

Definable equivalence relations and zeta functions of groups

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with an appendix by Raf Cluckers

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In memory of Fritz Grunewald.

Abstract

We prove that the theory of the p -adics \mathbb{Q}_p admits elimination of imaginaries provided we add a sort for $\mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p)$ for each n . We also prove that the elimination of imaginaries is uniform in p . Using p -adic and motivic integration, we deduce the uniform rationality of certain formal zeta functions arising from definable equivalence relations. This also yields analogous results for definable equivalence relations over local fields of positive characteristic. The appendix contains an alternative proof, using cell decomposition, of the rationality (for fixed p) of these formal zeta functions that extends to the subanalytic context.

As an application, we prove rationality and uniformity results for zeta functions obtained by counting twist isomorphism classes of irreducible representations of finitely generated nilpotent groups; these are analogous to similar results of Grunewald, Segal and Smith and of du Sautoy and Grunewald for subgroup zeta functions of finitely generated nilpotent groups.

1 Introduction

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This paper concerns the model theory of the p -adic numbers \mathbb{Q}_p and applications to certain counting problems arising in group theory. Recall that a theory (in the model-theoretic sense of the word) is said to have *elimination of imaginaries* (EI) if the following holds: for every model M of the theory, for every definable subset D of some M^n and for every definable equivalence relation R on D , there exists a definable function $f: D \rightarrow M^m$ for some m such that the fibers of f over $f(D)$ are precisely the equivalence classes of R . In other words, elimination of imaginaries states that every pair (D', E') (consisting of a definable set D' and a definable equivalence relation E' on it) reduces to a pair (D, E) where E is equality—here, as in descriptive set theory, we say that (D', E') *reduces to* (D, E) if there exists a \emptyset -definable map $f: D' \rightarrow D$ with $xE'y \iff f(x)Ef(y)$.

The theory of \mathbb{Q}_p (in the language of rings with a predicate for $\text{val}(x) \geq \text{val}(y)$) does not admit EI [61]: for example, no such f exists for the definable equivalence relation R on \mathbb{Q}_p given by xRy if $\text{val}(x - y) \geq 1$, because \mathbb{Q}_p/R is countably infinite but any definable subset of \mathbb{Q}_p^m is either finite or uncountable. Our first main theorem gives a p -adic EI result when we add for each n a sort S_n for the family of \mathbb{Z}_p -lattices in \mathbb{Q}_p^n .

Theorem 1.1 *The theory of \mathbb{Q}_p eliminates imaginaries in the language $\mathcal{L}_{\mathcal{G}}^-$.*

To be precise, we prove a version of this that holds for any finite extension of \mathbb{Q}_p (Theorem 2.6); see Section 2, p.10 for a precise definition of the language $\mathcal{L}_{\mathcal{G}}^-$.

Suppose we are given not just a single definable equivalence relation for some fixed \mathbb{Q}_p , but one for every \mathbb{Q}_p . For our applications to zeta functions below, we want to control the behavior of the elimination of imaginaries as we vary the prime p . Our second main result is that the theory of ultraproducts of \mathbb{Q}_p also eliminates imaginaries if we add similar sorts.

Theorem 1.2 *The theory of non-principal ultraproducts $\prod_p \mathbb{Q}_p/\mathcal{U}$ eliminates imaginaries in the language $\mathcal{L}_{\mathcal{G}}^-$ provided we add some constants.*

See Theorem 2.7 for a more precise statement of what constants are needed to eliminate imaginaries. This last result implies that the elimination of imaginaries in \mathbb{Q}_p is uniform in p ; see Corollary 2.9 for a precise statement of this uniformity.

In fact, we prove a more general result (Corollary 2.17), which yields EI both for \mathbb{Q}_p and for ultraproducts: given two theories T, \tilde{T} satisfying certain hypotheses, T has EI if \tilde{T} does. In our application, \tilde{T} is the theory of algebraically closed valued fields of mixed characteristic (ACVF $_{0,p}$) or equicharacteristic zero (ACVF $_{0,0}$) and T is

either the theory of a finite extension of \mathbb{Q}_p or the theory of an ultraproduct of \mathbb{Q}_p where p varies, with appropriate extra constants in each cases (in fact in the latter case Corollary 2.17 does not apply immediately but a variant does).

The notion of an invariant extension of a type plays a key part in our proof. If T is a theory, $M \models T$, $A \subseteq M$ and p is a type over A then an *invariant extension* of p is a type q over M such that $q|A = p$ and q is $\text{Aut}(M/A)$ -invariant. The theory ACVF is not stable; in [36, 37], Haskell, the first author and Macpherson used invariant extensions of types to study the stability properties of ACVF and to define notions of forking and independence. They proved that ACVF plus some extra sorts admits EI.

As an important consequence of Theorems 2.6 and 2.7, we prove the following rationality and uniformity result for zeta functions $S_p(t)$ counting the number of equivalence classes in some uniformly definable family of equivalence relations. (Here $t = t_1, \dots, t_r$ is a tuple of indeterminates and $S_p(t)$ is a power series in the t_i ; we obtain a zeta function in the more usual sense by setting $t_i = p^{-s_i}$, where the s_i are complex variables.)

Theorem 1.3 *The zeta functions $(S_p(t))_{p \text{ prime}}$ are uniformly rational.*

We also give a version of Theorem 1.3 for a uniformly definable family of equivalence relations over a local field of positive characteristic (Corollary 6.7).

See Section 6 for definitions and a precise statement (Theorem 6.1). Roughly speaking, uniform rationality means that each $S_p(t)$ can be expressed as a rational function with coefficients in \mathbb{Q} , where the denominator is a product of functions of the form $(1 - p^a t^b)$ or p^n with a, b, n independent of p , and the numerator is a polynomial in t such that each coefficient comes from counting the \mathbb{F}_p -points of a fixed variety over \mathbb{Z} . In particular, we obtain that $S_p(t)$ is rational not just for all sufficiently large primes, but for every prime; this is crucial for our applications to representation growth below, as well as to the following result, which deals with the abscissa of convergence.

Theorem 1.4 *Let $S_p(t)$ be as above and suppose we are in the one-variable case ($r = 1$, $t = t_1$). Define $\zeta_p(s) = S_p(p^{-s})$. Assume that the constant term of $\zeta_p(s)$ is 1 for all but finitely many primes and set $\zeta(s) = \prod_p \zeta_p(s)$. Then the abscissa of convergence of $\zeta(s)$ is rational (or $-\infty$).*

In fact, Theorem 6.1 yields a kind of “double uniformity”: the ultraproduct formalism allows us to vary not just the prime p , but also the choice of an extension L_p of \mathbb{Q}_p . For an application of this double uniformity, see the end of Section 8.

To describe the proof of Theorem 1.3, let us now come back to the meaning of our elimination of imaginaries result. It shows that any (D', E') can be reduced to

a (D, E) of a special kind—namely, the equivalence relation on $\mathrm{GL}_N(\mathbb{Q}_p)$ for some N whose equivalence classes are the left $\mathrm{GL}_N(\mathbb{Z}_p)$ -cosets. The quotients D/E have a specific geometric meaning—but can one explain abstractly in what way they are special? One useful observation is that we have reduced an arbitrary equivalence relation to a quotient by a definable group action. Another concerns volumes: the E -classes have volumes that are *motivically invertible* (in fact, each class is equivalent to a polydisk of an appropriate dimension and size).

Indeed, it is only the latter property of the geometric imaginaries that is actually used in the proof of Theorem 6.1. This proof relies on representing the number of classes of some definable equivalence relation E on some definable set D as an integral. The idea, going back to Denef and Igusa, is simple: the number of classes of E on D equals the volume of D , for any measure such that each E -class has measure one. The question is how to come up (definably) with such a measure. The setting is that we already have the Haar measure μ on \mathbb{Q}_p (normalized so that $\mu(\mathbb{Z}_p) = 1$), and for simplicity—one can easily reduce to this case—let us assume each E -class $[x]_E \in D/E$ has finite, nonzero measure. The problem then is to show that there exists a definable function $f: D \rightarrow \mathbb{Q}_p$ such that the measure of each E -class $[x]_E$ is of the form

$$\mu([x]_E) = |f(x)|, \tag{1.1}$$

and then replace μ with $|f|^{-1}\mu$. In practice, f is usually given explicitly (cf. [35, Section 2]). For more complicated equivalence relations, however, such as the one for representation zeta functions in Section 8, it is not clear *a priori* that such an f can be found, even in principle.

This point is beautifully brought out in work by Raf Cluckers; we are very pleased to have his permission to include it here as an Appendix. The Appendix contains a complete proof of the rationality results in Section 6 for fixed p which also extends to the analytic case, while avoiding an explicit elimination of imaginaries. It might be useful to say a word here about the two proofs. Given EI to the geometric sorts, we can represent E as the coset equivalence relation of $\mathrm{GL}_n(\mathbb{Z}_p)$. In this case we can take the measure in our p -adic integral to be the Haar measure on $\mathrm{GL}_n(\mathbb{Z}_p)$, where each class automatically has measure one. Equivalently, with respect to the additive Haar measure, we use the explicit factor $1/|\det(M)|$. In other words, for this canonical E , the reciprocal of the (additive) Haar measure of any E -class *is* represented by a definable function.

In the Appendix, any equivalence relation (D', E') is reduced to one with motivically invertible volumes. Indeed there exists a \emptyset -definable $D \subseteq D'$ such that $D \cap e$ has motivically invertible volume for each E' -class e (in particular, with $E = E'|_D$, the

natural map $D/E \rightarrow D'/E'$ is a bijection). This result is valid in the analytic case too, unlike geometric EI in its present formulation (see [38]).

We illustrate the power of Theorem 6.1 by using it to prove rationality results for certain zeta functions of finitely generated nilpotent groups (Theorems 7.2 and 1.5). Grunewald, Segal and Smith [35, Sec. 2] showed that subgroups of p -power index of such a group Γ can be parametrized p -adically if Γ is also torsion-free. More precisely, these subgroups can be *interpreted*: that is, placed in bijective correspondence with the set of equivalence classes of some definable equivalence relation on a definable subset D of some \mathbb{Q}_p^N . Let $b_n < \infty$ denote the number of subgroups of Γ of index n . Using p -adic integration over D and results of Denef and Macintyre, Grunewald, Segal and Smith showed that the p -local subgroup zeta function $\sum_{n=0}^{\infty} b_{p^n} t^n$ is a rational function of t , and that the degrees of the numerator and denominator of this rational function are bounded independently of p . Du Sautoy and others have calculated subgroup zeta functions explicitly in many cases [28, 30, 65] and studied uniformity questions. For instance, du Sautoy and Grunewald proved a uniformity result by showing that the p -adic integrals that arise in the calculation of subgroup zeta functions fall into a special class they call *cone integrals* [29]. See the start of Section 7 for further discussion of uniformity in the context of subgroup zeta functions.

We also consider situations where it is not clear how to find a definable function f satisfying Eqn. (1.1) and construct suitable definable p -adic integrals. The main one, and the original motivation for our results, is in the area of representation growth. This is analogous to subgroup growth: one counts not the number b_{p^n} of index p^n subgroups of a group Γ , but the number a_{p^n} of irreducible p^n -dimensional complex characters of Γ (modulo tensoring by one-dimensional characters if Γ is nilpotent). Here is our main result on representation zeta functions. Let $\zeta_{\Gamma,p}(s) = \sum_{n=0}^{\infty} a_{p^n} p^{-ns}$.

Theorem 1.5 *The p -local representation zeta functions $(\zeta_{\Gamma,p}(s))_{p \text{ prime}}$ of a finitely generated nilpotent group Γ are uniformly rational. Moreover, the global representation zeta function $\zeta_{\Gamma}(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ has rational abscissa of convergence.*

The results in Sections 7 and 8 both follow the same idea: we show how to interpret (uniformly and definably) in \mathbb{Q}_p the sets we want count. More precisely, in Section 7 we show how to interpret in $\text{Th}(\mathbb{Q}_p)$ the set of finite-index subgroups H of Γ and we show that the equivalence relations that arise are uniformly definable in p . This allows us to apply Theorem 6.1. The same idea is used in Section 8, but the details are more complicated. We show how to interpret in $\text{Th}(\mathbb{Q}_p)$ the set of pairs (N, σ) , where N is a finite-index normal subgroup of Γ and σ is an irreducible character of Γ/N , up to twisting by one-dimensional characters. The key idea is first to interpret triples

(H, N, χ) , where H is a finite-index subgroup of Γ , N is a finite-index normal subgroup of H and χ is a one-dimensional character of H/N —the point is that finite nilpotent groups are monomial, so any irreducible character is induced from a one-dimensional character of a subgroup. The equivalence relation of giving the same induced character can be formulated in terms of restriction, and shown to be definable. Inspecting these constructions shows that they are all uniform in p , so again Theorem 6.1 applies.

Since the first draft of this paper [42] was circulated, there has been considerable activity in the field of representation growth. Jaikin (*loc. cit.*) used the coadjoint orbit formalism of Howe and Kirillov to parametrize irreducible characters of p -adic analytic groups; rationality of the representation zeta function then follows from the usual arguments of semi-simple compact p -adic integration. Voll used similar ideas to parametrize irreducible characters of finitely generated torsion-free nilpotent groups, and showed that representation zeta functions are rational and satisfy a local functional equation [64] (in fact, he proved this for a very general class of zeta functions that includes representation zeta functions and subgroup zeta functions as special cases). Stasinski and Voll proved a uniformity result for representation zeta functions and calculated these zeta functions for some families of nilpotent groups [63, Thms. A and B]. Ezzat [34], [32], [33] and Snocken [62] calculated further examples of representation zeta functions of nilpotent groups. For work on representation growth for other kinds of group, see [49], [47], [4], [5], [3], [6], [1], [2], [8], [7].

The Kirillov orbit method has the advantage that it linearises the problem of parametrizing irreducible representations and simplifies the form of the imaginaries that appear. The disadvantage is that the proof of rationality only applies to $\zeta_{\Gamma,p}(s)$ for almost all p —one must discard a finite set of primes. *We stress that our result Theorem 1.5 is the only known proof of rationality of $\zeta_{\Gamma,p}(s)$ that works for every p .*

This paper falls naturally into two parts. The first part is model-theoretic: in Section 2 we establish an abstract criterion, Proposition 2.13, for elimination of imaginaries and apply it in Sections 4 and 5 to prove Theorems 2.6 and 2.7. Section 3 consists of a study of unary types in henselian valued fields, which is used extensively in Sections 4 and 5. In Section 6 we establish the general rationality result Theorem 6.1, we prove Theorem 6.3 and we show how the techniques developed in this paper can be used to prove transfer results between local fields of positive characteristic and mixed characteristic.

In the second part (Sections 7 and 8), we apply Theorem 6.1 to prove Theorems 7.2 and 1.5. The main tools are results from profinite groups; no ideas from model theory are used in a significant way beyond the notion of definability. We finish Section 8 by using the Kirillov orbit formalism and Theorem 6.1 to prove a double uniformity result (Theorem 8.12) for the representation zeta functions of groups of points of a smooth

unipotent group scheme.

Finally, the Appendix contains an alternative proof, Theorem A.2, of the rationality results of Section 6 for fixed p that generalizes to the analytic setting. An important application of this work of Cluckers is that it allows one to prove rationality of the subgroup and representation zeta functions associated to a compact semisimple p -adic analytic group. Here the methods of the main paper do not go through because one needs to use an extended language containing symbols for analytic functions, and elimination of imaginaries is not known to hold in this setting. We return to this point briefly below (see the paragraph after Remark 7.3), but we leave the full details to the interested reader.

Note: A draft of this paper [42] has been available for almost ten years now. Alongside the previous theorems concerning \mathbb{Q}_p for fixed p , the present version also contains new material on the model theory of ultraproducts of \mathbb{Q}_p , which allows us to prove the uniformity as p varies of the previous elimination of imaginaries and rationality theorems, as well as a transfer result between positive equicharacteristic and mixed characteristic. There is extra material on representation growth and a new Appendix on cell decomposition methods.

2 Elimination of imaginaries

2.1 Definition and first properties

We denote by \mathbb{N} ($\mathbb{N}_{>0}$) the non-negative (positive) integers, respectively. For standard model-theoretic concepts and notation such as dcl (definable closure) and acl (algebraic closure) we refer the reader to any introduction to model theory, e.g., the first chapter of [37].

Notation 2.1 If X is a definable (possibly ∞ -definable) set in some structure M and $A \subseteq M$, we will write $X(A) := \{a \in A : M \models X(a)\}$. If we want to make the parameters of X explicit, we will write $X(A; b)$.

We say that the definable set X is *coded* (in M) if it can be written as $R(M; b)$, where b is a tuple of elements of M , and where $b \neq b'$ implies that $R(M; b) \neq R(M; b')$. In this situation $\text{dcl}(b)$ depends only on X and is called a *code* for X . It is denoted $\langle X \rangle$. We say T *eliminates imaginaries* (EI) if every definable relation in every model of T is coded. Equivalently, if there are at least two constants, T eliminates imaginaries

if for any definable equivalence relation E there is a definable function whose fibers are exactly the equivalence classes of E (cf. [56, Lemme 2]).

For any theory T , by adding sorts for every definable quotient we obtain a theory T^{eq} that has elimination of imaginaries. These new sorts are called *imaginary sorts* and the old sorts from T are called the *real sorts*. Similarly, to any model M of T we can associate a (unique) model M^{eq} of T^{eq} that has the same real sorts as M . In general, we use the notation $\langle X \rangle$ to refer to the code of X with respect to T^{eq} . We will denote by dcl^{eq} the definable closure in M^{eq} and similarly for acl^{eq} .

We will consider many-sorted theories with a distinguished collection \mathcal{S} of sorts, referred to as the *dominant* sorts; we assume that for any sort S , there exists a \emptyset -definable partial function from a finite product of dominant sorts onto S (and this function is viewed as part of the presentation of the theory). The set of elements of dominant sorts in a model M is denoted $\text{dom}(M)$.

The following lemma and remark—which reduce elimination of imaginaries to coding certain functions—will not be used explicitly in the p -adic case, but they are an essential guideline as unary functions of the kind described in the remark are central to the proof of Proposition 2.13.

Lemma 2.2 (cf. [36, Remark 3.2.2]) *A theory T admits elimination of imaginaries if every function definable (with parameters) whose domain is contained in a dominant sort is coded in any model of T .*

Proof It suffices to show that every definable function f is coded. Pulling back by the given \emptyset -definable functions, it suffices to show that every definable function whose domain is contained in a product $M_1 \times \dots \times M_n$ of dominant sorts is coded. For $n = 1$, this is our assumption. For larger n , we use induction, regarding a definable function $f : M_1 \times \dots \times M_n \rightarrow M^k$ as the function f' mapping $c \in M_1$ to the code of the function $y \mapsto f(c, y)$. By compactness there are definable functions h_i covering f' . The codes of these h_i allow us to code f . \square

Remark 2.3 In Lemma 2.2, it suffices to consider definable functions $f = f_e$ on a definable subset D of a dominant sort, defined with a parameter e from an imaginary sort Y , such that *either* f is the identity on its domain (unary EI), *or* there exists an $A \subseteq M$ such that $e \in \text{dcl}^{\text{eq}}(A, f_e(c))$ for any $c \in D$ —i.e., f_e is determined by any one of its values—and $\text{tp}(e/A)$ implies the type of e over Ac for any $c \in D$.

Indeed, let e be imaginary. There exist $c_1, \dots, c_n \in \text{dom}(M)$ such that $e \in \text{dcl}^{\text{eq}}(c_1, \dots, c_n)$. Let $A_l = \text{dcl}^{\text{eq}}(e, c_1, \dots, c_l) \cap M$. We know that $e \in \text{dcl}^{\text{eq}}(A_n)$ and we want to show that $e \in \text{dcl}^{\text{eq}}(A_0) = \text{dcl}^{\text{eq}}(\text{dcl}^{\text{eq}}(e) \cap M)$; i.e., e is interdefinable with a tuple of real elements.

Let us proceed by reverse induction. Suppose $e \in \text{dcl}^{\text{eq}}(A_{l+1})$, let $A = A_l$ and let $c = c_{l+1}$. Then over Ac , e is interdefinable with a real tuple d ; so $d = f(e, c)$ and $e = h(d)$ for some A -definable functions f, h . By unary EI, any Ae -definable subset of a dominant sort is already A -definable. Thus, by hypothesis, $e = h(f(e, c'))$ for any $c' \models \text{tp}(c/A)$. Let D be an A -definable set with $c \in D$ and such that $e = h(f(e, c'))$ for any $c' \in D$. Then $f_e(y) = f(e, y)$ has a code in $A = A_l$ and we have $e \in \text{dcl}^{\text{eq}}(A_l)$.

Definition 2.4 We will say that a theory T *eliminates imaginaries up to uniform finite imaginaries (EI/UFI)* if for all $M \models T$ and $e \in M^{\text{eq}}$, there exists a tuple $d \in M$ such that $e \in \text{acl}^{\text{eq}}(d)$ and $d \in \text{dcl}^{\text{eq}}(e)$.

The theory T is said to *eliminate finite imaginaries (EFI)* if any $e \in \text{acl}^{\text{eq}}(\emptyset)$ is interdefinable with a tuple from M .

Let us now give a criterion for elimination of imaginaries from [39].

