of field elements of E_i . If $c \in \text{acl}(E_1 \cup E_2) \cap K$, then $c \in (L_1 L_2)^{\text{alg}}$.*

Proof. This follows immediately from the fact that there are no definable maps with infinite range from a geometric sort other than K to K (cf. Remark 2.34). \square

According to the group configuration theorem for stable theories, applied to the theory ACF over the model F , there exists a group G , ACF-definable over F , such that $a(\tau)$ is a group configuration for G ; in particular there exist $b_1, b_2 \in G$, $b_{12} = b_1 b_2$ such that $A(a_i) = F(b_i)^{\text{alg}}$. Since ACF-definable groups are (definably isomorphic to) algebraic groups, we can take G to be an algebraic group over F (cf. [HP94, Proposition 3.1]).

We work in the group $H \times G$. Let $c_i = (a_i, b_i)$. Since $a_i \models p|_C$, p is C -definable and $C = \text{acl}(C)$, $\text{tp}(c_i/C)$ has a C -definable extension q_i . Let $Z = \text{Stab}(q_2, q_3)$. This is a coset of $S := \text{Stab}(q_2)$ and $q := c_1^{-1} q_1$ is a generic type of S . Let $J = \{h \in H : (h, 1) \in S\}$.

Claim 6.1.2. *J is purely imaginary.*

Proof. Pick any $h \in J$, $D \supseteq C$ and $(a, b) \models q_2|_{\text{acl}(Dh)}$. Then $(h \cdot a, b) \models q_2|_{Dh}$. Since all geometric sorts but K are purely imaginary and $a \cap K \subseteq \text{acl}(b)$, we have $\text{acl}(Da) \cap K \subseteq \text{acl}(Db)$ and hence $\text{acl}(D, h \cdot a) \cap K \subseteq \text{acl}(Db)$. So $\text{acl}(Dh) \cap K \subseteq \text{acl}(D, a, h \cdot a) \cap K \subseteq \text{acl}(Db)$. This inclusion holds for any $(a, b) \models q_2|_{\text{acl}(Dh)}$. Since q_2 is D -definable, in particular D -invariant, it follows that $\text{acl}(Dh) \cap K \subseteq \text{acl}(D)$ and h is purely imaginary over any $D \supseteq C$. By compactness and Lemma 2.33, J is purely imaginary. \square

Note that S projects onto H ; this is because the projection contains a realization of $p_1|_C$ and p_1 generates H . Similarly, S projects dominantly onto G . Let $J' = \{g \in G : (1, g) \in S\}$. By an argument similar to the proof of the above claim, for all $h \in J'$ and $(a, b) \models q_2|_{Ch}$, since $b \in \text{acl}(a)$, we have that $h \in \text{acl}(Ca)$. Since q_2 is C -invariant, it follows that $h \in \text{acl}(C)$ and hence J' is finite. Note also that J' is normal in the projection of S to G and hence is a normal subgroup of G .

Now S can be viewed as the graph of a homomorphism $f : H \rightarrow G/J'$ with kernel J . Since G/J' is isomorphic to an algebraic group defined over F , replacing G by G/J' we may assume

that $f : H \rightarrow G$. Since $S \subseteq H \times G$ is the ∞ - C -definable graph of a function, is relatively definable—we have $(a, b) \notin S$ if and only if $(a, b') \in S$ for some $b' \neq b$ —and f is a C -definable function. \square

Remark 6.2. Assume that H is a connected ∞ -definable Abelian group with a (right) generic, then there is a universal map f as in Proposition 6.1, i.e. for any algebraic group G' and any definable homomorphism $f' : H \rightarrow G'$, there exists a unique definable homomorphism $g : f(H) \rightarrow G'$ with $f' = g \circ f$.

Instead of H being Abelian, it suffices to assume that Question 5.19 has a positive answer for all definable subgroups of H .

Proof. Let f be as in Proposition 6.1, then, since $J := \ker(f)$ is purely imaginary, $f'(J)$ is finite. Thus $\ker(f') \cap J$ is a finite index subgroup of J . If we choose $\ker(f)$ to be as small as possible, i.e. $\ker(f)/\ker(f)^0$ to be least possible, then $J \leq \ker(f')$. \square

The following example shows that the assumption that the group has a generic cannot be eliminated. Without it, it seems likely that one can find an isogeny between a quotient of H by a purely imaginary group and a quotient of an ind-algebraic group.

Example 6.3 (Versions of tori). The value group Γ of an algebraically closed valued field is a divisible ordered Abelian group. So every group definable in Γ is locally definably isomorphic to the group $V := \Gamma^n$. Nevertheless, interesting variants of V are known. If Δ is a finitely generated subgroup of V , the convex hull $C(\Delta)$ of Δ (with respect to the product partial ordering of V) is an ind-definable subgroup (a direct limit of definable sets) which is itself not definable. But the quotient $C(\Delta)/\Delta$ is canonically isomorphic to a group definable in Γ (cf. [PS04]).

These examples lift to the tori $T = \mathbb{G}_m^n$ over algebraically closed valued fields. Let $r : T \rightarrow V$ be the homomorphism induced by the valuation map. Let Δ be a finitely generated subgroup of T . Then $C(\Delta) := r^{-1}(C(r(\Delta)))$ is ind-definable, and $C(\Delta)/\Delta$ is definable in ACVF. For example, fix $t \in K$ with $\text{val}(t) = \tau > 0$, and define a group structure on $A(t) := \{x : 0 \leq \text{val}(x) < \tau\}$ by: $x \star y = x \cdot y$ if $x \cdot y \in A(t)$, $x \star y = x \cdot y/t$ otherwise. This group admits a homomorphism into a definably compact Γ -internal group, with stably dominated kernel. So $A(t)$ cannot have generic definable types.

Note that $A(t)$ contains no infinite purely imaginary subgroup. Moreover, if, for some finite group J , $A(t)/J$ is a Zariski-dense subgroup of an algebraic group G , then the group configuration of G is inter-algebraic with that of \mathbb{G}_m . It follows that G is isogenous to \mathbb{G}_m . But $A(t)$ has too many torsion points for that to be possible.

Corollary 6.4. *Let $C = \text{acl}(C)$ and H be a connected ∞ - C -definable stably dominated group. Then, there exists an algebraic group G over $C \cap K$ and a C -definable homomorphism $f : H \rightarrow G$, with boundedly imaginary kernel.*

Proof. We follow the notation and proof of Proposition 6.1. Note that, by Proposition 2.10.(4) and Proposition 2.9, q_i is stably dominated over C . There only remains to show that J is boundedly imaginary. So pick an $h \in J$ and $D \supseteq C$. Then for any $(a, b) \models q_2|Dh$, $(h \cdot a, b) \models q_2|C$. Now a is purely imaginary over Db , and hence so is $h \cdot a$. By Lemma 2.33, there exists $\beta_0 \leq 0 \leq \beta_1 \in \Gamma_D(a, b)$ such that $a \in \text{dcl}(D, \beta_0, \beta_1, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$. Since $\text{tp}(ab/D)$ is stably dominated, we have $\beta_0, \beta_1 \in \Gamma(D)$, and $a \in \text{dcl}(D, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$. Since $\text{tp}(h \cdot a/D) = \text{tp}(a/D)$, we also have $h \cdot a \in \text{dcl}(D, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$. Thus $h \in \text{dcl}(a, h \cdot a, D) \subseteq \text{dcl}(D, \beta_0\mathcal{O}/\beta_1\mathfrak{M})$. Since this holds for any $h \in J$ and $D \supseteq C$, by compactness and Corollary 2.37, J is boundedly imaginary. \square

A very similar proof yields the following:

Remark 6.5. Let H be a purely imaginary stably dominated definable group. Then H is boundedly imaginary.

Proof. Let p be the principal generic type of H ; with H and p defined over $C = \text{acl}(C)$. Then, for any $a \models p|C$, $\Gamma_C(a) \subseteq C$. Since a is purely imaginary over C , it follows from Lemma 2.33 and Corollary 2.37 that a is boundedly imaginary over C . Any element of B^0 is a product of two realizations of $p|C$, so B^0 is boundedly imaginary. It follows that some definable set B' containing B^0 is boundedly imaginary. Finitely many translates of B' cover B , so B is also boundedly imaginary. \square

6.1. Stably dominated subgroups of algebraic groups. All the schemes over \mathcal{O} that we consider will be assumed to be flat and reduced over \mathcal{O} . By an \mathcal{O} -variety (or variety over \mathcal{O}), we mean a (flat and reduced) scheme over \mathcal{O} , admitting a finite open covering by schemes isomorphic to $\mathcal{O}[X_1, \dots, X_n]/I$. Since \mathcal{O} is a valuation ring, $\text{Spec}(R)$ is flat over $\text{Spec}(\mathcal{O})$ if and only if no non-zero element of \mathcal{O} is a 0-divisor in R . So $\mathcal{O}[X_1, \dots, X_n]/I$ is flat if and only if $I = IK[X_1, \dots, X_n] \cap \mathcal{O}[X_1, \dots, X_n]$. Hence there are no infinite descending chains of \mathcal{O} -subvarieties. For any set $Z \subseteq \mathcal{O}_K^n$, let $I := \{f \in \mathcal{O}[X_1, \dots, X_n] : f|_Z = 0\}$ and $R = \mathcal{O}[X_1, \dots, X_n]/I$. Then $V = \text{Spec}(R)$ is flat over $\text{Spec}(\mathcal{O})$. If $Z = W(K) \cap \mathcal{O}^n$ for some affine variety W over K , then $V(\mathcal{O}) = Z$.

If V is a scheme over \mathcal{O} , we write V_K for $V \otimes_{\mathcal{O}} K$, the generic fiber, and V_k for $V \otimes_{\mathcal{O}} k$, the special fiber. Let $r : V(\mathcal{O}) \rightarrow V_k(k)$ be the natural map. If W is a variety over \mathcal{O} , and V' a subvariety of $W_K = W \otimes_{\mathcal{O}} K$, there exists a unique \mathcal{O} -subvariety V of W such that $V_K = V'$, and $V(\mathcal{O}) = V'(K) \cap W(\mathcal{O})$. For example, if $V' \subset \mathbb{A}^n$ is an affine variety over K , defined by a radical ideal $P \subset K[X]$, we let $V = \text{Spec}(\mathcal{O}[X]/P \cap \mathcal{O}[X])$. Then V_k is the zero set of the image of $P \cap \mathcal{O}[X]$ in $k[X]$. In this case, we denote the affine coordinate ring $K[X]/P$ by $K[V]$, and $\mathcal{O}[V] = \mathcal{O}[X]/(P \cap \mathcal{O}[X])$.