Lemma 2.5 *A theory T eliminates imaginaries if it eliminates imaginaries up to uniform finite imaginaries and for every set of parameters A , T_A eliminates finite imaginaries.*

Proof Let $e \in M^{\text{eq}} \models T^{\text{eq}}$. Then by EI/UFI, there exists $d \in M$ such that $e \in \text{acl}^{\text{eq}}(d)$ and $d \in \text{dcl}^{\text{eq}}(e)$. Hence e is a finite imaginary in T_d and there exists $d' \in M$ such that $e \in \text{dcl}^{\text{eq}}(dd')$ and $dd' \in \text{dcl}^{\text{eq}}(ed) = \text{dcl}^{\text{eq}}(e)$, i.e., e is coded by dd' . \square

2.2 Valued fields

If F is a field then we denote by $\overline{F}^{\text{alg}}$ the algebraic closure of F . Let L be a valued field, with valuation ring $\mathcal{O}(L)$, maximal ideal $\mathcal{M}(L)$ and residue field $k(L)$. We will find it convenient to consider the value group $\Gamma(L)$ in both an additive notation (with valuation $\text{val} : L \rightarrow \Gamma(L) \cup \{\infty\}$) and a multiplicative notation (with reverse order and absolute value $|\cdot|$), depending on the setting. We will consider valued fields in the geometric language whose sorts (later referred to as the geometric sorts) are as follows. We take a single dominant sort K , for L itself. The additional sorts S_n, T_n for $n \in \mathbb{N}$ are given by

$$S_n := \text{GL}_n(K)/\text{GL}_n(\mathcal{O}) = \text{B}_n(K)/\text{B}_n(\mathcal{O}),$$

the set of lattices in K^n , and

$$T_n := \text{GL}_n(K)/\text{GL}_{n,n}(\mathcal{O}) = \bigcup_{m \leq n} \text{B}_n(K)/\text{B}_{n,m}(\mathcal{O}) = \bigcup_{e \in S_n} e/\mathcal{M}e.$$

Here a *lattice* is a free \mathcal{O} -submodule of K^n of rank n , B_n is the group of invertible upper triangular matrices, $GL_{n,m}(\mathcal{O})$ is the group of matrices in $GL_n(\mathcal{O})$ whose m th column reduces mod \mathcal{M} to the column vector of k having a one in the m th entry and zeroes elsewhere, and $B_{n,m}(\mathcal{O}) := B_n(\mathcal{O}) \cap GL_{n,m}(\mathcal{O})$. There is a canonical map from T_n to S_n taking $f \in e/\mathcal{M}e$ to the lattice e .

It is easy to see, using elementary matrices, that $GL_n(K) = B_n(K)GL_n(\mathcal{O})$, justifying the equivalence of the first two definitions of S_n . Equivalently, it is shown in [36, Lemma 2.4.8] that every lattice has a basis in triangular form.

Note that there is an obvious injective \emptyset -definable function $S_m \times S_{m'} \rightarrow S_{m+m'}$, namely $(\lambda, \lambda') \mapsto \lambda \times \lambda'$, so we can identify any subset of a product of S_{n_i} with a subset of S_n , where $n = \sum_i n_i$.

Note also that S_1 can be identified with Γ by sending the coset $c\mathcal{O}^*$ to $v(c)$. Then k can be identified with the fiber of $T_1 \rightarrow S_1$ above the zero element of $S_1 = \Gamma$. More generally, let $\overline{\mathbb{B}} = \{\{x : \text{val}(x - a) \geq \text{val}(b)\} : a, b \in K\}$ and let $\mathbb{B} = \{\{x : \text{val}(x - a) > \text{val}(b)\} : a, b \in K\}$ be the sets of *closed* (respectively *open*) balls with center in K and radius in $v(K)$. Then $\overline{\mathbb{B}}$ embeds into $S_2 \cup K$ and \mathbb{B} into T_2 . Indeed, the set of closed balls of radius $+\infty$ is identified with K . The group $G(K)$ of affine transformations of the line acts transitively on the closed balls of nonzero radius; the stabilizer of $\mathcal{O} \in \overline{\mathbb{B}}$ is $G(\mathcal{O})$, so $\overline{\mathbb{B}} \setminus K \cong G(K)/G(\mathcal{O}) \subseteq GL_2(K)/GL_2(\mathcal{O})$. The group $G(K)$ also acts transitively on \mathbb{B} and the stabilizer of $\mathcal{M} \in \mathbb{B}$ is $G(K) \cap GL_{2,2}(\mathcal{O})$. We will write $\mathbb{B} := \overline{\mathbb{B}} \cup \mathbb{B}$ for the set of all balls. Note, however, that if Γ has a smallest positive element, the open balls are also closed balls.

In Sections 3 and 5, we will also consider the sort $RV := K^*/(1 + \mathcal{M})$ and the canonical projection $rv : K^* \rightarrow RV$. The structure on RV is given by its group structure and the structure induced by the exact sequence $k^* \rightarrow RV \rightarrow \Gamma$, where the second map is denoted val_{rv} —i.e., we have a binary predicate interpreted as $\text{val}_{rv}(x) \leq \text{val}_{rv}(y)$, a unary predicate interpreted as k^* , and the ring structure on k (adding a zero to k^*). This exact sequence induces on each fiber of val_{rv} the structure of a k -vector space (if we add a zero to the fiber). When $T \supseteq HF_0$ (the theory of henselian valued fields with residue field of characteristic zero), RV is stably embedded and the structure induced on RV is exactly the one described above. Note that we can identify RV with T_1 if we add a zero to each fiber of val_{rv} .

The theory of a structure is determined by the theory of the dominant sorts; so, for any field L we can speak of $\text{Th}(L)$ in the geometric sorts. We take the geometric language $\mathcal{L}_{\mathcal{G}}$ to include the ring structure on the sort K and the natural maps $GL_n(K) \rightarrow S_n(K)$ and $GL_n(K) \times K^n \rightarrow T_n(K)$.

In [36], it is shown that ACVF eliminates imaginaries in $\mathcal{L}_{\mathcal{G}}$. Let us now give the counterpart of this theorem for p -adic fields.

By \mathcal{L}_G^- , we denote the restriction of \mathcal{L}_G to the sorts K and S_n . For each subset $N \subseteq \mathbb{N}_{>0}$, we will also consider an expansion \mathcal{L}_G^N of \mathcal{L}_G^- by a constant a and for all $n \in N$ a tuple of constants c_n of length n in the field sort.

By a *uniformizer* of a valued field we mean an element a whose valuation is positive and generates the value group. By an (*unramified*) *n-Galois uniformizer* we mean a tuple c of elements of the valuation ring whose residues generate $T(k)/T(k)^n$, where T is the restriction of scalars of \mathbb{G}_m from $k[\omega_n]$ to k for ω_n some primitive n th root of unity. (Here $T(k)^n$ denotes the n th powers in $T(k)$.) Note that because $k[\omega_n]$ is of degree at most n over k , c is a tuple of length at most n . Adding some zeroes at the end of the tuple if necessary, we may assume it is a tuple of length exactly n .

Let PL_0 be the theory of pseudo-local fields of residue characteristic 0, i.e., henselian fields with value group a \mathbb{Z} -group (i.e., an ordered group elementarily equivalent to $(\mathbb{Z}, 0, <)$) and residue field a pseudo-finite field of characteristic 0. So PL_0 is the theory of ultraproducts $\prod \mathbb{Q}_p/\mathcal{U}$ of p -adics over non-principal ultrafilters on the set of primes. In fact, let \mathfrak{L}_p be a set of finite extensions of \mathbb{Q}_p and let $\mathfrak{L} = \bigcup_p \mathfrak{L}_p$. Any ultraproduct $\prod_{L \in \mathfrak{L}} L/\mathcal{U}$ of residue characteristic zero—i.e., such that the ultrafilter \mathcal{U} does not contain any set included in some \mathfrak{L}_{p_0} —is a model of PL_0 . Note that if \mathfrak{L}_p is nonempty for infinitely many p then there exists an ultrafilter \mathcal{U} on \mathfrak{L} such that $\prod_{L \in \mathfrak{L}} L/\mathcal{U}$ has residue characteristic zero.

Let L be a valued field, regarded as an \mathcal{L}_G^- -structure, and let p be its residue characteristic. A *proper expansion* of L to \mathcal{L}_G^N is a choice of a and c_n for each $n \in N$ such that:

1. a is a uniformizer;
2. if n is prime to p or $p = 0$ then c_n is an unramified n -Galois uniformizer if one exists, and 0 otherwise;
3. if $p \neq 0$, p divides n and $n > p$ then c_n is zero;
4. if $p \neq 0$, $p \in N$ and L is not a finite extension of \mathbb{Q}_p then $c_p = 0$;
5. if $p \neq 0$, $p \in N$ and L is a finite extension of \mathbb{Q}_p then the first component of c_p is a generator of L over \mathbb{Q}_p that is algebraic over \mathbb{Q} , and the other components are zero.

The point of 5. is to ensure we have a constant that generates L over \mathbb{Q}_p in the local field case when $p \in N$; the precise details of how we handle c_n if p divides n are not important because in the arguments below we are always free to make N larger if we wish.

Note that because there are only finitely many possibilities for the minimal polynomial of ω_n over k , the class of proper expansions to $\mathcal{L}_{\mathcal{G}}^N$ of models of PL_0 is elementary. Let us denote this class by PL_0^N . Note also that a residue characteristic zero ultraproduct of proper expansions to $\mathcal{L}_{\mathcal{G}}^N$ of $L \in \mathfrak{L}$ is a model of PL_0^N .

Here is a precise statement of the two main elimination of imaginaries results of this paper. The first is for finite extensions of \mathbb{Q}_p :

Theorem 2.6 *The theory of \mathbb{Q}_p eliminates imaginaries in $\mathcal{L}_{\mathcal{G}}^-$. The same is true for any finite extension L of \mathbb{Q}_p , provided one adds a constant symbol for a generator of $L \cap \overline{\mathbb{Q}}^{\text{alg}}$ over $\mathbb{Q}_p \cap \overline{\mathbb{Q}}^{\text{alg}}$.*

The second is for their ultraproducts of residue characteristic zero:

Theorem 2.7 *$\text{PL}_0^{\mathbb{N}}$ eliminates imaginaries in $\mathcal{L}_{\mathcal{G}}^{\mathbb{N}}$.*

Remark 2.8 1. Although the T_n are needed to obtain EI in algebraically closed fields, they are not needed here. Indeed, if a valued field L has a discrete valuation (i.e., the value group has a smallest positive element $\text{val}(\lambda_0)$), then for any lattice e , $\lambda_0 e$ is itself a lattice, and a coset h of $\lambda_0 e$ —a typical element of T_n —can be coded by the lattice in K^{n+1} generated by $h \times \{1\}$. Hence all elements of $T_n(L)$ are coded in $S_{n+1}(L)$.

2. In Theorem 2.6, we need to add constants for elements of a subfield F with a certain number of properties:

- (a) F contains a uniformizer;
- (b) $\text{res}(F) = k$;
- (c) $\overline{K}^{\text{alg}} = \overline{F}^{\text{alg}} K$ (in fact, we need that for every finite extension L of K there is a generator of $\mathcal{O}(L)$ whose minimal polynomial is over F).

It suffices to take $F = \mathbb{Q}[c]$, where c generates L over \mathbb{Q}_p . Note that we can choose such a c that belongs to $\overline{\mathbb{Q}}^{\text{alg}}$.

Note also that a proper expansion of some finite extension L of \mathbb{Q}_p to $\mathcal{L}_{\mathcal{G}}^{\mathbb{N}}$, contains a generator of L over \mathbb{Q}_p . Hence proper expansions of L eliminate imaginaries in $\mathcal{L}_{\mathcal{G}}^{\mathbb{N}}$.

3. In Theorem 2.7, we need to name elements of a subfield F satisfying (a), (c) as above, and:

- (d) $\text{res}(F)(k^*)^n = k$ for all n ;
- (e) k admits EI in the language of rings augmented by constants for elements of $\text{res}(F)$.

We can choose F to be generated by a uniformizer a and unramified n -Galois uniformizers c_n for all n . It is clear that such an F satisfies (a). Furthermore, $\overline{k}^{\text{alg}} = \overline{\text{res}(F)}^{\text{alg}} k$. Indeed, let ω_n be a primitive n th root of unity, and let $d_n = \sum_i c_{n,i} \omega_n^i$. The degree n extension of $k[\omega_n]$ is contained in $k[\omega_n, \sqrt[n]{d_n}]$ by Kummer theory and it contains the degree n extension of k .

Now (c) is a consequence of (a) and the above statement and (e) also follows as any extension of degree n is generated by an element in $\overline{\text{res}(F)}^{\text{alg}}$, so there is an irreducible polynomial of degree n with $\text{res}(F)$ -definable parameters; this is the hypothesis of [10, Proposition B.(3)]. Finally for any n , there is a d such that $\{x \in k : x^n = 1\} = \{x \in k : x^d = 1\}$ and k contains primitive d th roots of unity. Then $c_d \in k$ generates $k^*/(k^*)^d = k^*/(k^*)^n$, so (d) holds.

4. It would be nice to find a more precise description of the imaginaries if no constants are named. For finite extensions of \mathbb{Q}_p , this is done in Remark 4.6.

Before going any further, let us show that Theorem 2.7 allows us to prove a uniform version of Theorem 2.6.

Corollary 2.9 *Let \mathfrak{L}_p be any set of finite extensions of \mathbb{Q}_p and let $\mathfrak{L} = \bigcup_p \mathfrak{L}_p$. Let $E(x, y)$ be an $\mathcal{L}_G^{\mathbb{N}}$ -formula (where x, y range over definable sets X, Y). Then there exist integers m, l , a set of integers N , a prime p_0 and some \mathcal{L}_G^N -formula $\phi(x, w)$ such that the following uniform statement of elimination of imaginaries holds. For all $p \geq p_0$ and all proper expansions to \mathcal{L}_G^N of $L_p \in \mathfrak{L}_p$, $\phi(x, w)$ defines a function*

$$f_{L_p} : X \rightarrow S_m(L_p) \times K(L_p)^l$$

and

$$L_p \models (\forall x, x')(f_{L_p}(x) = f_{L_p}(x')) \iff (\forall y) E(x, y) \equiv E(x', y)$$

Proof Assume \mathfrak{L}_p is nonempty for infinitely many p , otherwise the statement is trivial. The formula $\forall y E(x, y) \equiv E(x', y)$ defines an equivalence relation in any ultraproduct L of fields in \mathfrak{L} . By Theorem 2.7, there is a formula $\phi(x, w)$ (which works for any proper expansion to \mathcal{L}_G^N of any such ultraproduct of residue characteristic zero) such that $\phi(x, w)$ defines a function f and $f(x) = f(x')$ if and only if $\forall y E(x, y) \equiv E(x', y)$.

Let us now assume there is an infinite set $I \subseteq \mathfrak{L}$ such that I has a nonempty intersection with infinitely many \mathfrak{L}_p and for every $L \in I$, there is a proper expansion of L to $\mathcal{L}_{\mathcal{G}}^{\mathbb{N}}$ such that we do not have $f(x) = f(x')$ if and only if $\forall y E(x, y) \equiv E(x', y)$ in L . Then there exists an ultrafilter on \mathfrak{L} containing I but containing no set included in some \mathfrak{L}_{p_0} and such that $\prod_{L \in \mathfrak{L}} L/\mathcal{U} \models \text{PL}_0^{\mathbb{N}}$; but we do not have $f(x) = f(x')$ if and only if $\forall y E(x, y) \equiv E(x', y)$ in this ultraproduct, a contradiction.

By compactness, for some N , this equivalence also holds in proper expansions to $\mathcal{L}_{\mathcal{G}}^N$. \square

Remark 2.10 1. In particular, whenever $E(x, y)$ is interpreted in L_p as an equivalence relation, $f_{L_p}(x)$ codes the E -class of X .

2. If \mathfrak{L}_p is finite for all p then, as $\bigcup_{p < p_0} \mathfrak{L}_p$ is finite, we can find using Theorem 2.6 and Remark 2.8.2 a ϕ and an N that work for all $L \in \bigcup_p \mathfrak{L}_p$ and not just for p big enough.

The proof of Theorems 2.6 and 2.7 uses elimination of imaginaries and the existence of invariant extensions in the theory of algebraically closed valued fields. Recall that a theory T has the invariant extension property if whenever $A = \text{acl}(A) \subseteq M \models T$ and $c \in M$, $\text{tp}(c/A)$ extends to an $\text{Aut}(M/A)$ -invariant type over M . This holds trivially for any finite field, and by inspection, for $\text{Th}(\mathbb{Z}, +, <)$ and, although we will only use a weaker version of the extension property (Corollary 3.10) in the proof of Theorem 2.6, we will show that the theory of a finite extension of \mathbb{Q}_p (with the geometric sorts) enjoys the stronger version (Remark 4.7).

2.3 Real elimination of imaginaries

To illustrate the idea of transferring imaginaries from one theory to the other, consider the following way of deducing EI for RCF (the theory of real closed fields) from EI for ACF (the theory of algebraically closed fields).

Example 2.11 Let F be a field considered in a language extending the language of rings. Assume for all $M \models \text{Th}(F)$:

- (i) (Algebraic boundedness): If $A \subseteq M$ then $\text{acl}(A) \subseteq \overline{A}^{\text{alg}}$;
- (ii) (Rigidity of finite sets): No automorphism of M can have a finite cycle of size > 1 . Equivalently, for each n , there exists \emptyset -definable functions $r_{i,n}(x_1, \dots, x_n)$ that are symmetric in the x_i , such that $S = \{r_{1,n}(S), \dots, r_{n,n}(S)\}$. (Here $r_{i,n}(S)$ denotes $r_{i,n}(x_1, \dots, x_n)$ when $S = \{x_1, \dots, x_n\}$, possibly with repetitions.);

(iii) (Unary EI): Every definable subset of M is coded.

Then $\text{Th}(F)$ eliminates imaginaries (in the single sort of field elements).

Proof Let $f : M \rightarrow M$ be a definable function. By Lemma 2.2, it suffices to prove that f is coded. Let H be the Zariski closure of the graph of f . Since the theory is algebraically bounded, the set $H(x) := \{y : (x, y) \in H\}$ is finite for any x , of size bounded by some n . Let $U_{n,i}$ be the set of x such that $f(x) = r_{i,n}(H(x))$. Then, by elimination of imaginaries in ACF, H —being a Zariski closed set—is coded in $\overline{M}^{\text{alg}}$. But the code is definable over M and hence is in the perfect closure of M . Replacing this code by some p^n th power in the characteristic p case, we can suppose it belongs to M . Moreover, each U_i (being unary) is coded; these codes together give a code for f . \square

Note that RCF satisfies the hypotheses of Example 2.11, but $\text{Th}(\mathbb{Q}_p)$ (in the field sort alone) does not. More precisely, as shown in the introduction, the value group cannot be definably embedded into \mathbb{Q}_p^n . Hence hypothesis (iii) fails for $\text{Th}(\mathbb{Q}_p)$ in the field sort alone.

Remark 2.12 If F is a field satisfying (i), (iii), then F has EI/UFI. This is an immediate consequence of Proposition 2.13 as hypotheses (ii) and (iv) of Proposition 2.13 are true if \tilde{T} is the theory of algebraically closed fields in the language of rings.

2.4 Criterion for elimination of imaginaries

Let \tilde{T} be a complete theory in a language $\tilde{\mathcal{L}}$. Assume \tilde{T} eliminates quantifiers and imaginaries. Let T be a complete theory in a language $\mathcal{L} \supseteq \tilde{\mathcal{L}}$; assume T contains the universal part of \tilde{T} .

In a model M of the theory T , three kinds of definable closure can be considered: the usual definable closure $\text{dcl}_{\mathcal{L}}$, the definable closure in M^{eq} , denoted $\text{dcl}_{\mathcal{L}}^{\text{eq}}$ and the imaginary definable closure restricted to real points $\text{dcl}_{\mathcal{L}}^{\text{eq}} \cap M$. As $\text{dcl}_{\mathcal{L}}^{\text{eq}} \cap M$ and $\text{dcl}_{\mathcal{L}}$ take the same value on set of real points, we will both denote them $\text{dcl}_{\mathcal{L}}$. One must take care however that if A contains imaginary elements, $A \not\subseteq \text{dcl}_{\mathcal{L}}(A)$.

As \tilde{T} eliminates imaginaries, these distinctions are not necessary in models of \tilde{T} and we will only need $\text{dcl}_{\tilde{\mathcal{L}}}$. One should note that, as \tilde{T} eliminates quantifiers, $\text{dcl}_{\tilde{\mathcal{L}}}$ is the closure under quantifier-free $\tilde{\mathcal{L}}$ -definable functions and hence that, for any $A \subseteq M$, $\text{dcl}_{\tilde{\mathcal{L}}}(A) \cap M \subseteq \text{dcl}_{\mathcal{L}}(A)$.

Analogous statements hold for $\text{acl}_{\tilde{\mathcal{L}}}$, $\text{acl}_{\mathcal{L}}$, $\text{acl}_{\mathcal{L}}^{\text{eq}}$, $\text{tp}_{\tilde{\mathcal{L}}}$, $\text{tp}_{\mathcal{L}}$, etc.

One should also be careful that if $M \models T$ is contained in some $\widetilde{M} \models \widetilde{T}$, there is no reason in general that M^{eq} should be contained in \widetilde{M} . In fact, the whole purpose of the following proof is to show that under certain hypotheses every element of M^{eq} is interdefinable with a tuple in M .

Since we are describing notation, let us also point out that we will write interchangeably $\text{dcl}(A, b) = \text{dcl}(Ab) = \text{dcl}(A \cup \{b\})$.

Proposition 2.13 *Assume \widetilde{T} and T have the properties given above. Let M be an $|L|^+$ -saturated and $|L|^+$ -homogeneous model of T and let $\widetilde{M} \models \widetilde{T}$ be such that $M|_{\widetilde{\mathcal{L}}} \leq_{\widetilde{\mathcal{L}}} \widetilde{M}$ and such that any automorphism of M extends to an automorphism of \widetilde{M} . If conditions (i)–(iv) below hold for any $A = \text{acl}_{\mathcal{L}}(A) \subseteq M$ and $c \in \text{dom}(M)$, then T admits elimination of imaginaries up to uniform finite imaginaries (see Definition 2.4).*

- (i) (Relative algebraic boundedness) For every $M' \prec M$, $\text{dcl}_{\mathcal{L}}(M'c) \subseteq \text{acl}_{\widetilde{\mathcal{L}}}(M'c)$.
- (ii) (Internalizing $\widetilde{\mathcal{L}}$ -codes) Let $e \in \text{dcl}_{\widetilde{\mathcal{L}}}(M)$. Then there exists a tuple e' of elements of M such that an automorphism of \widetilde{M} stabilizing M fixes e if and only if it fixes e' .
- (iii) (Unary EI) Every $\mathcal{L}(M)$ -definable unary subset of $\text{dom}(M)$ is coded in M .
- (iv) (Invariant types) There exists an $\text{Aut}(\widetilde{M}/A)$ -invariant type \widetilde{p} over \widetilde{M} such that $\widetilde{p}|_M$ is consistent with $\text{tp}_{\mathcal{L}}(c/A)$.

Moreover, for any $\widetilde{\mathcal{L}}(\widetilde{M})$ -definable function r whose domain contains \widetilde{p} , let $\partial_{\widetilde{p}}r$ be the \widetilde{p} -germ of r (where two $\widetilde{\mathcal{L}}(\widetilde{M})$ -definable functions r, r' have the same \widetilde{p} -germ if they agree on a realization of \widetilde{p} over \widetilde{M}). Then:

- (*) There exists a directed order I and a sequence $(\epsilon_i)_{i \in I}$, with $\epsilon_i \in \text{dcl}_{\widetilde{\mathcal{L}}}(A, \langle r \rangle)$ such that $\sigma \in \text{Aut}(\widetilde{M}/A)$ fixes $\partial_{\widetilde{p}}r$ if and only if σ fixes almost every ϵ_i —i.e., σ fixes all $i \geq i_0$, for some $i_0 \in I$.

Some comments on the proposition:

1. There are two ways to ensure that automorphisms of M extend to automorphisms of \widetilde{M} . The first is to take \widetilde{M} homogeneous enough. The other is to take \widetilde{M} atomic over M ; in the case of valued fields, we could take \widetilde{M} to be the algebraic closure of M .
2. In fact, we will only need (iv) for $|A| \leq |\mathcal{L}|$.

3. If \tilde{p} is definable then, for a uniformly defined family of functions r_b , $\partial_{\tilde{p}}r_b$ is an imaginary (and we could take ϵ_i to be that imaginary). Nevertheless, if \tilde{p} is not definable and say (ϵ_i) is countable then Condition (iv) implies that the germ is a Σ_2^0 -hyper-imaginary, i.e., an equivalence class of sequences indexed by I where the equivalence relation is given by a countable union of countable intersections of definable sets (although each definable set will involve only a finite number of indices, the countable union of countable intersections can involve them all). In the case of ACVF one also has that σ fixes $\partial_{\tilde{p}}r$ if and only if σ fixes cofinally many ϵ_i ; in this case the equivalence relation is also a countable intersection of countable unions of definable sets, so it is Δ_2^0 .
4. Hypotheses (ii) and (iii) are special cases of elimination of imaginaries. It would be nice to move (iii) from the hypotheses to the conclusion, i.e., assuming only (i), (ii) and (iv), to show that every imaginary is “equivalent” to an imaginary of \tilde{M} definable over M .

First let us clarify how $\text{Aut}(M)$ acts on $\text{dcl}_{\tilde{\mathcal{L}}}(M)$ as this action will be used implicitly throughout the proof. Any $\sigma \in \text{Aut}(M)$ can be extended to an automorphism $\tilde{\sigma} \in \text{Aut}(\tilde{M})$ and all these extensions are equal on $\text{dcl}_{\tilde{\mathcal{L}}}(M)$, hence we have a well-defined action of σ on $\text{dcl}_{\tilde{\mathcal{L}}}(M)$ and the notation $\text{Aut}(M/B)$ makes sense even if $B \subseteq \text{dcl}_{\tilde{\mathcal{L}}}(M)$. Similarly, if \tilde{p} is an $\text{Aut}(\tilde{M}/B)$ -invariant type, $\text{Aut}(M/B)$ acts on \tilde{p} -germs of $\tilde{\mathcal{L}}(M)$ -definable functions.

We begin our proof with elimination of finite sets:

Lemma 2.14 *Assume (ii) holds in Proposition 2.13. Then every finite set $E \subseteq M$ is coded.*

Proof By EI for \tilde{T} , the finite set E is coded by a tuple $e' \in \tilde{M}$; e' may consist of elements in $\text{dcl}_{\tilde{\mathcal{L}}}(M)$ but outside M . By (ii), there exists a tuple e of elements of M such that an automorphism of $\text{dcl}_{\tilde{\mathcal{L}}}(M)$ leaving M invariant fixes E if and only if it fixes e' if and only if it fixes e . Thus E and $e \in M$ are interdefinable. \square

Proof [Proposition 2.13] Let $e \in M^{\text{eq}}$. For some $c_1, \dots, c_n \in \text{dom}(M)$, we have $e \in \text{dcl}_{\tilde{\mathcal{L}}}^{\text{eq}}(c_1, \dots, c_n)$. Let $A_i = \text{dcl}_{\tilde{\mathcal{L}}}(e, c_1, \dots, c_i) \subseteq M$. The claim is that $e \in \text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A_0)$. We have $e \in \text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A_n)$ and show by reverse induction on $l \leq n$ that $e \in \text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A_l)$. Assume inductively that $e \in \text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(A_{l+1})$. Let $A = A_l, c = c_{l+1}$. It is easy to check that

$$A = \text{dcl}_{\tilde{\mathcal{L}}}(Ae).$$

As $e \in \text{acl}_{\mathcal{L}}^{\text{eq}}(A_{l+1})$, for some tuple $d \in A_{l+1} = \text{dcl}_{\mathcal{L}}(Ace)$, some $\mathcal{L}(A)$ -definable function f and some $\mathcal{L}(A)$ -definable, finite-set-valued function g , we have

$$e \in g(d), d = f(c, e).$$

Let $f_e(x) = f(x, e)$. Let $\bar{A} = \text{acl}_{\mathcal{L}}(A)$ and let $p = \text{tp}_{\mathcal{L}}(c/\bar{A})$.

Let $M_0 \prec M$ such that M_0^{eq} contains Ae . Note that for all c' in the domain of f_e , $f_e(c') \in \text{dcl}_{\mathcal{L}}(M_0c)$. By (i), there exists an $\tilde{\mathcal{L}}(M_0)$ -definable finite-set-valued function $\phi_{c'}$ such that $f_e(c') \in \phi_{c'}(c')$. By compactness, for some finite set I_0 and $\tilde{\mathcal{L}}(M_0)$ -definable finite-set-valued functions $(\phi_i)_{i \in I_0}$, the following holds: for any c' in the domain of f_e , $f_e(c') \in \phi_i(c')$ for some $i \in I_0$. Let $\phi(x) = \bigcup_{i \in I_0} \phi_i(x)$; so $f_e(c') \in \phi(c')$ for all c' in the domain of f_e . Hence if Φ is the set of all $\tilde{\mathcal{L}}(M)$ -definable, finite-set-valued functions ψ with a domain containing that of f_e and such that for all c' in the domain of f_e , $f_e(c') \in \psi(c')$, then Φ is nonempty.

Let \tilde{p} be an $\text{Aut}(\tilde{M}/\bar{A})$ -invariant type over \tilde{M} extending p , as in (iv). For $m \in \mathbb{N}$, let Φ_m be the set of all $\tilde{\mathcal{L}}(M)$ -definable functions $\phi \in \Phi$ such that for $c \models \tilde{p}$, $\phi(c)$ is an m -element set. Note that as \tilde{p} is a complete type, m does not depend on c . Let m be minimal such that Φ_m is nonempty. Clearly all $\phi \in \Phi_m$ share the same \tilde{p} -germ: if ϕ, ϕ' do not have the same \tilde{p} -germ, then $\phi''(x) := \phi(x) \cap \phi'(x)$ lies in $\Phi_{m'}$ for some $m' < m$. Pick $F_E \in \Phi_m$, defined over $E \subseteq M$. So F_E covers f_e , F_E is $\tilde{\mathcal{L}}(E)$ -definable, and the \tilde{p} -germ of F_E is invariant under $\text{Aut}(M/\bar{A}e)$.

Claim 2.15 The \tilde{p} -germ of F_E is invariant under $\text{Aut}(M/\bar{A})$.

Proof Let (ϵ_i) be a sequence as in (iv), coding the germ of F_E on \tilde{p} . Note that $\epsilon_i \in \text{dcl}_{\mathcal{L}}(M)$ (since F_E is $\tilde{\mathcal{L}}(E)$ -definable and $E \subseteq M$). By (ii), we may replace ϵ_i by an element of M , without changing $\text{Aut}(M/\epsilon_i)$; we do so.

Now, almost all ϵ_i must be in $\text{acl}_{\mathcal{L}}(\bar{A}e)$. For otherwise, by moving to a subsequence we may assume all ϵ_i are outside $\text{acl}_{\mathcal{L}}(\bar{A}e)$. So $\text{Aut}(M/\bar{A}e\epsilon_i)$ has infinite index in $\text{Aut}(M/\bar{A}e)$. By Neumann's Lemma, for any finite set X of indices i , there exists $\tau \in \text{Aut}(M/\bar{A}e)$ with $\tau(\epsilon_i) \neq \epsilon_i$, for all $i \in X$. By compactness (and homogeneity of M), there exists $\tau \in \text{Aut}(M/\bar{A}e)$ with $\tau(\epsilon_i) \neq \epsilon_i$ for all i . But then τ fails to fix the \tilde{p} -germ of F , contradicting the $\text{Aut}(M/\bar{A}e)$ -invariance of this germ.

So for almost all i , some finite set \mathcal{E}_i containing ϵ_i is defined over Ae . By Lemma 2.14, the finite set \mathcal{E}_i is coded in M . But $A = \text{dcl}_{\mathcal{L}}(Ae)$, so \mathcal{E}_i is defined over A . Hence $\epsilon_i \in \bar{A}$, i.e., ϵ_i is fixed by $\text{Aut}(M/\bar{A})$. This being the case for almost all i , the \tilde{p} -germ of F_E is invariant under $\text{Aut}(M/\bar{A})$. \square

Claim 2.16 $e \in \text{acl}_{\mathcal{L}}^{\text{eq}}(A)$.

Proof It suffices to show that if $((e_i, E_i) : i \in \mathbb{N})$ is an indiscernible sequence over \bar{A} with $e_0 = e$ and $E_0 = E$, then $e_i = e_j$ for some $i \neq j$. Let $c \models \tilde{p}|A(E_i)_{i \in \mathbb{N}}$ such that $c \models p$. By (iii) and because $A = \text{dcl}_{\mathcal{L}}(Ae)$, $\text{tp}_{\mathcal{L}}(c/A)$ implies $\text{tp}_{\mathcal{L}}(c/Ae)$; hence $\text{tp}_{\mathcal{L}}(e/A)$ implies $\text{tp}_{\mathcal{L}}(e/Ac)$. So $\text{tp}_{\mathcal{L}}(e_i/Ac) = \text{tp}_{\mathcal{L}}(e/Ac)$.

By Claim 2.15, the \tilde{p} -germs of the F_{E_i} are equal; so $F_{E_i}(c)$ is a finite set F that does not depend on i . But $f(c, e_i) \in F$, so $f(c, e_i)$ takes the same value on some infinite set I' of indices i . Hence so does the finite set $g(f(c, e_i))$. As $e \in g(f(c, e))$, it follows that $e_i \in g(f(c, e_i))$, so infinitely many e_i lie in the same finite set and $e_i = e_j$ for some $i \neq j$. \square

We have just shown that e lies in $\text{acl}^{\text{eq}}(A) = \text{acl}^{\text{eq}}(\text{dcl}^{\text{eq}}(e) \cap M)$. This concludes the proof of Proposition 2.13. \square

Let us now show that this first criterion can be turned into a criterion for elimination of imaginaries.

Corollary 2.17 *Let T and \tilde{T} be as in Proposition 2.13 and let us suppose moreover that:*

(v) (Weak rigidity) *For all $A = \text{acl}_{\mathcal{L}}(A)$ and $c \in \text{dom}(M)$, $\text{acl}_{\mathcal{L}}(Ac) \subseteq \text{dcl}_{\mathcal{L}}(Ac)$.*

Then T eliminates imaginaries.

Proof Let $e \in M^{\text{eq}}$ be an imaginary element. We have $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(c_1, \dots, c_n)$ for some $c_1, \dots, c_n \in \text{dom}(M)$. Let $A_i = \text{acl}_{\mathcal{L}}(e, c_1, \dots, c_i) \subseteq M$. Then $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(A_n)$; we show by reverse induction on $l \leq n$ that $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(A_l)$. We assume inductively that $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(A_{l+1})$. Let $A = A_l, c = c_{l+1} \in \text{dom}(M)$. It is easy to check that

$$A = \text{acl}_{\mathcal{L}}(Ae)$$

and that, for some tuple d ,

$$d \in A_{l+1} = \text{acl}_{\mathcal{L}}(Ace), e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(Ad).$$

By Proposition 2.13, $e \in \text{acl}_{\mathcal{L}}^{\text{eq}}(A_0)$, so $d \in \text{acl}_{\mathcal{L}}(Ac)$. By weak rigidity (v), $d \in \text{dcl}_{\mathcal{L}}(Ac)$. Thus $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(Ac)$.

Say $e = h(c)$, where h is an $\mathcal{L}^{\text{eq}}(A)$ -definable function. Then $h^{-1}(e)$ is an $\mathcal{L}(M)$ -definable subset of $\text{dom}(M)$, hence by (iii) it has a code $e' \in M$. Clearly e and e' are interdefinable over A . As $e' \in M$, we have $e' \in \text{dcl}_{\mathcal{L}}(Ae) = A$. So $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(A) = \text{dcl}_{\mathcal{L}}^{\text{eq}}(A_l)$. This finishes the induction and shows that $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(A_0)$.

Let a be a tuple from A_0 such that e is $\mathcal{L}^{\text{eq}}(a)$ -definable. Let a' be the (finite) set of conjugates of a over e . Then $\text{dcl}_{\mathcal{L}}^{\text{eq}}(e) = \text{dcl}_{\mathcal{L}}^{\text{eq}}(a')$ and, by Lemma 2.14, a' is coded, hence e is interdefinable with some sequence from M . \square

Keeping (v) out of Proposition 2.13 makes the proof of the EI criterion messier than strictly necessary. Nonetheless, distinguishing the case without (v) is important for ultraproducts of the p -adics where (v) fails.

The following lemma will be used to prove (v) in the p -adic case.

Lemma 2.18 *Assume that for any $a \in M$, there exists a tuple c from $\text{dom}(M)$ with $a \in \text{dcl}_{\mathcal{L}}(c)$ and such that $\text{tp}_{\mathcal{L}}(c/\text{acl}_{\mathcal{L}}(a))$ extends to an $\text{Aut}(M/\text{acl}_{\mathcal{L}}(a))$ -invariant type over M . Then (v) follows from:*

(v') *If $B \subseteq \text{dom}(M)$ then $\text{acl}_{\mathcal{L}}(B) \subseteq \text{dcl}_{\mathcal{L}}(B)$.*

Proof Let $A = \{a_i : i < \kappa\}$. For each i , pick a tuple c_i of elements of $\text{dom}(M)$ with $a_i \in \text{dcl}_{\mathcal{L}}(c_i)$, and extend $\text{tp}_{\mathcal{L}}(c_i/\text{acl}_{\mathcal{L}}(a_i))$ to an $\text{Aut}(M/\text{acl}_{\mathcal{L}}(a_i))$ -invariant type p_i . Let $A_0 = A$, and recursively let $A_{i+1} = A_i \cup \{b_i\}$, where $b_i \models p_i|_{\text{acl}_{\mathcal{L}}(A_i c)}$, and $A_\lambda = \bigcup_{i < \lambda} A_i$ for limit λ .

Claim 2.19 $\text{acl}_{\mathcal{L}}(Ac) \cap \text{dcl}_{\mathcal{L}}(A_i c) \subseteq \text{dcl}_{\mathcal{L}}(Ac)$.

Proof By induction on i . The limit case is trivial. To move from i to $i + 1$, let $d \in \text{acl}_{\mathcal{L}}(Ac) \cap \text{dcl}_{\mathcal{L}}(A_{i+1}c)$ and let $\sigma \in \text{Aut}(M/A_i c)$. As $\text{tp}_{\mathcal{L}}(b_i/\text{acl}_{\mathcal{L}}(A_i c))$ is invariant under σ , $d \in \text{acl}_{\mathcal{L}}(A_i c)$ and d is definable over $A_i c b_i$, we have $\sigma(d) = d$, i.e., d is definable over $A_i c$ and hence $d \in \text{dcl}_{\mathcal{L}}(Ac)$ by induction. \square

Now $A_\kappa \subseteq \text{dcl}_{\mathcal{L}}(A_\kappa \cap \text{dom}(M))$ and so $\text{dcl}_{\mathcal{L}}(A_\kappa c) = \text{dcl}_{\mathcal{L}}((A_\kappa c) \cap \text{dom}(M))$. By (v') this set contains $\text{acl}_{\mathcal{L}}(A_\kappa c)$ and hence $\text{acl}_{\mathcal{L}}(Ac)$. Applying Claim 2.19 with $i = \kappa$, we obtain (v). \square

3 Extensible 1-types in intersections of balls

The goal of this section is to establish some results about unary types in henselian fields (specifically, finite extensions of \mathbb{Q}_p and ultraproducts of such fields), which will be useful to prove that Proposition 2.13 can be applied to these fields.

In this section, we will not be considering valued fields in the geometric language as we need quantifier elimination and not elimination of imaginaries. Let \mathcal{R} be a set of symbols; we will be working in the countable language $\mathcal{L} := \{K, +, \cdot, ^{-1}, \text{val} : K \rightarrow \Gamma, r : K \rightarrow K_r, \dots\}_{r \in \mathcal{R}}$ where the K_r are new sorts, each r is such that $r|_{K^*}$ is a surjective group homomorphism $K^* \rightarrow K_r$ that vanishes on $1 + \mathcal{M}^\nu$ for some $\nu = \nu(r) \in \mathbb{N}$ and the \dots refer to additional constants on K and additional relations

on the sorts K_r and Γ . Let T be some theory of valued fields in this language that eliminates quantifiers. Throughout this section M will be a sufficiently saturated model of T and $\lambda_0 \in K(M)$ a uniformizer. We will write \bar{r} for the (possibly infinite) tuple of all $r \in \mathcal{R}$.

Assume Γ is definably well-ordered (every nonempty definable subset with a lower bound has a least element). Let

$$Q_{\mathcal{R}} = \{(u, v_r, \dots)_{r \in \mathcal{R}} : (\exists x \in K) \text{val}(x) = u, \forall r \in \mathcal{R}, r(x) = v_r\}.$$

We write $\text{val}(x) \gg \text{val}(y)$ if $\text{val}(x) > \text{val}(y) + m \text{val}(\lambda_0)$ for all $m \in \mathbb{N}$.

Finite extensions of the p -adics fit in this setting, by Prestel-Roquette [57], if we take the r_n to be the canonical projections $K^* \rightarrow K^*/(K^*)^n$. Note that every element of these finite groups is in $\text{dcl}^{\text{eq}}(\emptyset)$. In the case of ultraproducts of p -adic fields of residue characteristic zero and more generally of henselian valued fields with residue characteristic zero (denoted as HF_0), one map r suffices: the canonical projection $\text{rv} : K^* \rightarrow \text{RV}$.

Observe that $\text{val}(x - y) \gg \text{val}(x - z)$ implies $r(x - z) = r(y - z)$ for all $r \in \mathcal{R}$. Indeed $(y - z)/(x - z) = 1 + (y - x)/(x - z) \in 1 + \mathcal{M}^{\nu(r)}$.

Notation 3.1 If $b \in \mathbb{B}(M)$, $x \in \text{dom}(M)$ and $x \notin b$, the valuation $\text{val}(x - y)$ takes the same value for all $y \in b$. We denote it $\text{val}(x - b)$. By $\text{rad}(b)$ we denote the infimum of $\text{val}(y - y')$, $y, y' \in b$.

Moreover for all $r \in \mathcal{R}$, if $\text{val}(x - b) + \nu(r) \text{val}(\lambda_0) \leq \text{rad}(b)$, then $r(x - y) = r(x - y')$ for all $y, y' \in b$. We write $r(x - b) = r(x - y)$ in this case.

Definition 3.2 Let $f = (f_i)_{i \in I}$ be a family of A -definable functions for some $A \subseteq M^{\text{eq}}$. A partial type p over A is *complete over A relative to f* if the map $\text{tp}(c/A) \mapsto \text{tp}(f(c)/A)$ is injective on the set of complete types over A that extend p .

Remark 3.3 The partial type $p(x)$ is complete over A relative to f if and only if for every formula $\phi(x)$ over A , there exists a formula $\theta(u)$ over A such that $p \vdash (\phi(x) \iff \theta(f(x)))$.

For the rest of the section we are going to study generic types of intersections of balls. Let $\bar{b} = \{b_i : i \in I\}$ be a descending sequence of balls in $\mathbb{B}(M)$. Let $P = \bigcap_{i \in I} b_i$. Let $P_{\Gamma} = \{\gamma \in \Gamma : \forall i \in I \gamma > \text{rad}(b_i)\}$. For any A with $b_i \in \text{dcl}^{\text{eq}}(A)$, we define the generic type of P over $A \subseteq M^{\text{eq}}$ to be

$$q_P|A := P(x) \cup \{x \notin b : b \in \mathbb{B}(\text{acl}^{\text{eq}}(A)), b \text{ strictly included in } P\}.$$

In Section 4, we will also be considering the ACVF-generic of such an intersection P , i.e., the same notion of genericity but considered in algebraically valued fields. Note that if L is a valued field, $A \subseteq L$ and P is an intersection of balls in $\mathbb{B}(A)$, then the difference between the generic type of P over A in L and in $\overline{L}^{\text{alg}}$ is that the latter must also avoid balls that do not have a center or a radius in L but in $\overline{L}^{\text{alg}}$.

Remark 3.4 If P is a strict intersection, i.e., P is not equal to a ball or equivalently \bar{b} does not have a minimal element, then for an element to be generic in P over A it suffices to check that x is not contained in any ball $b \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ contained in P . Indeed, if $b \in \mathbb{B}(\text{acl}^{\text{eq}}(A))$, then the smallest ball containing all A -conjugates of b is strictly included in P and is definable over A .

In what follows, we will consider $A \subseteq M^{\text{eq}}$ containing all constants in K , and \bar{b} a decreasing sequence of balls in $\mathbb{B}(\text{dcl}^{\text{eq}}(A))$ (indexed by some ordinal). Unless otherwise mentioned, until Proposition 3.9 we will suppose that $P = \bigcap_i b_i$ is strict.