Lemma 6.6. *Let V be an irreducible affine \mathcal{O} -variety and assume $V(\mathcal{O}) \neq \emptyset$. Then $V(\mathcal{O})$ is Zariski dense in V_K .*

Proof. Let L be a large algebraically closed field extending K . Let $a \in V_K(L)$ be a K -generic point. Thus for any $b \in V(K)$ there exists a K -algebra homomorphism $h : K[a] \rightarrow K$ with $h(a) = b$. In particular, there exists such an h with $h(a) \in V(\mathcal{O})$. It follows that the maximal ideal \mathfrak{M} of K generates a proper ideal of $\mathcal{O}[a]$: otherwise, for some $m \in \mathfrak{M}$ and $f \in \mathcal{O}[X]$, $m \cdot f(a) = 1$; but then applying h , we would have $m \cdot f(h(a)) = 1$. Thus \mathfrak{M} extends to a maximal ideal \mathfrak{M}' of $\mathcal{O}[a]$, and thence to a maximal ideal \mathfrak{M}'' of some valuation ring \mathcal{O}_L of L , with $\mathcal{O}[a] \subset \mathcal{O}_L$. Thus the valuation on K can be extended to L in such a way that $a \in \mathcal{O}(L)$. By model completeness of ACVF, there exists $a' \in V_K(K)$, with coordinates in \mathcal{O} , outside any given proper K -subvariety of V . \square

Lemma 6.7. *Let V be a scheme over \mathcal{O} , with $\dim(V_K) = n$. Let q be a K -definable type of elements of $V(\mathcal{O})$ with $\dim_{\text{st}}(r_*q) = n$. Assume V is defined over $B = \text{acl}(B)$; then q is stably dominated over B .*

Proof. Say q is defined over $B' \supset B$. Let $a \models q|B'$. Note that $\text{trdeg}(B'(r(a))/B') \geq n$, so $\text{trdeg}(B(r(a))/B) \geq n$. It follows that $B(a)$ is an extension of B of transcendence degree n and residual transcendence degree n . The type $\text{tp}(a/B)$ is therefore stably dominated via r . Since $r(a) \downarrow_B B'$, q is its unique B -definable extension. \square

Let G be a group scheme over \mathcal{O} , with generic fiber G_K and special fiber G_k . Then $G(\mathcal{O}) \subseteq G_K(K)$, and we have a definable group homomorphism $r : G(\mathcal{O}) \rightarrow G_k(k)$. Note that $r(G(\mathcal{O}))$ is a definable, hence algebraic, subgroup of G_k ; it may be a proper subgroup. In turn, when G is not of finite type, $\dim(G_k)$ might be smaller than $\dim(G_K)$.

Proposition 6.8. *Let p be a definable type concentrating on $G(\mathcal{O})$. Assume that r_*p is a generic type of G_k and that $\dim(G_K) = \dim(G_k) < \infty$. Then p is a stably dominated generic type of $G(\mathcal{O})$. When G is of finite type, $G(\mathcal{O})$ has finitely many generic types.*

Proof. Consider translates $q = {}^g p$ of p , for some $g \in G(\mathcal{O})$. Clearly $r_*q = {}^{\tau(g)}(r_*p)$. By Lemma 6.7, q is stably dominated over $\text{acl}(B)$, where B is a base of definition of G . So p is a stably dominated generic.

When G is finite type, by Corollary 4.11, G^0 is definable and hence, the orbit of generic types is finite. \square

6.1.1. *Linear groups.* In this section, our goal is to show that all stably dominated ∞ -definable subgroups of affine algebraic groups, with their generics, may be obtained as the \mathcal{O} -points of a group scheme G over \mathcal{O} . This result was generalized in [Hal] to the non-affine setting. Note that Example 6.12 shows that we cannot always assume G to be of finite type.

We start with a version of the maximum modulus principle for generics of subgroups:

Proposition 6.9. *Let G be an affine algebraic group over K , H be a Zariski dense definable subgroup of $G(K)$ and p be a definable type of elements of H . Then the following are equivalent:*

- (1) p is the unique stably dominated generic of H .
- (2) For any regular function f on G , p attains the highest modulus of f on H ; i.e for some γ_f , for any $x \in H$, $\text{val}f(x) \geq \gamma_f$ and

$$\models (\text{d}_p x)(\text{val}f(x) = \gamma_f).$$

Note that, because any function constant on connected components of G is regular, connectedness of G follows from these assumptions.

Proof. Let us first prove that (1) implies (2). Since p is stably dominated, for any regular f there exists γ_f with $(\text{d}_p x)(\text{val}(f(x)) = \gamma_f)$. If $(a, b) \models p \otimes p$ then $a \cdot b \models p$; so $\text{val}(f(a \cdot b)) = \gamma_f$. By Corollary 2.39, for any $a, b \models p$ we have $\text{val}(f(a \cdot b)) \geq \gamma_f$. But since p is the unique generic, any element c of H is a product of two realizations of p . Thus $\text{val}(f(c)) \geq \gamma_f$.

Let us now prove that (2) implies (1). Note that, using quantifier elimination for algebraically valued fields, (2) characterizes p uniquely. Also, since γ_f does not depend on x , p is orthogonal to Γ , hence stably dominated, by Proposition 2.15. On the other hand, for any $h \in H$, replacing f by $f(h \cdot x)$, we see that (2) is invariant under H -translations. Thus if (2) holds of p , it holds of every translate, so every translate of p equals p . By Remark 3.12, p is the unique generic type of H . \square

Proposition 6.10. *Let G be an affine algebraic group scheme over \mathcal{O} . Let p be a stably dominated type of elements of $G(\mathcal{O})$. Assume:*

- (\star) *if $f \in K[G]$ and $\text{val}(f(x)) \geq 0$ for $x \models p$, then $f \in \mathcal{O}[G]$.*

*Then p is the unique generic of $H := G(\mathcal{O})$ if and only if r_*p is the unique generic type of $r(H)$.*

Proof. Let us assume that p is the unique generic of H . In an algebraic group, to show that a type is the unique generic is to show that any regular function f vanishing on the type, vanishes on the whole group. Let $f \in k[r(H)]$ vanish on r_*p . Lifting to \mathcal{O} , we have $F \in \mathcal{O}[G]$ with $\text{val}(F(a)) > 0$ for $a \models p$. By Proposition 6.9.(2), $\text{val}(F(a')) > 0$ for all $a' \in H$. So f vanishes on $r(H)$.