Lemma 3.5 *Suppose $A \subseteq K(M)$. Fix $a \in A$ with $a \in b_i$ for each i . Then $q_P|_A$ is complete relative to $\text{val}(x - a)$ and to $\bar{r}(x - a)$. Moreover, if $P(\text{dcl}^{\text{eq}}(A)) = \emptyset$ then $q_P|_A$ is complete.*

Proof Taking into account quantifier elimination, we must show the following: let $c, c' \in M$ be two realizations of $q := q_P|_A$ such that $(\text{val}(c - a), \bar{r}(c - a))$ has the same type over A as $(\text{val}(c' - a), \bar{r}(c' - a))$; then the substructures $A(c), A(c')$ generated by c, c' over A (which are simply the fields generated by c, c' over A) are isomorphic over A .

Extend the valuation from $K(M)$ to $L := \overline{K(M)}^{\text{alg}}$ —the algebraic closure of $K(M)$ —and extend each $r \in \mathcal{R}$ to a group homomorphism with kernel $\ker(r) \cdot (1 + \lambda_0^{\nu(r)} \mathcal{O}(L)) \subseteq L$. This is possible, as for all $a \in \ker(r)$ and $b \in \mathcal{O}(L)$, if $a(1 + \lambda_0^{\nu(r)} b) \in K(M)$, then $a(1 + \lambda_0^{\nu(r)} b) \in \ker(r)$: indeed either $a = 0$ or $1 + \lambda_0^{\nu(r)} b \in K(M)$, and in the latter case, $b \in K(M)$ and $\text{val}(b) \geq 1$, so $b \in \mathcal{O}(M)$ and thus $(1 + \lambda_0^{\nu(r)} b) \in \ker(r)$. By construction, the following still holds: for all $x, y, z \in L$, $\text{val}(x - y) \gg \text{val}(x - z)$ implies $r(x - z) = r(y - z)$.

Then it suffices to show that $\overline{A}^{\text{alg}}(c), \overline{A}^{\text{alg}}(c')$ are $\overline{A}^{\text{alg}}$ -isomorphic, by an isomorphism commuting with the extensions of the maps r (one can then restrict the isomorphism to $A(c)$). As $(\text{val}(c - a), \bar{r}(c - a))$ and $(\text{val}(c' - a), \bar{r}(c' - a))$ realize the same type over A , by taking a conjugate of c' over A , we may assume the tuples are equal.

Take any $d \in \overline{A}^{\text{alg}}$. If $d \notin b_i$ for some i , then $\text{val}(c - d) = \text{val}(c' - d)$. Moreover, for any $k \in \mathbb{N}$, $\text{val}(c - c') \geq \text{rad}(b_{i+k}) \geq \text{rad}(b_i) + k \text{val}(\lambda_0) > \text{val}(c - d) + k \text{val}(\lambda_0)$; and it follows that $\bar{r}(c - d) = \bar{r}(c' - d)$.

If $d \in b_i$ for each i , then the smallest ball $b \in \mathbb{B}(L)$ containing a and all the conjugates of d over A is (quantifier-free) A -definable in L . As Γ is definably well-ordered, the $K(M)$ -points of b form a ball $b' \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ which is included in P . Hence c and c' are not in b' nor, in fact, in any of the balls centered around b' with radius $\text{rad}(b') - k \text{val}(\lambda_0)$. It follows that $\text{val}(c - d) = \text{val}(c - a)$ and $\bar{r}(c - d) = \bar{r}(c - a)$, and similarly for c' . Hence $\text{val}(c' - d) = \text{val}(c - d)$ and $\bar{r}(c' - d) = \bar{r}(c' - d)$.

As any rational function g over A is a ratio of products of constant or linear polynomials, it follows that $\text{val}(g(c)) = \text{val}(g(c')), \bar{r}(g(c)) = \bar{r}(g(c'))$. This proves the first part of the lemma.

If P does not contain any point in A , then there cannot be any $d \in \bar{A}^{\text{alg}}$ such that $d \in b_i$ for each i . Indeed, let $d_{j \leq n}$ be the \mathcal{L} -conjugates of d over A ; then $e := 1/n \sum_j d_j \in \text{dcl}^{\text{eq}}(A)$ and for all i , $d_j \in b_{i+k}$ where k is such that $k \text{val}(\lambda_0) \geq \text{val}(n)$ and $\text{val}(e - d) \geq \text{val}(1/n) + \text{rad}(b_{i+k}) \geq \text{rad}(b_i)$. It follows that $e \in P(\text{dcl}^{\text{eq}}(A))$, a contradiction. But the hypothesis about $(\text{val}(x - a), r(x - a))$ is only used in the case $d \in P$. Thus the second assertion follows. \square

Remark 3.6 Suppose T extends HF_0 and $A \subseteq K(M)$. Without any assumption on P (it can be strict, a closed ball or an open ball), if $P(A) = \emptyset$ then P is a complete type. The exact same proof works as balls are convex in residue characteristic zero and the unique $r = \text{rv}$ we need has kernel $1 + \mathcal{M}$, i.e., $\text{val}(x - y) > \text{val}(x - z)$ alone implies $r(x - z) = r(y - z)$.

We now want to prove (in Proposition 3.9) that Lemma 3.5 is true without the assumption that $A \subseteq K(M)$.

Lemma 3.7 *Suppose $A \subseteq M^{\text{eq}}$ is such that P contains no $b \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$. Then $q_P|_A$ is a complete type.*

Proof Let us suppose A is countable. Then the partial type $P = \bigcap_{n=1,2,\dots} b_n$ is not isolated over A ; for if the formula $\theta(x)$ with parameters in A implies $x \in b_i$ for all i , then, as Γ is definably well-ordered, there is a smallest ball b containing θ . This ball is strictly contained in P and is A -definable, a contradiction. Then by the omitting types theorem, there exists a model M_0 such that $A \subseteq M_0^{\text{eq}}$ and $P(M_0) = \emptyset$. By Lemma 3.5, $q_P|_{K(M_0)}$ is a complete type, and, as K is dominant in M_0^{eq} , P is a complete type over M_0^{eq} and hence over A .

If A is not countable, let c and c' be generic in P over A and let $(M_0^{\text{eq}}, A_0) \prec (M^{\text{eq}}, A)$ be countable (in the language where we add a predicate for A) and contain c and c' . Let Q be the intersection of all A_0 -definable balls in M_0 that contain c ; then

Q is strict, it contains no A_0 -definable ball and also contains c' (all of this is expressed in the type of c, c' in the language with the new predicate). By the countable case, c and c' have the same type over A_0 in M_0^{eq} and hence, they have the same type over A in M^{eq} . \square

Lemma 3.8 *Let $q_{\mathcal{R}}$ be a complete type over A extending $Q_{\mathcal{R}}$. Suppose $q_{\mathcal{R}}$ implies both that $u \in P_{\Gamma}$ and that, for any $\gamma \in P_{\Gamma}(\text{dcl}^{\text{eq}}(A))$, $\gamma > u$. Then*

$$q_P|M \cup \bigcup_{a \in P(M)} q_{\mathcal{R}}(\text{val}(x-a), \bar{r}(x-a))$$

is consistent.

Proof We may assume M has an element a' with $a' \in b_i$ for each i . Note that $q_{\mathcal{R}}$ is consistent with $\{\gamma > u : \gamma \in P_{\Gamma}(M)\}$. Indeed, for any $\gamma \in P_{\Gamma}(M)$, if $q_{\mathcal{R}} \vdash u \geq \gamma$, then some $\psi \in q_{\mathcal{R}}$ is bounded below by γ ; but then the minimum $\gamma' \leq \gamma$ of ψ in M exists as Γ is definably well-ordered, γ' is in $P_{\Gamma}(\text{dcl}^{\text{eq}}(A))$ and $q_{\mathcal{R}} \vdash \gamma' \leq u$, contradicting our hypothesis.

Let c' be such that $(\text{val}(c'), \bar{r}(c')) \models q_{\mathcal{R}} \cup \{\gamma < u : \gamma \in P_{\Gamma}(M)\}$ and let $d = a' + c'$. Clearly $d \models q_P|M$; indeed $\text{val}(d - a') = \text{val}(c') \in P_{\Gamma}$ and thus $d \in b_i$ for all i . Now, let us assume there exist $b \in \mathbb{B}(\text{dcl}^{\text{eq}}(M))$ included in P and containing d . Taking a bigger ball, we can suppose that $a' \in b$, too; but then $\text{val}(d - a') = \text{val}(c') > \text{rad}(b) \in P_{\Gamma}(M)$ contradicting the choice of c' . Moreover for any $a \in P(M)$, $\text{val}(d - a') = \text{val}(c') \ll \text{val}(a - a')$. Thus $\text{val}(d - a) = \text{val}(d - a') = \text{val}(c')$ and $\bar{r}(d - a) = \bar{r}(d - a') = \bar{r}(c')$, and d realizes the given type. \square

Proposition 3.9 *Assume P is strict and fix $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ with $a \subseteq b_i$ for each i . Then $q_P|A$ is complete relative to $\text{val}(x-a)$ and to $\bar{r}(x-a)$. Moreover, if P does not contain any ball in $\mathbb{B}(\text{dcl}^{\text{eq}}(A))$ then $q_P|A$ is complete.*

Proof The second case is tackled in Lemma 3.7. So we can suppose that such an $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ exists. Let $c, c' \models q_P|A$ such that $q_{\mathcal{R}} := \text{tp}(\text{val}(c-a)\bar{r}(c-a)/A) = \text{tp}(\text{val}(c'-a)\bar{r}(c'-a)/A)$. Let $M_0 \prec M$ such that $A \subseteq M_0^{\text{eq}}$. It follows from Lemma 3.8 that there exists $c_0 \models q_P|M_0^{\text{eq}} \cup q_{\mathcal{R}}(\text{val}(x-a)\bar{r}(x-a))$. Taking conjugates of c and c' over A , we can suppose that $(\text{val}(c-a)\bar{r}(c-a)) = (\text{val}(c_0-a)\bar{r}(c_0-a)) = (\text{val}(c'-a)\bar{r}(c'-a))$ as these three tuples have the same type over A .

Then, as shown in the proof of Lemma 3.8, $c, c' \models q_P|M_0^{\text{eq}}$. By Lemma 3.5, c and c' have the type over M_0^{eq} and hence over A . \square

Corollary 3.10 *Let L be a finite extension of \mathbb{Q}_p , $M \models \text{Th}(L)$ and $A \subseteq M$ such that $\mathbb{B}(\text{acl}^{\text{eq}}(A)) \subseteq A$. Let $c \in \text{dom}(M)$. Then $\text{tp}(c/A)$ extends to an $\text{Aut}(M/A)$ -invariant type.*

Proof Let $W(c; A) = \{b \in \mathbb{B}(A) : c \in b\}$, $P = \bigcap_{b \in W(c; A)} b$. As the residue field of M is finite, P cannot reduce to a single ball (that ball would be the union of finitely many proper subballs, each in $\mathbb{B}(\text{acl}^{\text{eq}}(A))$, hence in A and one of them would contain c). Note that $c \models q_P|A$.

If there is no ball $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ contained in P , then let $q_{\mathcal{R}}$ be any $\text{Aut}(M/A)$ -invariant type extending $Q_{\mathcal{R}}$ which implies $u \in P_{\Gamma}$ and $\alpha > u$ for all $\alpha \in P_{\Gamma}(M)$. If such a ball a exists, we suppose $q_{\mathcal{R}}$ also extends $\text{tp}(v(c-a), \bar{r}(c-a)/A)$. By Lemma 3.8, $q^* := q_P|M(x) \cup \bigcup_{a \in P(M)} q_{\mathcal{R}}(\text{val}(x-a), \bar{r}(x, a))$ is consistent. Clearly q^* is $\text{Aut}(M/A)$ -invariant. It follows from Proposition 3.9 that q^* is complete and that it extends $\text{tp}(c/A)$. \square

Let N_n be the group of matrices of the form $I_n + b$, where I_n is the identity matrix in GL_n , and b is an upper triangular matrix with all entries having valuation $\gg 0$. Thus $N_n = \text{B}_n(\mathcal{O}) \cap \bigcap_m (I_n + \lambda_0^m \text{B}_n(\mathcal{O}))$.

Lemma 3.11 *There exists an $\text{Aut}(M)$ -invariant type $p|M$ of matrices $a \in N_n$, invariant under right multiplication: for all $A \subseteq M^{\text{eq}}$ and $b \in N_n(A)$, if $c \models p|A$, then $cb \models p|A$. The type p is complete relative to the absolute values and \bar{r} -values of the entries. Moreover, if $Q_{\mathcal{R}}$ has an $\text{Aut}(M)$ -invariant 1-type t extending the cut defined by $(\text{val}(\lambda_0), 2\text{val}(\lambda_0), \dots)$ on the left, then p can be taken to be complete.*

Proof Let $P = \bigcap_i (\lambda_0^i \mathcal{O})$ and let $q = q_P|M$; then q is $\text{Aut}(M)$ -invariant and complete relative to val and \bar{r} by Proposition 3.9 (as P contains 0). If t as above exists, then take $q := q_P|M \cup t(\text{val}(x), \bar{r}(x))$ which is consistent by Lemma 3.8, complete and $\text{Aut}(M)$ -invariant.

Let p be the type of upper-triangular matrices obtained by taking the $\frac{n(n+1)}{2}$ -th tensor power of q (where by tensor product, we mean the tensor product of types; see just below for a more explicit statement), using the lexicographic order on the matrix entries, and adding 1 on the diagonal: thus for all $A \subseteq M^{\text{eq}}$, if $a \in M_n$, then $I_n + a \models p|A$ if and only if $a_{11} \models q|A$, $a_{12} \models q|\text{dcl}^{\text{eq}}(Aa_{11})$, \dots , $a_{22} \models q|\text{dcl}^{\text{eq}}(Aa_{11} \dots a_{1,n})$, \dots , $a_{n,n} \models q|\text{dcl}^{\text{eq}}(Aa_{11}, \dots, a_{n,n-1})$, while $a_{ij} = 0$ for $i > j$.

The fact that p is an $\text{Aut}(M)$ -invariant (partial) type of elements of N_n is clear. As for the right translation invariance, let $I_n + b \in N_n(A)$ and $I_n + a \models p|A$; we have

to show that $(I_n + a)(I_n + b) = I_n + a + b + ab \models p|A$. Let $d = a + b + ab$. Then $d_{11} = a_{11} + b_{11} + a_{11}b_{11}$. We have

$$\text{val}(a_{11}b_{11}) > \text{val}(b_{11}) \gg \text{val}(a_{11}) \gg 0.$$

So $\text{val}(d_{11}) = \text{val}(a_{11})$ and hence d_{11} also realizes $q_P|A$. Furthermore, we also have $\bar{r}(d_{11}) = \bar{r}(a_{11})$; it follows that $d_{11} \models q|A$. Similarly

$$d_{12} = a_{12} + b_{12} + a_{11}b_{12} + a_{12}b_{22};$$

here a_{12} has strictly bigger valuation than any of the other summands, so again $\text{val}(d_{12}) = \text{val}(a_{12})$ and $\bar{r}(d_{12}) = \bar{r}(a_{12})$, thus $d_{12} \models q|\text{dcl}^{\text{eq}}(Aa_{11})$. But since $b \in \text{dcl}^{\text{eq}}(A)$, $d_{11} \in \text{dcl}^{\text{eq}}(Aa_{11})$, so $d_{12} \models q|\text{dcl}^{\text{eq}}(Ad_{11})$. Continuing in this way we see that $I_n + d \models p|A$. \square

Corollary 3.12 *Let R be a left coset of N_n in $B_n(K)$. There exists an $\text{Aut}(M/\langle R \rangle)$ -invariant type of elements of R .*

Proof Pick $g \in R$, let p be the right- N_n -invariant type of Lemma 3.11, and for all $A \subseteq M^{\text{eq}}$, let $p^g|A = \text{tp}(cg/Ag)$, where $c \models p|\text{dcl}^{\text{eq}}(Ag)$. Then $p^g|A = p^{hg}|A$ for $h \in N_n(\text{dcl}^{\text{eq}}(A))$, since p is right- N_n -invariant. Thus any automorphism fixing $\langle R \rangle$ must fix the global type $p^g|M$. \square

Corollary 3.13 *Let L be a finite extension of \mathbb{Q}_p and let $M \models \text{Th}(L)$, $e \in S_n(M)$, $E = \text{acl}^{\text{eq}}(e)$. Then there exists an $\text{Aut}(M/E)$ -invariant type of bases for e .*

Proof It was noted in Section 2.2 that any lattice e has a triangular basis; this basis can be viewed as the set of columns of a matrix in $B_n(K)$. Let b, b' be two such bases, and suppose $b' = \sigma(b)$, $\sigma \in \text{Aut}(M/E)$. Then as $e/\lambda_0^m e$ is finite for all m , the columns of b, b' must be in the same coset of $\lambda_0^m e$ for each m . Thus if we write $b' = ab$ with $a \in B_n(\mathcal{O})$, then $a = I_n$ modulo $\lambda_0^m \mathcal{O}$ for each m , so $a \in N_n$ and $\text{Aut}(M/E)$ preserves the coset $N_n b$. So it suffices to take the $\text{Aut}(M/\langle N_n b \rangle)$ -invariant type of elements of $N_n b$ guaranteed by Corollary 3.12. \square

Let us now suppose that T extends HF_0 . Using similar techniques, we can extend the previous results to the case when P is a closed ball (this case is only relevant to Section 5). For the last result, though, we will also need the residue field to be pseudo-finite.

Let b be a closed ball. We will write res_b for the map that sends $x \in b$ to $x + \text{rad}(b)\mathcal{M}$, the maximal open subball of b containing x .

Lemma 3.14 *Let $b \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ be a closed ball and q a complete type over A extending $\text{res}_b(b)$ such that $q \vdash x \neq b'$ for all $b' \in \text{res}_b(b)(\text{acl}^{\text{eq}}(A))$. Then $q_b|M \cup q(\text{res}_b(x))$ is consistent.*

Proof Let us first show that q is consistent with $\{x \neq b' : b' \in \text{res}_b(b)(M^{\text{eq}})\}$. If not, there is a finite number of balls $b_i \in \text{res}_b(b)(M^{\text{eq}})$ such that $q \vdash \bigvee_i x = b_i$. If we take a minimal number of such balls, each of them must realize q and hence be algebraic over A , a contradiction.

Now, let c be such that $\text{res}_b(c) \models q \cup \{x \neq b' : b' \in \text{res}_b(b)(M^{\text{eq}})\}$; then $c \models q_b|M$. Indeed $c \in b$ and if c is in $b' \in \mathbb{B}(M^{\text{eq}})$ such that $b' \subseteq b$, then $c \in \text{res}_b(b') \in \text{res}_b(b)(M^{\text{eq}})$, contradicting the choice of c . \square

Lemma 3.15 *Suppose $P = b$ is a closed ball. Then $q_b|A$, the generic type of b , is complete relative to res_b .*

Proof If $A \subseteq K(M)$ then, by the same considerations as in Lemma 3.5 (and, as $\text{HF}_0 \subseteq T$, taking $\bar{r} = \text{rv}$ is enough), it suffices to show that if c and c' are realizations of $q_b|A$ such that $\text{res}_b(c) = \text{res}_b(c')$ then for all $d \in \bar{A}^{\text{alg}}$, $\text{rv}(c-d) = \text{rv}(c'-d)$. If $d \in \text{res}_b(c)$, then $c \in \text{res}_b(c) = \text{res}_b(d) \in \mathbb{B}(\text{acl}^{\text{eq}}(A))$ as $d \in \bar{A}^{\text{alg}}$. This contradicts the fact that $c \models q_b|A$. Hence $d \notin \text{res}_b(c)$. As c and $c' \in \text{res}_b(c) = \text{res}_b(c')$, $\text{val}(c-c') > \text{val}(c-d)$ and $\text{rv}(c-d) = \text{rv}(c-d')$.

If A is not contained in K , let $c, c' \models q_b|A$ such that $q := \text{tp}(\text{res}_b(c)/A) = \text{tp}(\text{res}_b(c')/A)$. By Lemma 3.14, there exists $c_0 \models q_b|M \cup q$. Taking A -conjugates of c and c' , we can suppose that $\text{res}_b(c) = \text{res}_b(c_0) = \text{res}_b(c')$. Then, as seen in the proof of Lemma 3.14, $c, c' \models q_b|M$. By the previous paragraph c and c' have the same type over M and hence over A . \square

Corollary 3.16 *Suppose $P = b$ is a closed ball and let $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ be contained in b . Then $q_b|A$ is complete relative to $\text{rv}(x-a)$.*

Proof If $c, c' \models q_b|A$, then $\text{val}(c-a) = \text{val}(c'-a) = \text{rad}(b)$ and hence $\text{res}_b(c) = \text{res}_b(c')$ if and only if $\text{rv}(c-a) = \text{rv}(c'-a)$. Thus the corollary follows immediately from Lemma 3.15. \square

Corollary 3.17 *Suppose k is pseudo-finite, $k(A)$ contains the constants needed for k to have EI and $P = b$ is a closed ball that contains no ball $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$. Then any $x \in b$ generates a complete type over A .*

Proof By Lemma 3.15, it suffices to show that $\text{res}_b(b)$ is a complete type over A . But $\text{res}_b(b)$ is a definable 1-dimensional affine space over k —i.e., a $V := \gamma\mathcal{O}/\gamma\mathcal{M}$ -torsor where $\gamma := \text{rad}(b)$. Hence $H := \text{Aut}(\text{res}_b(b)/k, A)$ is a subgroup of a semi-direct product of V and the multiplicative group $\mathbb{G}_m(k)$. The subgroup $H \cap V$ (i.e., the group of translations of $\text{res}_b(b)$ that also are automorphisms over A and k) is ∞ -definable over A . Indeed it is the set $\{u \in V : \forall \bar{y} \forall x (x \in \text{res}_b(b) \wedge \bar{y} \in k) \implies (\phi(x, \bar{y}) \iff \phi(x + u, \bar{y}))\}$ for all A -formulas $\phi(x, \bar{y})$.

As k is a pseudo-finite field, V has no nontrivial proper definable subgroup. And because in a pseudo-finite field any ∞ -definable group is an intersection of definable groups, V has no nontrivial proper ∞ -definable subgroup either. If $H \cap V = V$ then H acts transitively on $\text{res}_b(b)$ (by translation) and, as $H \leq \text{Aut}(\text{res}_b(b)/A)$, we are done. On the contrary, if $H \cap V = \{1\}$, then H contains no translations and must either have exactly one fixed point or be the trivial group and hence fix all points in $\text{res}_b(b)$.

Suppose H has only one fixed point $a \in \text{res}_b(b)$ and let $\theta \in \text{Aut}(\text{res}_b(b)/A)$. For any $\sigma \in H$, $\theta^{-1} \circ \sigma \circ \theta \in H$ and hence $(\theta^{-1} \circ \sigma \circ \theta)(a) = a$, i.e., $\theta(a)$ is fixed by σ . As a is the only point fixed by H , $\theta(a) = a$ and $a \in \text{dcl}^{\text{eq}}(A)$: but this is a contradiction. It follows that H fixes every point in $\text{res}_b(b)$ and hence, because k is stably embedded, $\text{res}_b(b) \subseteq \text{dcl}^{\text{eq}}(k, A)$. But then we must also have $V \subseteq \text{dcl}^{\text{eq}}(k, A)$. Hence $(V, \text{res}_b(b))$ is A -definably isomorphic to a definable (regular) homogeneous space (G, R) of $k^{\text{eq}} = k$. As k is stably embedded, (G, R) is definable over $A' := k^{\text{eq}}(\text{dcl}^{\text{eq}}(A)) = k(\text{dcl}^{\text{eq}}(A))$.

Hence we only have to show that any A' -definable 1-dimensional affine space in a pseudo-finite field has an A' -point to obtain a contradiction. Let us consider k elementarily embedded in the fixed field of $L \models \text{ACFA}$ and let $\overline{A'}$ be the algebraic closure of A' in L . Note that A is algebraically closed in ACFA and is a model of ACF . By usual arguments (e.g., [45]) there exists an ACF $\overline{A'}$ -definable homogeneous space (G', S') and interalgebraic group configurations in (G, R) and (G', S') . Replacing G' with its identity component G'_0 and S' with the G'_0 -orbit of any $\overline{A'}$ -point in S' (there is such a point because $\overline{A'} \models \text{ACF}$), we can suppose that G' is connected. By some additional classical arguments (although the literature mainly concerns itself with groups and not homogeneous spaces at this point: see [45] again), there is an $\overline{A'}$ -definable subgroup H of $G \times G'$, such that $H_0 := \{x \in G : (x, 0) \in H\}$ and $H'_0 := \{x \in G' : (0, x) \in H\}$ are finite central subgroups and the left and right projections of H must have finite index in G (respectively G'). But as G and G' are connected, these projections must be the groups themselves. As G has no torsion (we are in characteristic 0), H_0 is trivial. Taking the quotient of (G', S') by H'_0 —i.e., considering the group G'/H'_0 acting on the H'_0 orbits of S' —the group H is in fact (the graph of) an isomorphism. In particular, as G has no proper definable subgroup, this implies that the action of G' on S' is also regular, i.e., S' is a G' -torsor.

Let (a, a') be generic in $R \times S'$, let X be the H -orbit of (a, a') and let $P = \text{tp}(aa'/\overline{A'})$. As P and X have the same dimension (equal to 1), P cannot be covered with infinitely many H -orbits (pseudo-finite fields have the (E) property of [43]) and as $\overline{A'}$ is algebraically closed (including imaginaries), X must contain P and hence is $\overline{A'}$ -definable. Moreover, it is quite easy to see that X is (the graph of) an isomorphism between R and S' . As S' contains $\overline{A'}$ -points, so does R . Let d be one of these points, and let $(d_i)_{i=1\dots n}$ be its A' -conjugates. Then $1/n \sum_i d_i \in R(A')$, and we have the A' -point we have been looking for. \square

To conclude this section, we summarize the classification of unary types in PL_0 .

Proposition 3.18 *Suppose T extends PL_0 and $k(A)$ contains the constants needed for k to have EI. Let $a \in \mathbb{B}(\text{dcl}^{\text{eq}}(A))$ with $a \subseteq b_i$ for each i . Then $q_P|A$ is complete relative to $\text{val}(x - a)$ and to $\overline{r}(x - a)$. Moreover, if P does not contain any ball in $\mathbb{B}(\text{dcl}^{\text{eq}}(A))$ then $q_P|A$ is complete.*

Proof If P is strict we can apply Proposition 3.9. If not, we apply Corollary 3.16 or Corollary 3.17. \square

4 The p -adic case

Let L be a finite extension of \mathbb{Q}_p . As stated in Remark 2.8.2, it can be shown that there exists a number field $F \subseteq L$ that contains a uniformizer λ_0 of L , such that $\text{res}(L) = \text{res}(F)$ and such that every finite extension L' of L is generated by an element α whose minimal polynomial is defined over F and such that α also generates the valuation ring $\mathcal{O}(L')$ over $\mathcal{O}(L)$. Let T_L denote the theory of L in $\mathcal{L}_G \cup \{P_n : n \in \mathbb{N}_{>0}\} \cup \{c\}$, where the predicates P_n stand for the nonzero n th powers (in the sort K) and c generates F over \mathbb{Q} . Then T_L is model complete (cf. [57]) and it is axiomatized by the fact that K is a henselian valued field with value group a \mathbb{Z} -group and residue field \mathbb{F}_p , by the isomorphism type of F and by the definition of the P_n predicates.

We now check the hypotheses of Corollary 2.17 for $T = T_L$ and $\tilde{T} = \text{ACVF}_{0,p,F}^G$ (the theory of algebraically closed valued fields of mixed characteristic in the geometric language with a constant for c ; the F in the subscript is there to recall that we added a constant for a generator of F to the theory). We use the same notation as in Proposition 2.13.

(i) **Relative algebraic boundedness:** By model completeness and the nature of the axioms—the only axioms that are not universal are the fact that the field is henselian and the definition of the predicates P_n ; but both state the existence of algebraic points— $\text{acl}_{\tilde{\mathcal{L}}}(M'c) \cap M$ is an elementary submodel of M , hence certainly \mathcal{L} -definably closed.

(ii) **Internalizing $\tilde{\mathcal{L}}$ -codes:** As $K(M)$ is henselian, $K(\text{dcl}_{\tilde{\mathcal{L}}}(M)) = K(M)$, hence if $e \in K$, there is nothing to do. For any element e of $S_n(\tilde{M})$ let us write $\Lambda(e) \subseteq K^n$ for the lattice represented by e . If $e \in S_n(\text{dcl}_{\tilde{\mathcal{L}}}(M))$, $\Lambda(e)$ has a basis in some finite extension L_0 of $L := K(M)$. Say $[L_0 : L] = m_0$; let L' be the join of all field extensions of L of degree m_0 . Then L' is a finite extension of L such that any $\sigma \in \text{Aut}(\tilde{M})$ stabilizing M , stabilizes L' ; let $[L' : L] = m$. By hypothesis, there is a generator a of L' over L whose characteristic polynomial over L is defined over F . One has an a -definable isomorphism $f_a : L' \rightarrow L^m$ (as vector spaces over L), with $f_a(\mathcal{O}(L')) = \mathcal{O}(L)^m$ (i.e., $\mathcal{O}(L')$ is a free $\mathcal{O}(L)$ -module of rank m). The morphism f_a further induces an isomorphism of the lattice $\Lambda(e)(L')$ with a lattice $f_a(\Lambda(e)(L')) = \Lambda(e')(L)$ for some $e' \in S_{nm}(M)$. As any a' of the (finitely many) that are $\text{Aut}(L'/F)$ -conjugate to a is also $\text{Aut}(L'/L)$ -conjugate to a , we see that $\Lambda(e')(L) = f_{a'}(\Lambda(e)(L'))$ as well. Thus e and e' are interdefinable in the sense required in (ii).

The argument for T_n is similar (alternatively, for finite extensions L' of L , the value group also has a least element, so we can apply Remark 2.8.1).

Remark 4.1 We have proved something slightly stronger than (ii): we also have $e \in \text{dcl}_{\tilde{\mathcal{L}}}(e')$. The inverse of f_a is a linear map $L^m \rightarrow L'$, say $g_a(\alpha_1, \dots, \alpha_m) := \sum \alpha_i a^i$. From the viewpoint of \tilde{M} , g_a is an a -definable linear map with $g_a(\Lambda(e')) = \Lambda(e)$ (as g_a is K -linear, this remains true for the lattices generated by the L - or L' - points of $\Lambda(e)$ and $\Lambda(e')$). Moreover this is also true for any of the finitely many conjugates of a . Thus $e \in \text{dcl}_{\tilde{\mathcal{L}}}(e')$.

The following corollary of this stronger version of (ii) is not needed for what follows but it does shed some light on the interaction between automorphisms of M and $\tilde{\mathcal{L}}$ -definable sets.

Corollary 4.2 *Let $A = \text{dcl}_{\mathcal{L}}(A) \subseteq M$. Let G be the group of automorphisms of \tilde{M} that stabilize M and fix A point-wise. Let $e \in \tilde{M}$, and assume $g(e) = e$ for all $g \in G$. Then $e \in \text{dcl}_{\tilde{\mathcal{L}}}(A)$.*

Proof We have $e \in \text{dcl}_{\tilde{\mathcal{L}}}(M)$, since $\text{Aut}(\tilde{M}/M)$ fixes e . Let e' be as in (ii). Then G fixes e' ; since G maps surjectively to $\text{Aut}(M/A)$, we have $e' \in A$. By Remark 4.1, $e \in \text{dcl}_{\tilde{\mathcal{L}}}(e')$. So $e \in \text{dcl}_{\tilde{\mathcal{L}}}(A)$. \square

(iii) Unary EI: In [60] P. Scowcroft proved a weak version of this, where the sets are classes of equivalence relations in two variables. We prove here that every unary subset can be coded in \mathbb{B} .

Let e be an imaginary code for a unary subset $D \subseteq K(M)$. Let $A = \text{acl}_{\mathcal{L}}^{\text{eq}}(e)$ and let $B = \mathbb{B}(A)$.

Claim 4.3 For all $c \in K(M)$, $\text{tp}_{\mathcal{L}}(c/B) \vdash \text{tp}_{\mathcal{L}}(c/A)$.

Proof Following the notation of Corollary 3.10, recall that $W(c; A) = \{b \in \mathbb{B}(A) : c \in b\}$. Let $P = \bigcap W(c; A) = \bigcap W(c; B)$, a strict intersection. Then $\text{tp}_{\mathcal{L}}(c/B) \vdash q_P|B = q_P|A$. By Proposition 3.9, either $q_P|A$ is a complete type and we are done, or there is some $a \in B$ such that $a \subseteq P$ and $q_P|A$ is complete relative to $\bar{r}(x-a)$ and $\text{val}(x-a)$. As $K^* = F(K^*)^n$ for all n , $\text{tp}_{\mathcal{L}}(\bar{r}(c-a)/A)$ follows from its type over F , i.e., over $\text{dcl}_{\mathcal{L}}(\emptyset)$. Moreover $\Gamma(\text{dcl}_{\mathcal{L}}(B)) = \Gamma(\text{dcl}_{\mathcal{L}}(A))$ (as elements of Γ are coded by balls). Thus, as Γ is stably embedded and has unary EI, $\text{tp}_{\mathcal{L}}(\text{val}(c-a)/B) \vdash \text{tp}_{\mathcal{L}}(\text{val}(c-a)/A)$ and we have the expected result. \square

As D is $\mathcal{L}(A)$ -definable, D is also $\text{Aut}(M/B)$ -invariant, so that by compactness D is definable over B . Hence $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(B)$. We conclude as in Corollary 2.17: there is a tuple a from \mathbb{B} with $a \in \text{acl}_{\mathcal{L}}(e)$ and $e \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(a)$; so $\text{dcl}_{\mathcal{L}}^{\text{eq}}(e) = \text{dcl}_{\mathcal{L}}^{\text{eq}}(a')$, where a' is the finite set of $\mathcal{L}(e)$ -conjugates of a . We already know that finite sets are coded (e.g., by (ii) and Lemma 2.14).

(iv) Invariant types and germs: The main ingredient for this proof is the Minimality of ACVF.

Let $A = \text{acl}_{\mathcal{L}}(A)$, $c \in K(M)$, $W(c; A) = \{b_i : i \in I\}$ and $P = \bigcap_i b_i$. The balls b_i are linearly ordered by inclusion, and we order I correspondingly: $i \leq j$ holds if $b_j \subseteq b_i$. As seen previously, P is a strict intersection. Let \tilde{p} be the ACVF generic of P .

If $r(x, b)$ is an $\tilde{\mathcal{L}}$ -definable function, let $X(b', b'') = \{x : r(x, b') \neq r(x, b'')\}$. Then $X(b', b'')$ is a finite Boolean combination of balls and there exists $i = i(b', b'')$ such that $X(b', b'') \cap P$ is contained in a proper subball of P if and only if for each $j \geq i$, $X(b', b'') \cap b_j$ is contained in a proper subball of b_j .

Define an equivalence relation E_i by $b'E_i b''$ if and only if $X(b', b'') \cap b_i$ is contained in a proper subball of b_i (i.e., $r(x, b')$ and $r(x, b'')$ have the same germ on the ACVF-generic of b_i). Let $e_i = b/E_i$. Then:

$$\begin{aligned}
& \sigma \in \text{Aut}(\widetilde{M}/A) \text{ fixes the germ of } r(x, b) \\
& \iff r(x, b) \text{ and } r(x, \sigma b) \text{ have the same } \widetilde{p}\text{-germ} \\
& \iff X(b, \sigma b) \cap P \text{ is contained in a proper subball of } P \\
& \iff \text{for some } i, \text{ for all } j \geq i, bE_j \sigma(b) \\
& \iff \text{for some } i, \text{ for all } j \geq i, \sigma \text{ fixes } e_j.
\end{aligned}$$

As for the consistency of $\widetilde{p}|M$ with $\text{tp}_{\mathcal{L}}(c/A)$: by definition of the ACVF generic, $\widetilde{p}|M$ is generated by P along with all formulas $x \notin b$, where $b \in \mathbb{B}(\text{dcl}_{\widetilde{\mathcal{L}}}(M))$ is a proper subball of P . As P is part of $\text{tp}_{\mathcal{L}}(c/A)$, it suffices to show that $\text{tp}_{\mathcal{L}}(c/A)$ does not imply any formula $x \in d$ with d an $\mathcal{L}(m)$ -definable finite union of balls $d_j \in \mathbb{B}(\text{dcl}_{\widetilde{\mathcal{L}}}(M))$ strictly included in P .

Claim 4.4 For all $b \in \mathbb{B}(\text{dcl}_{\widetilde{\mathcal{L}}}(M))$ such that $b(M) \neq \emptyset$, there exists $b' \in \mathbb{B}(M)$ such that $b(M) = b'(M)$.

Proof As Γ is definably well-ordered, $\inf\{\text{val}(a - c) : a, c \in b(M)\} = \gamma \in \Gamma(M)$. We can now take b' to be the ball of radius γ around any point in $b(M)$. \square

It follows that $d(M)$ is equal to a finite union d' of balls in $\mathbb{B}(M)$ and $\text{tp}_{\mathcal{L}}(c/A)$ implies $x \in d' \subseteq P$. But this would contradict Lemma 3.8.

(v) Weak rigidity: We use Lemma 2.18. The hypothesis that for all $a \in M$ there is a tuple $c \in K(M)$ such that $a \in \text{dcl}_{\mathcal{L}}(c)$ and $\text{tp}_{\mathcal{L}}(c/\text{acl}_{\mathcal{L}}(a))$ extends to an $\text{Aut}(M/\text{acl}_{\mathcal{L}}(a))$ -invariant type, holds trivially when $a \in K(M)$ and follows from Corollary 3.13 when $a \in \Lambda(M)$. If $a \in T_m(M)$ for some m then, as the value group has a least element, a is coded by an element of $S_{m+1}(M)$ (see Remark 2.8.1) and hence, applying Corollary 3.13 to the code in $S_{m+1}(M)$, we are done.

The assumption (v') of Lemma 2.18 is proved for \mathbb{Q}_p by van den Dries in [22]. Let us briefly recall his proof to check that it adapts to the finite extension of \mathbb{Q}_p case.

Let $B \subseteq K(M)$ (we can assume that $B = \text{dcl}_{\widetilde{\mathcal{L}}}(B) \cap K(M)$ is a field and contains F). Let $\sigma \in \text{Aut}(M/B)$ and let $B' = \text{fix}(\sigma)(\text{acl}_{\widetilde{\mathcal{L}}}(B) \cap M)$. It suffices to show that $B' \models T_L$. Indeed, by model completeness, $B' \prec M$ will then contain $\text{acl}_{\mathcal{L}}(B)$, hence $\text{acl}_{\mathcal{L}}(B)$ is rigid over B .

As noted in the proof of (i), in order to show that $B' \models T_L$, we only have to show that B' is henselian and that the definition of the P_n is preserved.

By the universal property of the henselization, B^h is contained in B' and thus B' is henselian. Moreover, let $x \in B' \cap P_n(M)$ and let $y \in K(M)$ such that $x = y^n$. Note first that $(y/\sigma(y))^n = x/\sigma(x) = 1$ and thus that $y/\sigma(y) \in \text{acl}_{\mathcal{L}}(\emptyset)$. Furthermore, for all $m \in \mathbb{N}$, there exists $q \in F$ such that $yq \in P_m(M)$ (because $K(M)$ is henselian, $\Gamma = \text{val}(F^*) + m\Gamma$ and $\mathcal{O} = \mathcal{O}(F) + \lambda_0^{m'}\mathcal{O}$). But then $y/\sigma(y) = yq/\sigma(yq) \in P_m(M)$ for all m . As $\bigcap_m P_m(\text{acl}_{\mathcal{L}}(\emptyset)) = \{1\}$, it follows that $y = \sigma(y)$, i.e., $y \in B'$.

Remark 4.5 As in [22], it follows from this proof that the restriction of T_L to the sort K has definable Skolem functions.

Proof of Theorem 2.6: By Corollary 2.17, we have EI to the sorts K, S_n, T_n . But as is explained in Remark 2.8.1, the sorts T_n are not actually needed.

We finish the section with some additional remarks.

Remark 4.6 If we do not want to add a constant c to the language, then it suffices to add “Galois-twisted S_n ”, interpreted as $S_n(K')$ for K' ranging over the finite extensions of K .

Indeed, by Theorem 2.6, any imaginary e is interdefinable over c with some tuple of real elements e' . So we have an e -definable function f_e with $f_e(c) = e'$ and a \emptyset -definable function h with $h(c, f_e(c)) = e$. As c is algebraic over \mathbb{Q} , restricting to e -conjugates of c , we can take the graph of f_e (a finite set) to be a complete type over e .

With the new sorts, it is clear that (ii) holds without adding a constant and f_e is coded by some tuple $d \in M$. Let us now show that d is a code for e . If e' is $\mathcal{L}(d)$ -conjugate to e there is some $\sigma \in \text{Aut}(M/d)$ such that $\sigma(e) = e'$. As σ fixes d , $\sigma(c)$ is also in the domain of f_e and hence $\text{tp}_{\mathcal{L}}(c'/e) = \text{tp}_{\mathcal{L}}(c/e)$, i.e., $e' = \sigma(e) = \sigma(h(c, f_e(c))) = h(\sigma(c), f_e(\sigma(c))) = e$. This implies that d is a code for e .

Remark 4.7 Let $A = \text{acl}_{\mathcal{L}}(A) \subseteq M \models T_L$. Then every type over A extends to an $\text{Aut}(M/A)$ -invariant type.

Proof Let $c \in M$; then $c = f(a_1, \dots, a_n)$, where $a_i \in \text{dom}(M)$, and f is \emptyset -definable. It suffices to extend $\text{tp}_{\mathcal{L}}(a_1, \dots, a_n/A)$ to an $\text{Aut}(M/A)$ -invariant type. If $\text{tp}_{\mathcal{L}}(c/M)$ and $\text{tp}_{\mathcal{L}}(d/Mc)$ are $\text{Aut}(M/A)$ -invariant, then so is $\text{tp}_{\mathcal{L}}(cd/M)$; so it suffices to show that $\text{tp}_{\mathcal{L}}(a_i/A_i)$ extends to an $\text{Aut}(M/A_i)$ -invariant type for each i , where $A_i := \text{dcl}_{\mathcal{L}}(A_{i-1}a_{i-1})$. But (by hypothesis (v) of Corollary 2.17) we have that $A_i = \text{acl}_{\mathcal{L}}(A_i)$, so Corollary 3.10 applies. \square

Remark 4.8 Rigidity of finite sets fails for the theory of the p -adics in the geometric language, i.e., $\text{acl}_{\mathcal{L}} \neq \text{dcl}_{\mathcal{L}}$.

Proof As the value group is stably embedded, one can find a non-trivial automorphism σ fixing the value group in a sufficiently saturated model. By definability of the angular component function ac , it follows that x and $\sigma(x)$ have the same angular coefficient. Take $a \in \mathcal{O}$ with $\sigma(a) \neq a$. Let $\gamma = \text{val}(\sigma(a) - a)$, $\text{ac}(\sigma(a) - a) =: \alpha$. Then $\text{val}(\sigma^2(a) - \sigma(a)) = \gamma$, $\text{ac}(\sigma^2(a) - \sigma(a)) = \alpha$, etc. As $p \cdot \alpha = 0$ in the residue field, $(\sigma^p(a) - a) = \sum_{i=0}^{p-1} (\sigma^{i+1}(a) - \sigma^i(a))$ has valuation $\delta > \gamma$. Thus in the ring $\mathcal{O}/\delta\mathcal{O}$, the image of a is not a fixed point, but has an orbit of size p under σ . This set of size p is not rigid. \square

Remark 4.9 The same techniques developed here to prove elimination of imaginaries in \mathbb{Q}_p can also be used to give an alternative proof for elimination of imaginaries in real closed valued fields (see [53]). Hypothesis (i) of Corollary 2.17 also follows from the fact that the algebraic closure is a model, (ii) follows as in the p -adic case, (iii) follows from the description of 1-types given in [53, Proposition 4.8]; and so does the existence of the type in (iv). The rest of (iv) is proved exactly as here and so is (v).