The converse uses (\star). Assume that r_*p is generic in $r(H)$. We want to show that Proposition 6.9.(2) holds to be able to conclude. Let $F \in K[G]$. Since p is stably dominated, for some γ , for any $a \models p$, $\text{val}(F(a)) = \gamma$. If $\gamma = 0$, then, by (\star), any K -multiple of F lies in $\mathcal{O}[G]$, so

$F = 0$. Otherwise, we may assume $\gamma = 1$. By (\star) , it follows that $F \in \mathcal{O}[G]$. Suppose there exists $a' \in G(\mathcal{O})$ with $\text{val}(F(a')) = \text{val}(c) < 0$ —we may take $a' \in G(\mathcal{O}_0)$, a fixed submodel. Then $c^{-1}F \in \mathcal{O}[G]$, and $\text{val}(c^{-1}F(a)) > 0$ for $a \vDash p$, i.e. $\text{res}(c^{-1}F(a)) = 0$. By genericity of $r_\star p$, $\text{res}(c^{-1}F)$ vanishes on $r(H)$; so $\text{val}(c^{-1}F(a')) < 0$ for all $a' \in G(\mathcal{O})$; a contradiction. \square

Theorem 6.11. *Let G be an affine pro-algebraic group over K . Let H be a Zariski dense ∞ -definable connected stably dominated subgroup of G . Then there exists a pro-affine group variety \mathbb{H} over \mathcal{O} and an isomorphism $\phi: G \rightarrow \mathbb{H}_K$, such that $\phi(H) = \mathbb{H}(\mathcal{O})$.*

If G is an affine algebraic group and H is definable, \mathbb{H} can be chosen to be an affine group variety over \mathcal{O} .

Proof. Let $K_0 = (K_0)^a$ of K be some subfield of K over which G, H are defined, $\mathcal{O}_0 = \mathcal{O} \cap K_0$ its valuation ring, $R_0 := K_0[G]$ be the affine coordinate ring of G and p be the stably dominated generic of H . Note that since p generates H , p is Zariski dense in G . It follows that equalities in $R = K_0[G]$ can be determined by evaluating at any realization of $p|K_0$. We define $R = \{r \in K_0[G] : (d_p x)\text{val}(r(x)) \geq 0\}$. This is an \mathcal{O}_0 -subalgebra of R_0 .

Claim 6.11.1. $R \otimes_{\mathcal{O}_0} K_0 = R_0$.

Proof. Pick any $0 \neq r \in R_0$. Since p is stably dominated, it is orthogonal to Γ . Thus for some $c \in K_0$, for $a \vDash p|K_0$, $|r(a)| = |c|$. If $c = 0$, then r vanishes on p and hence on G , i.e. $r = 0 \in R_0$, contradicting the choice of r . So $c \neq 0$, and $c^{-1}r \in R$. This shows that the natural map $R \otimes_{\mathcal{O}_0} K_0 \rightarrow R_0$ is surjective. Injectivity is clear since R has no \mathcal{O}_0 -torsion. \square

Let $\mathbb{H} = \text{Spec}(R)$. We have $\mathbb{H}_K := \mathbb{H} \times_{\mathcal{O}_0} \text{Spec}(K) \simeq \text{Spec}(R \otimes_{\mathcal{O}_0} K) = G$. We identify \mathbb{H}_K with G . So p is the type of elements of $G(K)$ and in fact, by definition of R , of $G(\mathcal{O}) = \mathbb{H}(\mathcal{O})$.

The morphisms $x \mapsto x^{-1}: G \rightarrow G$ and $(x, y) \mapsto x \cdot y: G^2 \rightarrow G$ correspond to two operations:

$$i: R_0 \rightarrow R_0, i(r)(g) = r(g^{-1})$$

and

$$c: R_0 \rightarrow R_0 \otimes_{K_0} R_0, c(r) = \sum_{i=1}^n r_i \otimes s_j, r(g \cdot h) = \sum_{i=1}^n r_i(g) s_i(h).$$

Note that any \mathcal{O} -subalgebra R' of R_0 is \mathcal{O} -torsion-free, hence a flat \mathcal{O} -module. Thus the maps $R' \otimes_{\mathcal{O}} R' \rightarrow R' \otimes_{\mathcal{O}} R \rightarrow R \otimes_{\mathcal{O}} R$ are injective. We identify $R' \otimes_{\mathcal{O}} R'$ with its image in $R \otimes_{\mathcal{O}} R$. Let us say that an \mathcal{O} -subalgebra R' of R_0 is *Hopf* if $i(R') \subseteq R'$ and $c(R') \subseteq R' \otimes_{\mathcal{O}} R'$.

Claim 6.11.2. R is Hopf

Proof. Let us first consider co-inversion. Let $g = i(r)$ for some $r \in R$. Clearly $\text{val}(g(x)) = \text{val}(r(x^{-1})) \geq 0$ for any $x \vDash p$. Hence $g \in R$.

Let us now consider co-multiplication. Pick any $(a, b) \vDash p^{\otimes 2}$ and $r \in R$. Write $c(r) = \sum_j g_j \otimes h_j$ for finitely many $g_j, h_j \in R_0$. By [HHM08, Lemma 12.4], we may assume that $\text{val}(r(a \cdot b)) = \min_j (\text{val}(g_j(a)) + \text{val}(h_j(b)))$. We may also assume that no $g_j(a)$ is zero. As in Claim 6.11.1, we renormalize so that $\text{val}(g_j(a)) = 0$ for all j . Since $a \cdot b \vDash p$, it follows that $\min_j \text{val}(h_j(b)) = \text{val}(r(a \cdot b)) \geq 0$. Since both a and b realize $p|K_0$, for generic $c \vDash p|K_0$ we have $\text{val}(g_j(c)) = 0$ and $\text{val}(h_j(c)) \geq 0$, i.e. $g_j, h_j \in R$. \square

Claim 6.11.3. *The type p is the unique generic of $\mathbb{H}(\mathcal{O})$.*

Proof. Note that, by definition, for all $x \in \mathbb{H}(\mathcal{O})$ and $r \in R$, $\text{val}(r(x)) \geq 0$. Pick any $r \in R$. As in Claim 6.11.1, we find $c \in K_0$ such that $(d_p x)\text{val}(r(x)) = \text{val}(c)$ and $c^{-1}r \in R$. Thus, for all $x \in \mathbb{H}(\mathcal{O})$, $\text{val}(r(x)) \geq \text{val}(c)$. The claim now follows by Proposition 6.9. \square

Since they have the same generic, it follows that $\mathbb{H}(\mathcal{O}) = H$. Let us now prove that \mathbb{H} is a pro-affine group variety. Let \mathcal{F} be the family of finitely generated \mathcal{O}_0 -subalgebras of R that are Hopf. If $R' \in \mathcal{F}$ then $\text{Spec}(R')$ is an affine group variety over \mathcal{O} . So it suffices to show that R is the direct limit (i.e. the union) of \mathcal{F} . Note that if S is generated by R' and R'' as an \mathcal{O}_0 -algebra, then S is closed under c if R' and R'' are. Indeed $c : R \rightarrow R \otimes_{\mathcal{O}_0} R$ is an \mathcal{O}_0 -algebra homomorphism, so $\{r : c(r) \in S \otimes_{\mathcal{O}_0} S\}$ is an \mathcal{O}_0 -subalgebra of S , hence equal to S since it contains R' and R'' . Moreover, by the commutativity rules between c and i , the \mathcal{O}_0 -algebra $i(R')$ is also closed under c and hence, since the \mathcal{O}_0 -algebra generated by $R' \cup i(R')$ is closed under i , it is Hopf. It follows that it suffices, given $r \in R$, to find a finitely generated \mathcal{O}_0 -subalgebra R' of R closed under c with $r \in R'$.

Fix $r \in R$ and write $c(r) = \sum_{i=1}^n a_i \otimes b_i$, with n least possible, and $a_1, \dots, a_n, b_1, \dots, b_n \in R$.

Claim 6.11.4. *Let $(g_1, \dots, g_n) \in p^{\otimes n} | K_0$. The matrix $b = (b_i(g_j))_{1 \leq i, j \leq n}$ is invertible over K_0 .*

Proof. If not, there exists a non-zero vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha \cdot b = 0$. We may assume $\alpha_n = 1$. Since g_1, \dots, g_n , and α has weight at most $n-1$, some g_j must be independent (in the underlying algebraically closed field) from α . Therefore, for any $g \in p | K_0$, $\sum_i \alpha_i b_i(g) = 0$ and hence $\sum_i \alpha_i b_i(g) = 0$. Thus b_1, \dots, b_n are K -linearly dependent, contradicting the minimality of n . \square

The expression $r(x \cdot y) = \sum_{i=1}^n a_i(x) \cdot b_i(y)$ shows that $\{r(g \cdot y) : g \in G(K_0)\}$ spans a finite-dimensional K_0 -subspace of $K[G]$. Similarly $\{r(g \cdot y \cdot h) : g, h \in G(K_0)\}$ spans a finite-dimensional K_0 -subspace V of R_0 . Let R' be the \mathcal{O}_0 -algebra generated by $V \cap R$.

Claim 6.11.5. $c(V) \subseteq V \otimes_{\mathcal{O}_0} V$.

Proof. Let $(g_1, \dots, g_n, x) \in p^{\otimes n+1} | K_0$. The $r(x \cdot g_j) = \sum_i a_i(x) b_i(g_j)$ are in V by definition. By Claim 6.11.4, $a_i \in V$. By the symmetric argument, we also have $b_i \in V$ and hence $r \in A := \{r \in V : c(r) \in V \otimes_{\mathcal{O}_0} V\}$. Note that, by definition, A is closed by left and right multiplication by $G(K_0)$, so $V \subseteq A$ and thus $c(V) \subseteq V \otimes_{\mathcal{O}_0} V$. \square

By the renormalization argument of Claim 6.11.1, $V \otimes_{\mathcal{O}_0} V \subseteq V \otimes_{\mathcal{O}_0} R = R \otimes_{\mathcal{O}_0} V$ and hence $c(V \cap R) \subseteq (V \otimes_{\mathcal{O}_0} V) \cap (R \otimes_{\mathcal{O}_0} V) \cap (R \otimes_{\mathcal{O}_0} V) \cap (R \otimes_{\mathcal{O}_0} V) = (R \cap V) \otimes_{\mathcal{O}_0} (R \cap V)$. It immediately follows that $c(R') \subseteq R' \otimes_{\mathcal{O}_0} R'$. Since $r \in R'$, we are done.

The finite type statement follows by compactness. \square

In Theorem 6.11, we cannot always assume G to be finite type:

Example 6.12. Let $K_0 = \mathbb{C}(t)^a$, with $\text{val}(t) > 0$. Let $H_n := \{(x, y) \in \mathbb{G}_a \times \mathbb{G}_m : \text{val}(y - \sum_{i=1}^n 1/n! (t-x)^n) \geq \text{val}(t^{n+1})\}$, a definable subgroup of $(\mathbb{G}_a \times \mathbb{G}_m)(\mathcal{O})$, isomorphic to $(\mathbb{G}_a \times \mathbb{G}_m)(\mathcal{O})$. The group $H := \cap H_n$ is also connected stably dominated; but its generic is dominated via the map $(x, y) \mapsto \text{res}(x)$. The Zariski closure of H has dimension 2, but the generic type of H has a residual part of transcendence degree one.