5 The asymptotic case

Recall that HF_0 denotes the theory of henselian fields of residue characteristic 0 and PL_0 is the theory of henselian fields with value group a \mathbb{Z} -group and residue field a pseudo-finite field of characteristic 0. Our goal is now to prove that any completion T_F of PL_0 in the language \mathcal{L}_G with constants added for some subfield $F \subseteq K$ (see Remark 2.8.2) eliminates imaginaries. We will be using Proposition 2.13 with $T = T_F$ and $\tilde{T} = \text{ACVF}_{0,0,F}$. We still follow the notation of this proposition.

It is worth noting that we will not, in general, be able to use Corollary 2.17 as there are some ultraproducts of p -adics where (v) is false. Indeed, it is shown in [9, Theorem 7] that there exist a characteristic zero pseudo-finite field L , $A \subseteq L$, and $b \in L$ such that b has a finite orbit over A . Then A can be identified with the set $A' := \{at^0 : a \in A\} \subseteq L((t)) \models \text{PL}_0$ and b is algebraic but not definable over A' . It is easy to build a counter-example to (v) using A' and b .

(i) Relative algebraic boundedness: The proof is not as simple as in the p -adic case and needs some preliminary lemmas and definitions.

Definition 5.1 We will say that that T is *algebraically bounded (with respect to \tilde{T}) within the sort S* if for all $M \models T$ and $A \subseteq \text{dom}(M)$, $S(\text{acl}_{\mathcal{L}}(A)) \subseteq S(\text{acl}_{\tilde{\mathcal{L}}}(A))$.

Even if S is stably embedded, one must beware that this is, in general, slightly different from saying that $\text{Th}_{\mathcal{L}}(S)$ (the theory induced by T on the sort S) is algebraically bounded (with respect to $\text{Th}_{\tilde{\mathcal{L}}}(S)$), as in the latter case, one requires that $S(\text{acl}_{\mathcal{L}}(A)) \subseteq S(\text{acl}_{\tilde{\mathcal{L}}}(A))$ holds for all $A \subseteq S$.

Lemma 5.2 *Let $T_F \supseteq \text{HF}_0$ be such that $k^*/(k^*)^n$ is finite and $k^* = (k^*)^n \text{res}(F)$. Then:*

1. *If $A = \text{acl}_{\tilde{\mathcal{L}}}(K(A)) \cap M$, then $\Gamma(A) = \text{val}(K(A))$;*
2. *If $\text{Th}_{\mathcal{L}}(k)$ and $\text{Th}_{\mathcal{L}}(\Gamma)$ are algebraically bounded, then T is algebraically bounded within k and Γ .*

Proof

1. For any $a \in K(M)^*$ and $\gamma \in \Gamma(M)$ such that $n\gamma = \text{val}(a)$ for some $n \in \mathbb{N}$, there exist $x \in K(M)$ such that $\text{val}(ax^{-n}) = 0$ and $c \in F$ such that $\text{res}(ax^{-n}c^{-1}) \in k^n$. As M is an equicharacteristic zero henselian field, $ax^{-n}c^{-1} \in (K(M)^*)^n$ and hence $ac^{-1} \in (K(M)^*)^n$, i.e., there exists $a' \in \overline{F(a)}^{\text{alg}} \cap M \subseteq \text{acl}_{\tilde{\mathcal{L}}}(a) \cap M$ such that $\text{val}(a') = \gamma$. As $\Gamma(\text{acl}_{\tilde{\mathcal{L}}}(K(A))) = \mathbb{Q} \otimes \langle \text{val}(K(A)) \rangle$, the statement follows.
2. Delon shows in [20, Theorem 2.1] that in the three-sorted language (K, k, Γ) with val and res , field quantifiers can be eliminated up to formulas of the form

$$\phi^*(x, r) = \exists y \in K \bigwedge_i y_i x_i \in (K^*)^{n_i} \wedge \text{val}(y_i) = 0 \wedge \phi(r, \text{res}(y)),$$

where r is a tuple of variables from k . It follows immediately that if $A \subseteq K(M)$ then $\Gamma(\text{acl}_{\mathcal{L}}(A)) \subseteq \text{acl}_{\mathcal{L}}(\text{val}(A)) \subseteq \text{acl}_{\tilde{\mathcal{L}}}(\text{val}(A)) \subseteq \text{acl}_{\tilde{\mathcal{L}}}(A)$, where the first inclusion follows from field quantifier elimination and the second from algebraic boundedness of $\text{Th}_{\mathcal{L}}(\Gamma)$.

The presence of the ϕ^* makes it a little more complicated for k , but $\phi^*(a, r)$ implies that $a_i y_i \in (K^*)^{n_i}$ for some y_i such that $\text{val}(y_i) = 0$ and hence that $n_i | \text{val}(a_i)$. By the first statement, there exist $b_i \in \text{acl}_{\tilde{\mathcal{L}}}(A) \cap M$ such that $n \text{val}(b_i) = \text{val}(a_i)$, thus $\phi^*(a, r) \iff \exists y \in k \bigwedge_i y_i \text{res}(a_i b_i^{-n}) \in (k^*)^{n_i} \wedge \phi(r, y)$. It is now clear that $k(\text{acl}_{\mathcal{L}}(A)) \subset \text{acl}_{\mathcal{L}}(\text{res}(\text{acl}_{\tilde{\mathcal{L}}}(A) \cap M))$ and we can conclude as for Γ . \square

In the next three lemmas, we will suppose that the hypotheses of the previous lemma apply to T .

Lemma 5.3 *For all $A \subseteq K(M)$, $\text{RV}(\text{acl}_{\mathcal{L}}(A)) \subseteq \text{RV}(\text{acl}_{\tilde{\mathcal{L}}}(A))$, i.e., T is algebraically bounded within RV .*

Proof Let $c \in \text{RV}(\text{acl}_{\mathcal{L}}(A))$ and let $\gamma = \text{val}_{\text{rv}}(c)$. Then by Lemma 5.2, $\gamma \in \mathbb{Q} \otimes \text{val}(A)$. It follows that there exist $c' \in K(\text{acl}_{\tilde{\mathcal{L}}}(A) \cap M)$ and $n \in \mathbb{N}$ such that $\text{val}(c') = n\gamma$. Then $c^n / \text{rv}(c') \in k(\text{acl}_{\mathcal{L}}(A)) \subseteq k(\text{acl}_{\tilde{\mathcal{L}}}(A))$ —also by Lemma 5.2—and hence $c \in \text{acl}_{\tilde{\mathcal{L}}}(A)$. \square

Lemma 5.4 *For any $A = \text{acl}_{\tilde{\mathcal{L}}}(K(A)) \cap M$, $\mathbb{B}(\text{acl}_{\mathcal{L}}(A)) = \mathbb{B}(A)$. Moreover, any ball $b \in \mathbb{B}(\text{acl}_{\mathcal{L}}(A))$ contains a point in A .*

Proof Let $b \in \mathbb{B}(\text{acl}_{\mathcal{L}}(A))$ and let Q be the intersection of all balls in $\mathbb{B}(A)$ that contain b . As Q is $\text{Aut}(\widetilde{M}/A)$ -invariant, it suffices to show that b contains Q (and hence is equal to Q) to show it is $\text{Aut}(\widetilde{M}/A)$ -invariant and thus in $\text{dcl}_{\tilde{\mathcal{L}}}(A) \cap M = A$.

If $Q(A) = \emptyset$, it follows from Remark 3.6 that Q is a complete type over A in M , so Q is contained in b . Hence we can assume that we have a point $a \in Q(A)$. We can suppose $a \notin b$, or, because $\text{rad}(b) \in \Gamma(\text{acl}_{\mathcal{L}}(A)) \subseteq \Gamma(\text{acl}_{\tilde{\mathcal{L}}}(A) \cap M) = \Gamma(A)$, we would be done.

If Q is a closed ball that strictly contains b , then b is contained in a unique maximal open subball b' of Q . This subball b' is interdefinable over A (in \widetilde{M}) with $\text{rv}(b - a) \in \text{RV}(\text{acl}_{\mathcal{L}}(A)) \subseteq \text{RV}(\text{acl}_{\tilde{\mathcal{L}}}(A) \cap M) = \text{RV}(A)$, where the first inequality follows from Lemma 5.3. Hence b' is in A , contains b and is strictly contained in Q , contradicting the definition of Q .

Finally, if Q is a strict intersection or an open ball, then $\text{val}(b - a) \in \Gamma(\text{acl}_{\mathcal{L}}(A)) = \Gamma(A)$, thus the closed ball of radius $\text{val}(b - a)$ around a would be in A , would contain b and would be strictly contained in Q , a contradiction.

As for the second point, once we know that $b \in \mathbb{B}(A)$, since $\text{acl}_{\tilde{\mathcal{L}}}(A)$ is a model of ACVF, b contains a point c in $K(\text{acl}_{\tilde{\mathcal{L}}}(A)) = \overline{K(A)}^{\text{alg}}$ and, as balls are convex in residue characteristic zero, the average of the $\text{Aut}(\widetilde{M}, A)$ -conjugates of c is in $b(\text{dcl}_{\tilde{\mathcal{L}}}(A) \cap M) = b(A)$. \square

Lemma 5.5 *For any $A \subseteq \text{dcl}_{\tilde{\mathcal{L}}}(K(A)) \cap M$, $\text{acl}_{\mathcal{L}}(A) \subseteq \text{acl}_{\tilde{\mathcal{L}}}(A)$. In particular, for any $M' \prec M$ and $c \in K(M)$, $\text{acl}_{\mathcal{L}}(M'c) \subseteq \text{acl}_{\tilde{\mathcal{L}}}(M'c)$.*

Proof Let $C = \text{acl}_{\tilde{\mathcal{L}}}(A) \cap M$, so that $C = \text{acl}_{\tilde{\mathcal{L}}}(K(C)) \cap M$, and let $e \in \text{acl}_{\mathcal{L}}(A)$. If $e \in K \subseteq \mathbb{B}$, then Lemma 5.4 applies to e —viewed as a ball with an infinite radius—and we have $e \in C \subseteq \text{acl}_{\tilde{\mathcal{L}}}(A)$.

The remaining sorts S_n and T_n can be viewed as $B_n(K)/H$ (or a union of such in the case of T_n) where H is an $\tilde{\mathcal{L}}$ -definable subgroup. Note that there exists an

increasing sequence of $\tilde{\mathcal{L}}$ -definable subgroups $(G_i)_{i=1\dots m}$ of $B_n(K)$ with $G_0 = \{1\}$ and $G_m = B_n(K)$ such that for every i , there exists an $\tilde{\mathcal{L}}$ -definable morphism $\phi_i : G_i \rightarrow G$ with kernel G_{i-1} , where G is either the additive group $\mathbb{G}_a(K)$, or the multiplicative group $\mathbb{G}_m(K)$, and such that for every point $a \in G(C)$, $\phi^{-1}(a)$ contains a point in $G_i(C)$. It suffices to show by induction on i that if $H_i := G_i \cap H$ is an $\tilde{\mathcal{L}}$ -definable subgroup of G_i and $e \in (G_i/H_i)(\text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(C))$ then e is $\tilde{\mathcal{L}}(C)$ -definable.

Let $\phi : G_i \rightarrow G$, where $G = \mathbb{G}_a(K)$ or $G = \mathbb{G}_m(K)$, be a group homomorphism with kernel G_{i-1} . Then $e \in (G_i/H_i)(\text{acl}_{\tilde{\mathcal{L}}}^{\text{eq}}(C))$ can be viewed as an almost $\mathcal{L}(C)$ -definable coset $eH_i \subseteq G_i$ —i.e., a finite union of these cosets is $\mathcal{L}(C)$ -definable—and $\phi(eH_i)$ is an almost $\mathcal{L}(C)$ -definable coset of $\phi(H_i)$. Moreover, the group $H := \phi(H_i)$ is an $\tilde{\mathcal{L}}$ -defined subgroup of G . If $G = \mathbb{G}_a$, H has the form $y\mathcal{O}$ or $y\mathcal{M}$ and if $G = \mathbb{G}_m$, $H = \mathcal{O}^*$ or $H = 1 + I$ where I is some proper ideal of \mathcal{O} . Thus the cosets of H are either balls or annuli of the form $y\mathcal{O}^*$. In both cases, $\phi(eH_i)$ has a point $a \in C$ (in the ball case, apply Lemma 5.4, and in the other case, this is because such an annulus is equal to some $\text{val}^{-1}(\gamma)$ where $\gamma \in \Gamma(\text{acl}_{\tilde{\mathcal{L}}}(C)) = \Gamma(C) = \text{val}(K(C))$, by Lemma 5.2).

Let $a' \in \phi^{-1}(a) \cap G_i(C) = (a'G_{i-1}) \cap G_i(C)$; then $a'^{-1}(eH_i \cap a'G_{i-1})$ is a coset of $H_{i-1} = H_i \cap G_{i-1}$ in G_{i-1} that is almost $\mathcal{L}(C)$ -definable. By induction, $a'^{-1}(eH_i \cap a'G_{i-1})$ is $\tilde{\mathcal{L}}(C)$ -definable, but then $(eH_i \cap a'G_{i-1})$ is also $\tilde{\mathcal{L}}(C)$ -definable and hence eH_i —the only coset of H_i that contains $eH_i \cap a'G_{i-1}$ —is $\tilde{\mathcal{L}}(C)$ -definable. \square

(ii) Internalizing $\tilde{\mathcal{L}}$ -codes: Let $L = \prod \mathbb{Q}_p/\mathcal{U}$ be a non-principal ultraproduct. Provided we have a subfield of constants F such that every finite extension of L is generated by an element whose minimal polynomial is over F and which also generates the valuation ring over $\mathcal{O}(L)$, the proof for finite extensions of \mathbb{Q}_p goes through for $\text{Th}(L)$.

(iii) Unary EI: In the following lemmas, we will consider a theory T_F extending PL_0 where we have added constants F containing a uniformizer λ_0 , such that $\text{res}(F)$ contains the necessary constants for k to have EI and for all $n \in \mathbb{N}_{>0}$, $k^* = (k^*)^n \text{res}(F)$. Let $M \models T_F$ be saturated and homogeneous enough.

We will first study the imaginaries in RV . For all $\gamma \in \Gamma(M)$, let us write $\text{RV}_\gamma := \text{val}_{\text{rv}}^{-1}(\gamma)$. Let H be a (small¹) subgroup of $\Gamma(M)$ containing $1 := \text{val}(\lambda_0)$, and let $\text{RV}_H = \bigcup_{\gamma \in H} \text{RV}_\gamma$ where a point 0_γ is added to every RV_γ . The structure induced by $T_{F,H}$ on RV_H is that of an enriched family of (1-dimensional) k -vector spaces and we view it as a structure with one sort for each $\text{RV}_\gamma \cup \{0_\gamma\}$. As H is a group, RV_H is closed under tensor products and duals.

¹With respect to the saturation and homogeneity of M .

These k -linear structures are studied in [40]. Let us recall some of the definitions there.

Definition 5.6 Let $A = (V_i)_{i \in I}$ be a k -linear structure.

1. We say that A *has flags* if for any vector space V_i in A with $\dim(V_i) > 1$, there are vector spaces V_j and V_l in A with $\dim(V_j) = \dim(V_i) - 1$, $\dim(V_l) = 1$ and a \emptyset -definable exact sequence $0 \rightarrow V_l \rightarrow V_i \rightarrow V_j \rightarrow 0$.
2. We say that A *has roots* if for any 1-dimensional V_i and any $m \geq 2$, there exist V_j and V_l in A and \emptyset -definable k -linear embeddings $f : V_j^{\otimes m} \rightarrow V_l$ and $g : V_i \rightarrow V_l$ such that $\text{Im}(g) \subseteq \text{Im}(f)$.

Lemma 5.7 *The theory of RV_H with the structure induced by $T_{F,H}$ eliminates imaginaries.*

Proof It follows from [40, Proposition 5.10] that it suffices to show that RV_H has flags and roots. As every RV_a is 1-dimensional, the structure trivially has flags. But it does not have roots. Let us extend H to some H' such that $\text{RV}_{H'}$ has roots.

Let $R = \{r \in \mathbb{N}_{>0} : k(M) \text{ contains nontrivial } r\text{th roots of unity}\}$, let $L = K(M)[\lambda_0^{1/r} : r \in R]$ and let $H' = \langle H, 1/r : r \in R \rangle \subseteq \text{val}(L)$. Note that L is a ramified extension of $K(M)$ and that $\text{res}(L) = k(M)$, hence $\text{RV}_H(M) = \text{RV}_H(L)$. Now RV_1 has r th roots in $\text{RV}_{H'}$ for any r . Indeed, if $r \in R$ then $\text{RV}_{1/r}$ is an r th root and if $r \notin R$, then as the map $\text{RV} \rightarrow \text{RV} : x \mapsto x^r$ is injective, V_1 is its own r th root.

Let us show that for any $\gamma \in H'$ and any $r \geq 2$, RV_γ has an r th root. As $\gamma \in H'$, there exists $n \in \mathbb{N}$ such that $n\gamma \in H \subseteq \Gamma(M)$, a \mathbb{Z} -group. Hence there exist $\alpha \in H$ and $m \in \mathbb{N}$ such that $n\gamma = r n \alpha + m$. Let RV_β be an n rth root of RV_1 ; then $\text{RV}_\alpha \otimes \text{RV}_\beta^{\otimes m}$ is an r th root of RV_γ . By [40, Proposition 5.10], $\text{RV}_{H'}$ has elimination of imaginaries.

Any automorphism σ of RV_H can be extended to an automorphism of $\text{RV}_{H'}$. Indeed, if $h \in \text{RV}_{H'}$ then $\text{val}_{\text{rv}}(h) = \gamma + n/r$ where $\gamma \in H$, $n \in \mathbb{Z}$ and $r \in R$, and $h \text{rv}(\lambda_0)^{-n/r} \in \text{RV}_H$. Taking $\tilde{\sigma}(h) := \sigma(h \text{rv}(\lambda_0)^{-n/r}) \text{rv}(\lambda_0)^{n/r}$ will work. Moreover, we can find an automorphism of $\text{RV}_{H'}$ fixing only RV_H . Consider the homomorphism $\phi : H' \rightarrow k(M)$ sending $\gamma + n/r$ to d_r^n where $(d_r)_{r \in \mathbb{N}} \in k(M)$ is such that for all r and l , we have $d_r^r = 1$, $d_r \neq 1$ if $r \in R$ and $d_{l_r}^l = d_r$. Then $\theta : h \mapsto h\phi(\text{val}_{\text{rv}}(h))$ is a group automorphism of $\text{RV}_{H'}$ inducing the identity on both k and H' hence an automorphism of the full structure of $\text{RV}_{H'}$. It is easy to see that θ fixes only RV_H .

Note that because each fiber is a sort, if $X \subseteq \text{RV}_H^l$ for some $l \in \mathbb{N}$ and X is definable in RV_H , then it is defined by the same formula in $\text{RV}_{H'}$. Hence it is coded by some $x \in \text{RV}_{H'}$. But as there are isomorphisms of $\text{RV}_{H'}$ fixing only RV_H , we must

have $x \in \text{RV}_H$, and as automorphisms of RV_H extend to $\text{RV}_{H'}$, x is also a code for X in RV_H . \square

Proposition 5.8 *The theory induced by T_F on the sort RV (see Section 2.2) eliminates imaginaries to the sorts RV and Γ .*

Proof First let us show that for all $n \in \mathbb{N}_{>0}$, RV/RV^n is finite and $\text{RV} = \text{RV}^n \text{rv}(F)$. Let $a \in \text{RV}$. As Γ is a \mathbb{Z} -group, there exist $y \in \text{RV}$ and $r \in \mathbb{N}$ such that $r < n$ and $\text{val}_{\text{rv}}(a) = \text{val}_{\text{rv}}(y^n) + \text{val}(\lambda_0^r)$. Hence $\text{val}_{\text{rv}}(ay^{-n} \text{rv}(\lambda_0)^{-r}) = 0$, i.e., $ay^{-n} \text{rv}(\lambda_0)^{-r} \in k^*$. As $k^* = (k^*)^n \text{res}(F)$, there exists $m \in \text{res}(F)$ such that $ay^{-n}m^{-1} \text{rv}(\lambda_0^{-r}) \in (k^*)^n$, i.e., $a \in m \text{rv}(\lambda_0^r) \text{RV}^n$.

Moreover, for any $A \subseteq \text{RV}(M)$, $\text{val}_{\text{rv}}(\text{dcl}_{\mathcal{L}}(A)) \subseteq \mathbb{Q} \otimes \text{val}_{\text{rv}}(A)$. Indeed, let $\gamma \in \Gamma(M) \setminus \mathbb{Q} \otimes \text{val}_{\text{rv}}(A)$ and $d \in \bigcap (k(M)^*)^n \setminus \{1\}$; then there exists a group homomorphism $\phi_d : \Gamma(M) \rightarrow k^*(M)$ such that $\phi_d(\text{val}_{\text{rv}}(A)) = \{1\}$, $\phi_d(\gamma) = d$ and $\psi_d : t \mapsto t\phi_d(\text{val}_{\text{rv}}(t))$ defines an automorphism of $\text{RV}(M)$ fixing A , k and Γ , which sends any $x \in \text{val}_{\text{rv}}^{-1}(\gamma)$ to $dx \neq x$. Hence $\text{val}_{\text{rv}}^{-1}(\gamma)$ cannot contain any point definable over A .

Let us now code finite sets. For any tuple $\gamma \in \Gamma$, let RV_γ denote $\prod_i \text{RV}_{\gamma_i}$.

Claim 5.9 In the theory induced by T_F on the sorts $\text{RV} \cup \Gamma$, finite sets are coded.