It follows that H cannot be the \mathcal{O} points of a group scheme of finite type, nor can it be the connected component of a definable group.

Lemma 6.13. *Let G be a group scheme over \mathcal{O} . For each n , let $\phi_n(g) := g^n$, and assume $\phi_n : G \rightarrow G$ is a finite morphism. Then $G(\mathcal{O})$ is connected.*

Proof. By properness, $\phi_n : G(\mathcal{O}) \rightarrow G(\mathcal{O})$ is surjective. So any finite quotient has order prime to n . This holds for all n , so $G(\mathcal{O})$ has no finite quotients. \square

6.2. Abelian varieties.

Definition 6.14. Let V be an affine variety over K . A definable subset W is *bounded* if for any regular function f on V , $\{\text{val}(f(x)) : x \in W\}$ has a lower bound. For a general K -variety V , a definable subset W is *bounded* if there exists an open affine covering $V = \bigcup_{i=1}^m U_i$, and a bounded $W_i \subseteq U_i$, with $W = \bigcup_i W_i$.

This definition is due to [Ser89, §6.1]. The assumption there that the valuation is discrete is inessential.

If V' is a closed subvariety of V and W is bounded on V , then clearly $W \cap V'$ is bounded on V' . In the affine case, if V has coordinate ring $K[f_1, \dots, f_n]$, it suffices that the f_i have bounded valuation. In the case of projective space \mathbb{P}^n , a standard covering by bounded sets is given in projective coordinates by: $U_i = \{(x_0 : \dots : x_n) : x_i = 1 \text{ and } (\forall j \neq i) \text{val}(x_j) \geq 0\}$. Complete varieties are bounded as subsets of themselves.

Let $C \models \text{ACVF}$, so $C = \text{dcl}(F)$ where $F = C \cap K$; F is an algebraically closed valued field. Let \mathfrak{F} be the family of all C -definable functions on V into Γ . Recall (see Proposition 2.15) that a C -definable type r is stably dominated if and only if for any $f \in \mathfrak{F}$ there exists $f(r) := \gamma \in \Gamma(M)$ such that if $c \models r|C$ then $f(c) = \gamma$.

Definition 6.15. Let q be a C -definable type extending a type q_0 over C , let $(p_t : t \models q_0)$ be a family of stably dominated Ct -definable types concentrating on some variety V over F and U be an open affine of V . We say that the family $(p_t)_{t \models q_0}$ is *uniformly bounded at q (on U)* if for any regular function f on U defined over $D \supseteq C$, there exists $\alpha \in \Gamma$ such that if $t \models q|D\alpha$ and $c \models p_t|Dt\alpha$ then $c \in U$ and $\text{val}(f(c)) \geq \alpha$.

Lemma 6.16. *Let p_t and q be as above. Assume p_t concentrates on a bounded $W \subseteq V$. Then $(p_t)_{t \models q_0}$ is uniformly bounded at q .*

Proof. The types p_t (when $t \models q_0$) concentrate on one of the bounded affine sets W_i in Definition 6.14. Let U be the corresponding affine U_i . Any regular function on U_i is bounded on all of W_i , hence in particular on generic realizations of the p_t . \square

Recall Definition 2.27.

Lemma 6.17. *Let p_t and q be as above. Assume $(p_t)_{t \models q_0}$ is uniformly bounded at q on U . Then there exists a unique C -definable type $p_\infty := \lim_q p_t$ such that for any regular function f on U , if $a \models p_\infty$ then $\text{val}(f(a)) = \lim_q \text{val}(f(p_t))$.*

Moreover, p_∞ is stably dominated and if $h : V \rightarrow W$ is an isomorphism of varieties, then $\lim_q h_ p_t = h_* \lim_q p_t$.*

Proof. By assumption, we cannot have $\lim_q \text{val}(f(p_t)) = -\infty$. The set $\{f : \lim_q \text{val}(f(p_t)) = +\infty\}$ is a prime ideal I ; part of the condition on p_∞ is that $(f = 0) \in p_\infty$ if and only if $f \in I$. Let $V' \subseteq U$ be the zero set of I . The affine coordinate ring of V' is $\mathcal{O}_V(U)/I$. An element of $F(V')$ can, therefore, be written g/h with $g, h \in \mathcal{O}_V(U)$ and $h \notin I$; equivalently $\lim_q q_t(h) \neq \pm\infty$. Hence we can define a valuation $\overline{\text{val}}$ on $F(V')$ by $\overline{\text{val}}(g/h) = \lim_q g(p_t) - \lim_q h(p_t)$. This determines a valued field extension F^+ of F and hence gives a complete type p_∞ of elements of V' . It is clear that p_∞ is definable; and that that $\Gamma(F^+) = \Gamma(F)$, so p_∞ is orthogonal to Γ , and hence, By Proposition 2.15, it is stably dominated. Note that $\text{val}(f(p_\infty)) = \lim_q \text{val}(q_t(f))$ for any $f \in F(V')$. In particular the choice of U is immaterial. The functoriality is evident. \square

Recall Definition 2.28.

Lemma 6.18. *Let G be a bounded C -definable subgroup of an algebraic group \tilde{G} over F . Let H_t be a certifiably stably dominated C -definable family of subgroups of G , where t is a Γ -tuple,*

forming a directed system under inclusion and let $H := \bigcup_t H_t$. Assume G/H is Γ -internal. Then H is stably dominated. Moreover G/H is definably compact.

Proof. Let $q(t)$ be a C -definable type cofinal in the partial ordering $H_t \subseteq H_{t'}$. Let p_t be the principal generic type of H_t . Note that $H = \bigcup_{t=q|C} H_t^0$. By Lemma 6.16 the family (p_t) is uniformly bounded at q , so $p_\infty = \lim_q p_t$ exists.

Claim 6.18.1. *Let H and H' be connected stably dominated definable subgroups of G . Let p and p' be their generic types. Then $H \subseteq H'$ if and only if $p \star p' = p'$.*

Proof. If $H \subseteq H'$, then $p \star p' = p'$ by definition of genericity for p . Conversely if $p \star p' = p'$ then a generic of H is a product of two realizations of p' ; in particular it lies in H' . But any element of H is a product of two generics, hence any element of H lies in H' . \square

By the functoriality of Lemma 6.17, we have $\lim_q {}^a p_t = {}^a(\lim_q p_t)$ for any $a \in G$. Let $s \vDash q|C$ and $t \vDash q|Cs$. If $a \vDash p_s|Cs$, then ${}^a p_t = p_t$, and hence ${}^a p_\infty = p_\infty$. Thus $H_s^0 \subseteq \text{Stab}(p_\infty)$ and thus $H \subseteq \text{Stab}(p_\infty)$. On the other hand, as p_∞ is stably dominated and G/H is Γ -internal, the function $x \mapsto x \cdot H$ is constant on p_∞ , so p_∞ lies in a single coset $x \cdot H$. It follows that $p_\infty \star p_\infty^{-1}$ is generic in H , so H is stably dominated.

Now let r be a definable type on Γ , and let $h : \Gamma \rightarrow G/H$ be a definable function. For any $t \in \Gamma$, let p'_t be the generic type of $h(t)$, viewed as a coset of H ; this is a translate of the generic type of H . Let $p'_\infty = \lim_r p'_t$. Then p'_∞ is stably dominated, so it concentrates on a unique coset of H , corresponding to an element $e \in G/H$. Tracing through the definitions we see that $e = \lim_r h$. Since r and h are arbitrary, G/H is definably compact. \square

The following is an immediate consequence of Theorem 5.15 and Lemma 6.18.

Corollary 6.19. *Let A be an Abelian variety over K . Then there is a definably compact group C defined over Γ , and a definable homomorphism $\phi : A \rightarrow C$ with stably dominated kernel H . In particular A has a unique maximal stably dominated connected ∞ -definable subgroup—which is definable.*

If A is an Abelian scheme over \mathcal{O} , i.e. A_K is an Abelian variety with good reduction, then by Proposition 6.8, $A_K(K) = A(\mathcal{O})$ is stably dominated. Since $A_K(K)$ is divisible, it is connected. Thus, in this case, the definably compact quotient C is trivial.

In general, if F is locally compact, then the set $C(F)$ of points of C lifting to F -points will be a finite subgroup of the definably compact group C . On the other hand if $F = \mathbb{Q}_p((t))$, $C(F)$ can be a finite extension of \mathbb{Z} .

The stably dominated group H is dominated via a group homomorphism $h : H \rightarrow \mathfrak{h}$ to a stable—i.e. a k -internal—group \mathfrak{h} . After base change to a finite extension, \mathfrak{h} becomes isomorphic to an algebraic group over k . It would be interesting to compare this with the classical theory of semi-stable reduction.

6.3. Definable fields. Before we prove the classification of fields definable in algebraically closed valued fields, let us prove a variation on a result of Zilber.

Lemma 6.20. *Let (G, \cdot) and $(A, +)$ be infinite Abelian groups definable in an algebraically closed field K . Assume that G acts definably on A by group automorphisms and that the action is irreducible: i.e. there are no proper non-trivial L -invariant subgroups. Then A has a definable K -vector space structure and G acts linearly on A .*

Proof. Quotienting by the kernel of the action, we may assume that G acts faithfully. By Schur's lemma $R := \text{End}_G(A)$ is a division ring. Since G is Abelian, we can identify G with a multiplicative subgroup of R^* . As in the proof of [Mar02, Theorem 7.8.9], we can show

that the ring generated by G —which is a field—is definable and hence by [Poi88] it is definably isomorphic to K . So we have obtained a definable action of K^* on A by group automorphisms— a definable K -vector space structure— and the action of G factorizes through the action of K^* . \square

Theorem 6.21. *Let F be an infinite field definable in ACVF. Then F is definably isomorphic to the residue field or the valued field.*

Proof. If F is stable, then by [HHM06, Lemma 2.6.2], it is definable in the residue field (over some new parameters), which is a pure algebraically closed field. Hence, by [Poi88], it is definably isomorphic to the residue field. There are no infinite fields definable over Γ , since Γ is a pure ordered divisible Abelian groups.

Let us assume that F is unstable. By Proposition 5.26, there exists a non-definably compact Γ -internal definable homomorphic image of F^* . By [PS99, Theorem 1.2], V contains a definable one-dimensional torsion-free group. Since Γ is a pure ordered divisible Abelian group, such V contains a group isomorphic to $(\Gamma, +)$ —see [EE09, Theorem 1.5] for a more general statement. Hence F is not boundedly imaginary.

Let D be a subrng of F , with $(D, +)$ connected stably dominated, and such that F is the field of fractions of D (cf. Proposition 5.23). There exists a surjective definable map $D \times D \rightarrow F$ —namely $(x, y) \mapsto x/y$ for non-zero y , $(x, 0) \mapsto 0$. Since F is not boundedly imaginary, neither is D . Let f be the universal homomorphism of Remark 6.2 from $(F, +)$ into an algebraic group. Let I be the purely imaginary kernel. For any $c \in F$, $d \mapsto f(c \cdot d)$ is another homomorphism into an algebraic group, so it must factor through f ; thus if $f(d) = 0$ then $f(c \cdot d) = 0$, i.e. I is an ideal of F . If $I = F$, then $(D, +)$ is a purely imaginary stably dominated and by Remark 6.5, D is boundedly imaginary, a contradiction. It follows that $I \neq F$, so $I = (0)$. Hence f is an isomorphism onto a subgroup of an algebraic group G .