Proof Let $X \subseteq \text{RV}^i \times \Gamma^j$ be finite. As Γ is ordered, we can suppose that there are tuples γ and $\gamma' \in \Gamma$ such that $X \subseteq \text{RV}_\gamma \times \{\gamma'\}$. By Lemma 5.7, the projection of X on RV_γ is coded (over γ) by some $x \in \text{RV}_{\langle 1, \gamma \rangle}$. It is easy to see that $x\gamma\gamma'$ is a code for X . \square

To prove elimination of imaginaries in RV to the sorts RV and Γ , it suffices, by Lemma 2.2, to code $\mathcal{L}(A)$ -definable functions $f : \text{RV} \rightarrow R$, where R is either RV or Γ , for any $A \subseteq \text{RV}(M)$. Let us first consider the case $R = \text{RV}$. Let D be the domain of f and X its graph.

Lemma 5.10 *If there exist n and $m \in \mathbb{Z}$ such that for all $x \in D$, $n \text{val}_{\text{rv}}(f(x)) - m \text{val}_{\text{rv}}(x)$ is constant, then f is coded.*

Proof Let $\gamma_f = n \text{val}_{\text{rv}}(f(x)) - m \text{val}_{\text{rv}}(x) \in \Gamma(\text{dcl}_{\mathcal{L}}(\langle f \rangle))$. For all $y \in k^*$ and $x, z \in \text{RV}$, let $y \cdot (x, z) = (y^n x, y^m z)$. This defines an action of k^* on any RV_γ where γ is a 2-tuple. Let $y \in \bigcap_n (k^*)^n$ and $\gamma \in \Gamma(M)^2$ be such that $n\gamma_2 - m\gamma_1 = \gamma_f$ and $\gamma_1 \notin \mathbb{Q} \otimes \langle \text{val}_{\text{rv}}(A) \rangle$. By a similar automorphism construction as above, there is $\psi \in \text{Aut}(\text{RV}(M)/A)$ such that for all $x \in \text{RV}_\gamma$, $\psi(x) = y \cdot x$ and hence $x \in X$ implies $y \cdot x \in X$. By compactness, there exists $N \in \mathbb{N}_{>0}$ such that for any $x \in \text{RV}$ with

$\text{val}_{\text{rv}}(x) \notin \mathbb{Q} \otimes \langle \text{val}_{\text{rv}}(A) \rangle$ and for any $y \in (k^*)^N$, if $x \in X$ then $y \cdot x \in X$. Let $X' = \{x \in X : \forall y \in (k^*)^N, y \cdot x \in X\}$. Then it suffices to code X' and $X \setminus X'$. Note that $(x, y) \in X \setminus X'$ implies $\text{val}_{\text{rv}}(x) \in \mathbb{Q} \otimes \langle \text{val}_{\text{rv}}(A) \rangle$.

Claim 5.11 Suppose that X is stable under the action of $(k^*)^N$. Then f is coded.

Proof Let $E \subseteq \text{rv}(F)$ intersect all the classes of RV modulo $\text{RV}^{(Nn)}$. Fix $\gamma \in \Gamma$. For any $x \in D_\gamma := D \cap \text{RV}_\gamma$, there exist $y \in \text{RV}^N$ and $e \in E$ such that $x = y^n e$. As X is $(k^*)^N$ -stable, one can check that $g_\gamma(e) := y^{-m} f(x)$ depends only on e and γ . One can also check that $\text{val}_{\text{rv}}(g_\gamma(e)) = 1/n(\gamma_f + m \text{val}_{\text{rv}}(e)) \in \Gamma(\text{dcl}_{\mathcal{L}}(\langle f \rangle)) =: H$ and g_γ is in fact a function (with a finite graph G_γ) definable in RV_H . By Lemma 5.7 and compactness, there is a definable function $g : \Gamma \rightarrow \text{RV}_H^l$ for some $l \in \mathbb{N}$ such that $g(\gamma)$ codes g_γ (over H). It is quite clear that g is $\mathcal{L}(\langle f \rangle)$ -definable, but as $X = \bigcup_{\gamma \in \Gamma} (k^*)^N G_\gamma$, f is also $\mathcal{L}(H \langle g \rangle)$ -definable.

Now, as Γ has Skolem functions, we can definably order $\text{Im}(g)$, and, because RV_H^l is internal to k and the induced theory on k is simple, $\text{Im}(g)$ must be finite (a simple theory cannot have the strict order property). Thus $\text{Im}(g) \subseteq \text{acl}_{\mathcal{L}}(\langle f \rangle)$. For any $e \in \text{Im}(g)$, $g^{-1}(e) \subseteq \Gamma$ is coded. Let d be the tuple of all codes of fibers and corresponding images; then $d \in \text{acl}_{\mathcal{L}}(\langle f \rangle)$ and $\langle f \rangle \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(\gamma d)$ for some $\gamma \in H = \Gamma(\text{dcl}_{\mathcal{L}}(\langle f \rangle))$. We can conclude by coding the finite set of $\langle f \rangle$ -conjugates of γd (by Lemma 5.9). \square

Claim 5.12 Suppose that for all $x \in D$, $\text{val}_{\text{rv}}(x) \in \mathbb{Q} \otimes \langle \text{val}_{\text{rv}}(A) \rangle$. Then f is coded.

Proof By compactness, D must be contained in only finitely many RV_{γ_i} . All of these γ_i are $\mathcal{L}(\langle f \rangle)$ -definable and hence f lies inside RV_H , where $H := \Gamma(\text{dcl}_{\mathcal{L}}(\langle f \rangle))$. By Lemma 5.7, f is coded by some d over H , hence there is some tuple $\gamma \in H$ such that $d\gamma$ codes f . \square

Now, Claim 5.11 allows us to code X' and Claim 5.12 allows us to code $X \setminus X'$. This concludes the proof of Lemma 5.10. \square

Let us now show that we can reduce to Lemma 5.10. As $f(x) \in \text{dcl}_{\mathcal{L}}(Ax)$, we have $\text{val}_{\text{rv}}(f(x)) \in \mathbb{Q} \otimes \langle \text{val}_{\text{rv}}(Ax) \rangle$. By compactness, for all i in some finite set I , there exist $n_i, m_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{Q} \otimes \text{val}_{\text{rv}}(A) \cap \Gamma(M)$ such that for all $x \in D$, there exists $i \in I$ with $g_i(x) := n_i \text{val}_{\text{rv}}(f(x)) - m_i \text{val}_{\text{rv}}(x) = \gamma_i$. Define $E_{i,\gamma}$ to be the fiber of g_i above γ . Then $D \subseteq \bigcup_{i \in I} E_{i,\gamma_i}$. Let us assume that $|I|$ is minimal such that this inclusion holds.

Claim 5.13 The set $X := \{(\gamma_i)_{i \in I} \in \Gamma : D \subseteq \bigcup_{i \in I} E_{i,\gamma_i}\}$ is finite.

Proof We proceed by induction on $|I|$. Let us assume X is infinite, and pick any $x \in D$. By the pigeonhole principle, there exists $i_0 \in I$ and an infinite set $Y \subseteq X$ such that for all $(\gamma_i)_{i \in I} \in Y$, $x \in E_{i_0, \gamma_{i_0}}$, i.e., $g_{i_0}(x) = \gamma_{i_0}$. It follows that for all $(\gamma_i)_{i \in I}$ and $(\delta_i)_{i \in I} \in Y$, $\gamma_{i_0} = \delta_{i_0}$ and $E_{i_0, \gamma_{i_0}} = E_{i_0, \delta_{i_0}} =: E$. By minimality of $|I|$, $D \setminus E$ is nonempty and the set $\{(\gamma_i)_{i \in I \setminus \{i_0\}} \in \Gamma : D \setminus E \subseteq \bigcup_{i \in I \setminus \{i_0\}} E_{i, \gamma_i}\}$ is finite by induction, but it contains $\{(\gamma_i)_{i \in I \setminus \{i_0\}} : (\gamma_i)_{i \in I} \in Y\}$ which is infinite, a contradiction. \square

Then any $(\gamma_i)_{i \in I} \in X$ is in $\text{acl}_{\mathcal{L}}(\langle f \rangle)$, $f_i := f|_{E_{i, \gamma_i}}$ satisfies the conditions of Lemma 5.10 and it suffices to code each f_i . Indeed let d be the tuple of the codes for those functions; then $d \in \text{acl}_{\mathcal{L}}(\langle f \rangle)$ and, as $f = \bigcup_{i \in I} f_i$, $\langle f \rangle \in \text{dcl}_{\mathcal{L}}^{\text{eq}}(d)$. The code of the finite set of $\langle f \rangle$ -conjugates of d —which exists by Claim 5.9—is a code for f .

Finally, if $R = \Gamma$, then for all $\gamma \in \Gamma(M)$, $f^{-1}(\gamma) \subseteq \text{RV}$ is coded by the case $R = \text{RV}$. Hence f is interdefinable with a function from Γ to $\text{RV}^l \times \Gamma^m$ for some l and m . So we have to code functions from Γ to Γ (which we already know how to code) and from Γ to RV . Let $g : \Gamma \rightarrow \text{RV}$ be a definable function and let $h = g \circ \text{val}_{\text{RV}}$. Then $h : \text{RV} \rightarrow \text{RV}$ is coded as we have just shown and, as for all $\gamma \in \Gamma$, $h(\text{val}_{\text{RV}}^{-1}(\gamma)) = \{g(\gamma)\}$, a code for h is also a code for g . This concludes the proof of Proposition 5.8. \square

Remark 5.14 1. Let $B_m = \text{RV}/(k^*)^m$. We have a homomorphism $B_m \rightarrow \Gamma$ with finite kernel $k^*/(k^*)^m$. Hence B_m^m maps injectively into Γ , and our assumptions on constants imply that there is a set of \emptyset -definable representatives for the cosets of B_m^m in B_m . Thus the theory (and imaginaries) of B_m reduce to those of Γ .

2. On the other hand, it can be shown that every unary definable subset D of RV is a finite union of pullbacks from B_m for some m and subsets of $\text{val}_{\text{RV}}^{-1}(a)$ for a lying in some finite subset F_D of Γ . This m is uniform in families, and F_D can be defined canonically as the set of $a \in \Gamma$ such that $\text{val}_{\text{RV}}^{-1}(a)$ is not a pullback from B_m . This gives another proof of unary EI in RV (with the stated constants), given EI in any RV_H .

A similar (but slightly more complicated) decomposition is also true in higher dimension (e.g., adapt [41, Lemma 3.25] to our case by replacing Γ by a suitable B_m). Moreover, EI in RV also follows from this decomposition.

Let us come back to unary EI in T_F (in fact, the proof given here would work in any theory $T \supseteq \text{HF}_0$ such that Γ is definably well-ordered and RV has unary EI). We will proceed as in the case of finite extensions of \mathbb{Q}_p . First let us show that the analogue of Claim 4.3 is still true in this case.

Claim 5.15 Let $A = \text{acl}_{\mathcal{L}}^{\text{eq}}(A)$, let $B = \mathbb{B}(A)$ and let $c \in K(M)$. Then $\text{tp}_{\mathcal{L}}(c/B) \vdash \text{tp}_{\mathcal{L}}(c/A)$.

Proof As any element in RV is coded by a ball, RV is stably embedded and has unary EI, the claim is true if $c \in \text{RV}(M)$. Recall that $W(c; A) := \{b \in \mathbb{B}(A) : c \in b\}$. If $P := \bigcap W(c; A) = \bigcap W(c; B)$ does not contain any ball in B then P is a complete type over A and B (by Proposition 3.18) and we are done. If P does contain a ball $b \in B$, then, by Proposition 3.18, P is complete relative to $\text{rv}(x - b)$. But $\text{tp}_{\mathcal{L}}(\text{rv}(x - b)/B) \vdash \text{tp}_{\mathcal{L}}(\text{rv}(x - b)/A)$ and we are also done. \square

Unary EI in T_F follows as for finite extensions of \mathbb{Q}_p .

(iv) Invariant types and germs: The same proof as for finite extensions of \mathbb{Q}_p (nearly) works as we only used there that Γ is definably well-ordered. The one difference is that P can be a closed ball. But in that case \tilde{p} , the ACVF generic of P , is definable, thus the \tilde{p} -germ of any r is an imaginary element e , and one may take $I = \{0\}$ and $e_0 = e$. Moreover, the inconsistency of $\text{tp}(c/A)$ and $\tilde{p}|M$ would—by Claim 4.4—contradict Lemma 3.14.

Corollary 5.16 *Let $T_F \supseteq \text{HF}_0$ be an \mathcal{L} -theory such that $\text{Th}_{\mathcal{L}}(k)$ and $\text{Th}_{\mathcal{L}}(\Gamma)$ are algebraically bounded, Γ is definably well-ordered, RV has unary EI, K has a finite number of extensions of any given degree and $k^*/(k^*)^n$ is finite. Suppose also that we have added constants for a field $F \subseteq K$ such that $k^* = (k^*)^n \text{res}(F)$ and any finite extension of K is generated by an element whose minimal polynomial is over F and which also generates the valuation ring over $\mathcal{O}(K)$. Then T_F has EI/UFI in the sorts K and S_n .*

In particular this is true of ultraproducts of the p -adics (if we add some constants as in Remark 2.8.2).

Proof By Proposition 2.13 we have EI/UFI in the sorts K , S_n and T_n but as noted earlier the sorts T_n are not needed when the value group has a smallest positive element. \square

Elimination of finite imaginaries: As we already know that RV eliminates imaginaries, it suffices to show that every finite imaginary in PL_0 (over arbitrary parameters) can be coded in RV (the proof is adapted from [39, Lemma 2.10]).

Definition 5.17 If $C \subseteq C'$, we say that C' is *stationary* over C if $\text{dcl}^{\text{eq}}(C') \cap \text{acl}^{\text{eq}}(C) = \text{dcl}^{\text{eq}}(C)$. A type $p = \text{tp}(c/C)$ is *stationary* if cC is stationary over C .

Remark 5.18 1. It is clear that if C'' is stationary over C' and C' is stationary over C , then C'' is stationary over C .

2. If $\text{tp}(c/C)$ generates a complete type over $\text{acl}^{\text{eq}}(C)$, then $\text{tp}(c/C)$ is stationary. Indeed, let $x \in \text{dcl}^{\text{eq}}(Cc) \cap \text{acl}^{\text{eq}}(C)$; then there is a C -definable function f such that $f(c) = x$. As $\text{tp}(c/C)$ generates a complete type over $\text{acl}^{\text{eq}}(C)$, there is a C -definable set D such that for all $c' \in D$, $f(c') = x$, hence $x \in \text{dcl}^{\text{eq}}(C)$.

Lemma 5.19 *Let T be a theory extending PL_0 (in the geometric language with possibly new constants). For all $M \models T$ and $A \subseteq M$, there exists $C \preceq M$ containing $\text{RV}(M) \cup A$ and stationary over $\text{RV}(M) \cup A$.*

Proof Let us first prove the following claim.

Claim 5.20 Let $B = \text{dcl}_{\mathcal{L}}(B) \subseteq M$ such that $\text{RV}(M) \subseteq B$ and $b \in \mathbb{B}(M)$. Then there exists a tuple $c \in K(M)$ with $\text{tp}_{\mathcal{L}}(c/B)$ stationary, $b \in \text{dcl}_{\mathcal{L}}(c)$ and $b(M) \cap c \neq \emptyset$.

Proof Let us first suppose that $b \in \text{RV}(M)$, i.e., that b is of the form $c(1 + \mathcal{M})$. Let $P \subseteq b$ be a minimal (for inclusion) intersection of balls in $\mathbb{B}(B)$. For any $c \models P$ we have $b = \text{rv}(c)$, hence it suffices to show that P is a complete stationary type over B .

As P does not strictly contain any ball in $\mathbb{B}(B)$ by definition, it cannot contain balls in $\mathbb{B}(\text{acl}_{\mathcal{L}}^{\text{eq}}(B))$ (we are in residue characteristic zero) and by Proposition 3.18, P is a complete type over $\text{acl}_{\mathcal{L}}^{\text{eq}}(B)$. By Remark 5.18.2, P is stationary over B .

Now if $b \in \mathbb{B}(M)$, pick any $r \in \text{RV}(M)$ such that $\text{val}_{\text{rv}}(r) = \text{rad}(b)$. Applying the claim to r , we find $c \in K(M)$ such that $\text{tp}_{\mathcal{L}}(c/B)$ is stationary and $\text{rad}(b) \in \text{dcl}_{\mathcal{L}}(c)$. It now suffices to find a point $d \in b$ whose type is stationary over $\text{dcl}_{\mathcal{L}}(Bc)$, but we can proceed as in the first case. Then $b \in \text{dcl}_{\mathcal{L}}(cd)$ and $\text{tp}(cd/B)$ is stationary. \square

Starting with $B := \text{dcl}_{\mathcal{L}}(\text{RV}(M) \cup A)$, and applying the claim iteratively, we find $C \supseteq A \cup \text{RV}(M)$ such that C is stationary over $A \cup \text{RV}(M)$, $\text{dcl}_{\mathcal{L}}(C) = C$, $\mathbb{B}(M) \subseteq \text{dcl}_{\mathcal{L}}(K(C))$ and every ball in $\mathbb{B}(M)$ has a point in C .

Claim 5.21 We have $C \subseteq \text{dcl}_{\mathcal{L}}(K(C))$.

Proof Let $e \in C$. If $e \in K$ then the result is trivial, thus we only have to consider $e \in S_n$ or $e \in T_n$. Let us consider the same decomposition of S_n and T_n as in the proof of Lemma 5.5 and show by induction on i that for all $e \in (G_i/H_i)(M)$, e is $\mathcal{L}(K(C))$ -definable.

If we write e as eH_i then, as proved in Lemma 5.5, $\phi_i(eH_i)$ is either a ball or a set of the form $y\mathcal{O}^*$ and hence is definable over $\mathbb{B}(M)$ and has a point $a' \in K(C)$. Let $a \in \phi_i^{-1}(a')(C)$. Then $a^{-1}eH_i \cap G_{i-1}$ is a coset of H_{i-1} in G_{i-1} which is $\mathcal{L}(K(C))$ -definable by induction, and hence so is eH_i . \square

As $\text{dcl}_{\mathcal{L}}(C) = C$, we have $K(C) = K(C)^h \models \text{HF}_0$ and, as $\text{rv}(K(C)) = \text{RV}(M)$, it follows from field quantifier elimination in HF_0 in the language with sorts K and RV , that $K(C) \preceq K(M)$. But this implies that $C = \text{dcl}_{\mathcal{L}}(K(C)) \preceq M$. \square

Lemma 5.22 *Let T be a theory that extends PL_0 (in the geometric language) and let $A \subseteq M \models T$. Then every finite imaginary sort of T_A is in definable bijection with a finite imaginary sort of RV (with the structure induced by T_A).*

Proof Let $Y = D/E$ be a finite imaginary sort (in T_A) and let $\pi : D \rightarrow Y$ be the canonical surjection. As the field sort is dominant, we can assume that D is a definable subset of K^n for some n . Let $C \supseteq A$ be as in Lemma 5.19. As Y is finite and $C \prec M$, $Y(C) = Y(M)$ and there exists a finite set $H \subseteq K^n(C)$ meeting every E -class. Let W be some finite set in $\text{RV}(C)$, of bigger cardinality than H , and $h : W \rightarrow H$ any surjection. Note that any such surjection is $\mathcal{L}(C)$ -definable. Composing, we have an $\mathcal{L}(C)$ -definable surjection $\psi : W \rightarrow Y$. But there are only finitely many maps $W \rightarrow Y$, hence they are all algebraic over $\text{RV}(C) \cup A = \text{RV}(M) \cup A$ and by stationarity of C over $\text{RV}(M) \cup A$, ψ is $\mathcal{L}(\text{RV}(M) \cup A)$ -definable. Let $e \in \text{RV}(M)$ be such that ψ and W are $\mathcal{L}(Ae)$ -definable.

Let W be defined by the $\mathcal{L}(Ae)$ -formula $\phi(x, e)$ and ψ by the $\mathcal{L}(Ae)$ -formula $\psi(x, y, e)$ (which implies that for any e' , $\psi(M, M, e')$ is the graph of a function with domain $\phi(M, e')$). Then the formulas $\phi(x, z)$ and $\psi(x, y, z)$ define, respectively, a subset D' of $\text{RV}^{|e|+1}$ and a surjection $\psi : D' \rightarrow Y$. Let E' be defined by $E'((x, z), (x', z')) \iff \psi(x, z) = \psi(x', z')$. Then we have an $\mathcal{L}(A)$ -definable bijection $D'/E' \rightarrow Y$ and, as RV is considered with the structure induced by T_A , D'/E' is a finite imaginary sort of RV . \square

Proof of Theorem 2.7: Let $K \models \text{PL}_0$ and let $T = \text{Th}(K)$ (with constants added as in Corollary 5.16). As we have already proved EI/UFI in Corollary 5.16, by Lemma 2.5 it is enough to show that for any A , T_A eliminates finite imaginaries in the sorts K, S_n . Let $e \in \text{acl}_{\mathcal{L}}^{\text{eq}}(A)$; then, by Lemma 5.22, there exists an RV -imaginary e' interdefinable over A with e . By EI in RV to the sorts RV and Γ (Proposition 5.8), there exists a tuple $d \in \text{RV} \cup \Gamma$ such that e' is interdefinable with d , hence e is interdefinable with d over A . We have shown that any finite imaginary of T_A is coded (over A) in $\text{RV} \cup \Gamma = S_1 \cup T_1$ which are themselves coded in $S_1 \cup S_2$.

For a more canonical treatment of the parameters F in the pseudo-finite case, see [10]—it would be interesting to adapt *op. cit.* to the pseudo-local setting.

6 Rationality

Let $r \in \mathbb{N}$. For all tuples $l \in \mathbb{N}^r$, when $t = (t_i)_{1 \leq i \leq r}$, we write t^l for $\prod_{i \leq r} t_i^{l_i}$. We say a power series $\sum_{l \in \mathbb{N}^r} a_l t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$ with each $a_l \in \mathbb{N}$ is *rational* if it is equal to a rational function in t_1, \dots, t_r with coefficients from \mathbb{Q} . In this section we prove that certain zeta functions that come from counting the equivalence classes of definable equivalence relations are rational.