Let L be the limit stably dominated subgroup of (F^*, \cdot) and let $H := L \ltimes (F, +)$. Let $h : H \rightarrow G$ be the homomorphism of Proposition 6.1. The kernel is a purely imaginary subset of $H \subseteq F^2$ so it is finite. But H has no finite non-trivial normal subgroups, so h is an embedding and we identify H with its image in G .

We now proceed as in [Pil89], with some local changes of reasoning. The Zariski closure \overline{F} of F is a commutative algebraic group and the Zariski closure \overline{L} acts on F by conjugation. Let $B < \overline{F}$ be a \overline{L} -invariant subgroup of \overline{F} . Then $B \cap F$ is an ideal and thus is trivial. Quotienting by B , we would be able to embed F in a variety of strictly lower dimension than \overline{F} , a contradiction. Applying Lemma 6.20, we see that \overline{F} is a vector space over the valued field K and that the action of \overline{L} is linear. We also denote this action by \cdot since it extends the action of L on F by multiplication. If $r_i \in L$ and $\sum r_i = 0$, then $\sum r_i \cdot y = (\sum r_i) \cdot y = 0$ for any $y \in F$, and by Zariski density of F , for $y \in \overline{F}$. Thus the action may be extended to an action of the ring R generated by L on W , again extending the action by multiplication of R on F . Note that R contains the subrng D constructed in Proposition 5.23 and hence its fraction field is F . Finally, if $0 \neq r \in R$ then r acts on F as an invertible linear transformation, since the image contains $r \cdot F = F$ and hence, by Zariski density, \overline{F} . Thus we can extend the action to a K -linear action of F on \overline{F} which extends the action by multiplication. We also denote that action by \cdot .

Let Z be some non-zero orbit of L on F . Then Z is definably isomorphic to L , and, since F^*/L is Γ -internal, it has the same C -minimal dimension as F . So F contains a non-empty open subset U of \overline{F} . Pick any $c \in U$, then for any $\alpha \in \mathcal{O}$ with $\text{val}(\alpha - 1)$ sufficiently large, $\alpha c \in U$ and thus there exists $h \in L$ such that $h \cdot c = \alpha c$. Since F is a field, h is uniquely defined. For any $b \in L$, we have $\alpha(b \cdot c) = b \cdot (\alpha c) = b \cdot (h \cdot c) = h \cdot (b \cdot c)$ so the action by h and scalar multiplication

by α agree on Z . Since Z has the same C -minimal dimension as \overline{F} , the linear span of Z is \overline{F} and those two linear functions actually agree on \overline{F} .

Let $E = \{\alpha \in K : (\exists b \in F)(\forall x \in W)\alpha x = b \cdot x\}$. This is clearly a subfield of K , and it contains a neighborhood of 1. Hence it contains a neighborhood N of 0. If $0 \neq x \in K$ then $x^{-1}N \cap N$ is open and contains some non-zero u , so $x \cdot u \in N$ and $x = (x \cdot u)/u \in E$. It follows that $E = K$. Moreover, the map sending any $\alpha \in K$ to $b \in F$, such that the action by b coincides with scalar multiplication by α , is an embedding of rings. Since F has bounded dimension, for some m no definable subset of F admits a definable map onto K^m ; so $\dim_K F < m$. Since K is algebraically closed we have $K \simeq F$. \square

We naturally expect any infinite non-Abelian definably simple group definable in ACVF to be isomorphic to an algebraic group defined over the residue field or the valued field. A proof along the above lines may be possible assuming a positive solution to Problem 1.9 for ACVF, along with an interpretation of an ordered proper semi-group structure on $H \backslash G/H$.

The existing metastable technology yields a proof of the Abelian case:

Proposition 6.22. *Let A be a non-trivial Abelian group definable in ACVF. Then there exist definable subgroups $B < C \leq A$ and C/B definably isomorphic (with parameters) to an algebraic group over the residue field or a definable group over the value group.*

Proof. Let $\lambda : A \rightarrow \Lambda$ be as in Theorem 5.15. If $\lambda(A) \neq 0$ we can take $C = A$ and $B = \ker(\lambda)$. If $\lambda(A) = 0$, then A is limit stably dominated. In particular, A contains a non-zero stably dominated definable Abelian group C and, by Proposition 4.6, C has a non trivial stable quotient. Since all stable definable sets in ACVF are internal to k and k is a pure algebraically closed fields, it follows that this quotient is (isomorphic to) an algebraic group over k . \square

6.4. Residually Abelian groups.

Example 6.23. In ACVF, there exists connected stably dominated non-Abelian groups, with Abelian stable part:

- (1) Let $A = \mathbb{G}_a^2$, and let $\beta : A^2 \rightarrow \mathbb{G}_a$ be a non-symmetric bilinear map defined over the prime field, e.g. $\beta((a_1, a_2), (b_1, b_2)) = a_1 b_2 - a_2 b_1$. Let t be an element with $\text{val}(t) > 0$ and consider the following group law on A^2 : $(a, b) \star (a', b') = (a + a', b + b' + t\beta(a, a'))$.
- (2) If we work in $\mathbb{G}_a(\mathcal{O}/t^2\mathcal{O})$ where $\text{val}(t) > 0$, we can also just take $a \star b = a + b + t\beta(a, b)$. This dimension 1 example is due to Simonetta [Sim03]. This group does not lift to an algebraic group over \mathcal{O} .

However, we may ask if any connected stably dominated group of weight 1 definable is nilpotent.

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