For any finite extension L_p of \mathbb{Q}_p , it is natural here to consider the invariant Haar measure μ_{L_p} on $\mathrm{GL}_N(L_p)$. In terms of the additive Haar measure $\mu_{L_p}^{N^2}$ on $L_p^{N^2}$, μ_{L_p} can be defined thus: for any continuous $f: \mathrm{GL}_N(L_p) \rightarrow \mathbb{C}$ with compact support, $\int f(x) d\mu_{L_p}(x) = \int f(x) |\det(x)|^{-n} d\mu_{L_p}^{N^2}(x)$. As $\det(x)$ is uniformly definable for all L_p , Denef's results on definability of p -adic integration [16] extend immediately to $d\mu_{L_p}$ and the motivic counterpart of this result—see [17], although the result we will be needing is already implicit in older work by Denef and Pas (see, e.g., [55])—also extends to $d\mu_{L_p}$.

By left invariance, $\mu_{L_p}(A \cdot \mathrm{GL}_N(\mathcal{O}(L_p))) = \mu_{L_p}(\mathrm{GL}_N(\mathcal{O}(L_p)))$, a number that depends only on the normalization. We choose a normalization for $\mu_{L_p,+}$ and μ_{L_p} such that for any $A \in \mathrm{GL}_N(L_p)$, we have

$$\mu_{L_p}(A \cdot \mathrm{GL}_N(\mathcal{O}(L_p))) = 1. \quad (6.1)$$

Let K be a number field and let \mathcal{O}_K denote its ring of integers. For each prime p , let \mathfrak{L}_p be a set of finite extensions of \mathbb{Q}_p , each containing K , and let $\mathfrak{L} = \bigcup_p \mathfrak{L}_p$. We will say that $(R_{L_p})_{L_p \in \mathfrak{L}}$ and $(E_{L_p})_{L_p \in \mathfrak{L}}$ are *uniformly K -definable in \mathfrak{L}* , or just *uniformly K -definable*, if there exist two $\mathcal{L}_{\mathcal{G}}(K)$ -formulas ϕ and θ —i.e., $\mathcal{L}_{\mathcal{G}}$ -formulas with parameters in K —independent of L_p such that for all $L_p \in \mathfrak{L}$, $R_{L_p} = \phi(L_p)$ and $E_{L_p} = \theta(L_p)$. If $K = \mathbb{Q}$ then we often write *uniformly \emptyset -definable in \mathfrak{L}* instead of uniformly \mathbb{Q} -definable in \mathfrak{L} . If in addition $\mathfrak{L}_p = \{\mathbb{Q}_p\}$ for all p , then we often write *uniformly \emptyset -definable in p* instead of uniformly \mathbb{Q} -definable in \mathfrak{L} .

By a *(uniformly K -)definable family* $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{Z}^r}$ of subsets of L_p^N we mean a (uniformly K -)definable subset R_{L_p} of $L_p^N \times \mathbb{Z}^r$ —where $\mathrm{val}(L_p^*)$ is identified with \mathbb{Z} —and we write $R_{L_p,l}$ for the fiber above l of the projection from R_{L_p} to \mathbb{Z}^r . By a *(uniformly K -)definable family* $E_{L_p} = (E_{L_p,l})_{l \in \mathbb{Z}^r}$ of equivalence relations on R_{L_p} we mean a (uniformly K -)definable equivalence relation E_{L_p} on R_{L_p} such that for every $x, y \in R_{L_p}$, if $x E_{L_p} y$ then there exists $l \in \mathbb{Z}^r$ such that $x, y \in R_{L_p,l}$. We then have a (uniformly K -)definable equivalence relation $E_{L_p,l}$ on $R_{L_p,l}$ for every l , and by a slight abuse of notation we can regard $(E_{L_p,l})_{l \in \mathbb{Z}^r}$ as a (uniformly K -)definable family of subsets of L_p^{2N} . The set \mathbb{N}^r is a (uniformly K -)definable subset of \mathbb{Z}^r , so it makes

sense to talk of (uniformly K -)definable families $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{N}^r}$, etc.

Now we come to the main result of this section (cf. [29, Thms. 1.3 and 1.4]).

Theorem 6.1 *Let \mathfrak{L}_p and \mathfrak{L} be as above (note that we do not assume \mathfrak{L}_p is nonempty for infinitely many p). For all $L_p \in \mathfrak{L}$, let $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{N}^r}$ be a family of subsets of L_p^N and let $E_{L_p} = (E_{L_p,l})_{l \in \mathbb{N}^r}$ be a family of equivalence relations on $(R_{L_p,l})_{l \in \mathbb{N}^r}$ such that $(R_{L_p})_{L_p \in \mathfrak{L}}$ and $(E_{L_p})_{L_p \in \mathfrak{L}}$ are uniformly K -definable in \mathfrak{L} . Suppose that for each $l \in \mathbb{N}^r$ and each $L_p \in \mathfrak{L}$, the set of equivalence classes $R_{L_p,l}/E_{L_p,l}$ is finite. Let $a_{L_p,l} = |R_{L_p,l}/E_{L_p,l}|$. Then the power series*

$$S_{L_p}(t) := \sum_{l \in \mathbb{N}^r} a_{L_p,l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$$

is rational for every $L_p \in \mathfrak{L}$.

Moreover, there exist $k, n, d \in \mathbb{N}$, there exist tuples $(a_j)_{j \leq k}$ of integers and $(b_j)_{j \leq k}$ of elements of \mathbb{N}^r , and for all tuples $l \in \mathbb{N}^r$ with $|l| := \sum_{i \leq r} l_i \leq d$ there exist $q_l \in \mathbb{Q}$ and varieties X_l over \mathcal{O}_K , such that the following holds:

- (1) for all j , a_j and b_j are not both 0; and
- (2) for all $p \gg 0$ and all $L_p \in \mathfrak{L}_p$, we have

$$S_{L_p}(t) = \frac{\sum_{|l| \leq d} q_l |X_l(\text{res}(L_p))| t^l}{|\text{res}(L_p)|^n \prod_{j=1}^k (1 - |\text{res}(L_p)|^{a_j} t^{b_j})}. \quad (6.2)$$

Suppose we are given power series $S_{L_p}(t) = \sum_{l \in \mathbb{N}^r} a_{L_p,l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$ for each $L_p \in \mathfrak{L}$. We say the power series $(S_{L_p}(t))_{L_p \in \mathfrak{L}}$ are *uniformly rational for $p \gg 0$* if there exists a prime p_0 such that the $S_{L_p}(t)$ are of the form given in (6.2) for all $L_p \in \mathfrak{L}$ such that $p > p_0$.

Remark 6.2 1. Assume \mathfrak{L}_p is finite for all p (this is the case in most of our applications in Sections 7 and 8). Let \mathcal{L}_{rg} be the language of rings. At the cost of replacing the X_l by quantifier-free $\mathcal{L}_{\text{rg}}(\mathcal{O}_K)$ -definable sets, we can make (6.2) hold for every L_p , where k, n, d , the a_j , the b_j , the q_l and the X_l are all independent of the choice of L_p . In this case, we say the power series $(S_{L_p}(t))_{L_p \in \mathfrak{L}}$ are *uniformly rational*. In particular, suppose we are given definable R_{p_0} and E_{p_0} as above, but just for a single prime p_0 and a single L_{p_0} . Then taking $\mathfrak{L} = \{L_{p_0}\}$, we obtain that the power series

$$S_{L_{p_0}} := \sum_{l \in \mathbb{N}^r} a_{L_{p_0},l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$$

is rational, and is of the form (6.2) if we allow the X_l to be quantifier-free $\mathcal{L}_{\text{rg}}(\mathcal{O}_K)$ -definable sets (in fact, we can take X_l just to be a single point).

2. Often in this kind of rationality theorem, we can take $q_l = 1$ for all l . There are two reasons why more complicated rational coefficients appear here. The first reason is to turn the X_l into varieties instead of definable sets and the other reason is to get rid of the residual constant symbols that appear due to elimination of imaginaries.
3. Given uniformly rational power series $(S_{L_p}(t))_{L_p \in \mathfrak{L}}$, set $\varphi_{L_p}(s) = S_{L_p}(\text{res}(L_p)^{-s})$, where s is a complex parameter. Then $\varphi_{L_p}(s)$ has the form

$$\varphi_{L_p}(s) = \frac{\sum_{|l| \leq d} q_l |X_l(\text{res}(L_p))| |\text{res}(L_p)|^{-ls}}{|\text{res}(L_p)|^n \prod_{j=1}^k (1 - |\text{res}(L_p)|^{a_j - sb_j})}, \quad (6.3)$$

where the X_l , etc., are as in Theorem 6.1. It then follows by change of variable that for any $s_0 \in \mathbb{Z}$, the function $\varphi_{L_p}(s - s_0)$ (regarded as a function of s) also has the form (6.3). (The only slight subtlety here is that the change of variable might lead to a factor of the form $\text{res}(L_p)^n$ in the denominator where $n < 0$; but in this case, we can delete the factor and replace each X_l with $X_l \times \mathbb{A}^n$.)

Proof [Theorem 6.1] By uniform EI (Corollary 2.9)—and the fact that elimination of imaginaries still holds after adding new constants for K —there exist integers m_1 and m_2 , some $N \subseteq \mathbb{N}_{>0}$ and some $\mathcal{L}_{\mathcal{G}}^N(K)$ -formula $\phi(x, w)$ such that for all $p \gg 0$, for all proper expansions to $\mathcal{L}_{\mathcal{G}}^N$ of $L_p \in \mathfrak{L}_p$, ϕ defines a function $f'_{L_p} : R_{L_p} \rightarrow L_p^{m_2} \times S_{m_1}(L_p)$ such that for every $x, y \in R_{L_p}$, $x E_{L_p} y \iff f'_{L_p}(x) = f'_{L_p}(y)$. Let $f'_{L_p} = (f''_{L_p}, f_{L_p})$ where $f''_{L_p} : R_{L_p} \rightarrow L_p^{m_2}$ and $f_{L_p} : R_{L_p} \rightarrow S_{m_1}(L_p)$. For $l \in \mathbb{N}^r$, let $\mathcal{E}_{L_p, l} = \{f'_{L_p}(x) : x \in R_{L_p, l}\}$ and $\mathcal{E}_{L_p} = \bigcup_l \mathcal{E}_{L_p, l}$; so $E_{L_p, l} \subseteq L_p^{m_2} \times S_{m_1}(L_p)$ is finite, and it is the series $\sum_l |E_{L_p, l}| t^l$ we wish to understand. Let $\pi_{L_p} : \mathcal{E}_{L_p} \rightarrow S_{m_1}(L_p)$ be the projection, and let $F_{L_p, l} = \pi_p(\mathcal{E}_{L_p, l})$.

It follows from Lemma 5.5 and the fact that on the valued field sort the model-theoretic algebraic closure in ACVF coincides with the field-theoretic algebraic closure, that the size of the fiber $e_p(x) := |(\pi_{L_p})^{-1}(x)|$ is bounded by some positive integer D uniformly for $p \gg 0$. We may thus partition $F_{L_p, l}$ into finitely many pieces $F_{L_p, l}^\nu = \{x \in F_{L_p, l} : e_p(x) = \nu\}$; then

$$\sum_l |\mathcal{E}_{L_p, l}| t^l = \sum_{\nu \leq D} \nu \sum_l |F_{L_p, l}^\nu| t^l,$$

so it suffices to prove that the series for $F_{p,l}^\nu$ has the form (6.2).

Fix ν and let $F_{L_p,l} = F_{L_p,l}^\nu$; we need to retain only the information that $(F_{L_p,l})_{L_p \in \mathfrak{L}}$ is a family of finite subsets of $S_m(L_p)$, uniformly K -definable in \mathfrak{L} . We can identify each element of $S_m(L_p)$ with an element of $\mathrm{GL}_m(L_p)/\mathrm{GL}_m(\mathcal{O}(L_p))$, i.e., with a left coset of $\mathrm{GL}_m(\mathcal{O}(L_p))$; let $G_{L_p,l}$ be the union of these cosets. By Eqn. (6.1), we have

$$\mu_{L_p}(G_{L_p,l}) = |F_{L_p,l}|.$$

Thus

$$\sum_l |F_{L_p,l}| t^l = \sum_l \mu_{L_p}(G_{L_p,l}) t^l \in \mathbb{Q}[[t_1, \dots, t_r]].$$

Uniform rationality now follows from [17, Theorem 1.1 and Theorem 3.1], up to the fact that because the sets $G_{L_p,l}$ are \mathcal{L}_G^N -definable, the varieties X_l are over $\mathcal{O}_K[\overline{X}]$ where \overline{X} is specialized in $\mathrm{res}(L_p)$ to any tuple $(k_n : n \in N$ and k_n is the residue of an unramified n -Galois uniformizer). Let

$$C_n(L_p) = \{k_n \in \mathrm{res}(L_p) : k_n \text{ is the residue of an unramified } n\text{-Galois uniformizer}\}.$$

If $\mathrm{res}(L_p)[\omega_n]$ is of degree $d = d_{n,L_p}$ over $\mathrm{res}(L_p)$, then

$$|C_n(L_p)| = \frac{\phi(n)(|\mathrm{res}(L_p)|^d - 1)}{n},$$

where ϕ is the Euler totient function. Let $C = \prod_{n \in N} C_n$ and for all $c \in C(L_p)$, let $X_{c,l}(L_p)$ be the L_p -points of the specialization of X_l to c . Then $Y_l := \prod_{c \in C} X_{c,l}$ is an $\mathcal{L}_{\mathrm{rg}}(\mathcal{O}_K)$ -definable set such that

$$|X_l(\mathrm{res}(L_p))| = \sum_{d|n} 1_{d_{n,L_p}=d} \frac{-n}{\phi(n)} \frac{|Y_l(\mathrm{res}(L_p))|}{1 - |\mathrm{res}(L_p)|^d},$$

where $1_{d_{n,L_p}=d} = 1$ if $d_{n,L_p} = d$ and 0 otherwise. Replacing $|X_l(\mathrm{res}(L_p))|$ with this expression, we obtain a rational function of the right form where the X_l are $\mathcal{L}_{\mathrm{rg}}(\mathcal{O}_K)$ -definable, but, by [17, Theorem 2.1], X_l may be assumed to be a \mathcal{O}_K -variety for $p \gg 0$.

For L_p such that p is too small, we can still prove the rationality of S_{L_p} by the same argument using results for finite extensions of p -adic fields instead of those for ultraproducts: replace Corollary 2.9 by Theorem 2.6, Lemma 5.5 by the proof of (i) in Section 4 and [17, Theorem 1.1] by [16, Theorem 1.5 and Theorem 1.6.1]. \square

It follows from the uniform formula Eqn. (6.2) we gave for S_{L_p} in Theorem 6.1 that there exist a tuple of polynomials $(P_i)_{1 \leq i \leq r}$ from $\mathbb{Q}[X]$ and a polynomial $Q \in \mathbb{Q}[X]$ such that we have the following uniform growth estimate on $a_{L_p, l}$: for all $p \gg 0$ and all $L_p \in \mathfrak{L}$,

$$a_{L_p, l} \leq Q(q)P_1(q)^{n_1} \cdots P_r(q)^{n_r}, \quad (6.4)$$

where $q = |\text{res}(L_p)|$. If \mathfrak{L} is finite for all p then Eqn. (6.4) holds for every $L_p \in \mathfrak{L}$.

Below we consider uniformly \emptyset -definable families that arise in the following way. Take \mathcal{L}_p to be $\{\mathbb{Q}_p\}$ for all p . To simplify the notation in this case, we use subscripts p instead of \mathbb{Q}_p (hence we write \mathcal{D}_p and $S_p(t)$ below rather than $\mathcal{D}_{\mathbb{Q}_p}$ and $S_{\mathbb{Q}_p}(t)$). Let $\mathcal{D}_p \subseteq \mathbb{Q}_p^N$, let \mathcal{E}_p be an equivalence relation on \mathcal{D}_p and suppose $(\mathcal{D}_p)_{p \text{ prime}}$ and $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p . Suppose that $f_{p,1}, \dots, f_{p,r}: \mathcal{D}_p \rightarrow \mathbb{Q}_p - \{0\}$ are uniformly \emptyset -definable functions such that for every $l \in \mathbb{Z}^r$, the subset $\{x \in \mathcal{D}_p : |f_{p,i}(x)| = p^{-l_i}\}$ is a union of E_p -equivalence classes. Set $D_p = \{(x, |f_{p,1}(x)|, \dots, |f_{p,r}(x)|) : x \in \mathcal{D}_p\} \subseteq \mathbb{Q}_p^N \times \mathbb{Z}^r$ and define an equivalence relation $E_p \subseteq D_p \times D_p$ by $(x, s_1, \dots, s_r) E_p (x', s'_1, s'_r)$ if $x \mathcal{E}_p x'$ and $s_i = s'_i$ for all i . Then we can regard $(D_p)_{p \text{ prime}}$ as a uniformly \emptyset -definable family of sets and $(E_p)_{p \text{ prime}}$ as a uniformly \emptyset -definable family of equivalence relations on $(D_p)_{p \text{ prime}}$.

We now consider the abscissa of convergence of the zeta function in the one-variable case (under the assumption that $\mathcal{L}_p = \{\mathbb{Q}_p\}$ for all p), and give a proof of Theorem 1.4. Recall that if $\zeta(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a zeta function then the abscissa of convergence α of $\zeta(s)$ is the infimum of the set of $s \in \mathbb{R}$ such that the series for $\zeta(s)$ is convergent. Moreover, if $s \in \mathbb{C}$ then $\zeta(s)$ converges if $\text{Re}(s) > \alpha$ and diverges if $\text{Re}(s) < \alpha$.

We give a more precise statement of Theorem 1.4.

Theorem 6.3 *Let $\mathcal{L}_p = \{\mathbb{Q}_p\}$ for every prime p . Assume the notation and hypotheses of Theorem 6.1 and define $\zeta_p(s) = S_p(p^{-s})$ (cf. Remark 6.2.3). Assume that the constant term of $\zeta_p(s)$ is 1 for all but finitely many primes and set $\zeta(s) = \prod_p \zeta_p(s)$. Then the abscissa of convergence of $\zeta(s)$ is rational (or $-\infty$).*

Let $(\zeta_p(s))_{p \text{ prime}}$ be a family of zeta functions each of the form $\zeta_p(s) = \sum_{n=0}^{\infty} a_{p,n} p^{-ns}$. Consider the formal product $\zeta(s)$ given by $\zeta(s) = \prod_p \zeta_p(s)$. To ensure this makes sense, we assume that the constant term $a_{p,0}$ is 1 for all but finitely many primes. To prove Theorem 6.3, we need to control the behavior of the p -local factors $\zeta_p(s)$. Our proof is similar to parts of Avni's proof that the abscissa of convergence of the representation zeta function of an arithmetic lattice in a semisimple group is rational (see [6, proof of Thm. 6.4], and cf. also [29, Lem. 4.6 (1)]), but the details are slightly different because we allow the coefficients q_l in Eqn. (6.2) to be negative.

We need an estimate on the size of the varieties $X_l(\mathbb{F}_p)$ in Eqn. (6.2). Recall the concept of an *Artin set* [6, Defn. 4.6]; as noted in *loc. cit.*, an infinite Artin set A has positive analytic density, which implies that $\prod_{p \in A} \left(1 + \frac{1}{p}\right)$ diverges.

Lemma 6.4 *Let X be a variety defined over \mathbb{Z} . Then there is a partition of the set of primes into disjoint Artin sets A_1, \dots, A_r for some r , and there exist $c > 0$ and $(d_i, \mu_i) \in \mathbb{N} \times \mathbb{Q}_{>0}$ such that for every i and for every prime $p \in A_i$,*

$$|X(\mathbb{F}_p) - \mu_i p^{d_i}| < c p^{d_i - 1/2}. \quad (6.5)$$

Proof This follows from [6, Cor. 4.7], taking the parameter n and the formula $\phi(x, y)$ of *loc. cit.* to be 0 and a formula $\phi(x)$ that defines X , respectively. Note that the quantity $N_{d, \mu}$ in *loc. cit.* is 1 if Eqn. (6.5) holds for a given $p \in A_i$, and 0 if it does not, so Eqn. (6.5) holds for sufficiently large p . By increasing c if necessary, we can make Eqn. (6.5) hold for all p . \square

We recall two standard facts.

- (I) If $(x_n)_n$ is a sequence of non-negative real numbers then $\prod_n (1 + x_n)$ converges if and only if $\sum_n x_n$ converges.
- (II) The abscissa of convergence of a finite product of zeta functions with non-negative coefficients is the maximum of the abscissae of convergence of the factors.

Let A be a set of primes with positive analytic density (in particular, this implies that A is infinite). Let $t \in \mathbb{N}_{>0}$, let $d_1, \dots, d_t \in \mathbb{Z}$, let e_1, \dots, e_t be distinct positive integers and let q_1, \dots, q_t be nonzero real numbers. Let $k \in \mathbb{N}$, let $n \in \mathbb{N}$ and let $a_1, \dots, a_k \in \mathbb{Z}$, $b_1, \dots, b_k \in \mathbb{N}_{>0}$. Let $u \in \mathbb{N}$ and let g_1, \dots, g_u be nonzero integers. Let $\epsilon_1, \dots, \epsilon_t > 0$, let $\mu_1, \dots, \mu_t \geq 0$ and let $f_1, \dots, f_t: A \rightarrow \mathbb{R}$ such that $|f_i(p)| \leq \mu_i p^{d_i - \epsilon_i}$ for all $p \in A$. Consider the p -local zeta function

$$\zeta_p(s) := 1 + \frac{\sum_{l=1}^t q_l (p^{d_l} + f_l(p)) p^{-e_l s}}{p^n \prod_{m=1}^u (1 - p^{g_m}) \prod_{j=1}^k (1 - p^{a_j - b_j s})}. \quad (6.6)$$

We assume that the coefficients of $\zeta_p(s)$ (as a power series in p^{-s}) are non-negative.

We wish to determine the abscissa of convergence α of $\zeta(s) := \prod_{p \in A} \zeta_p(s)$. For each $p \in A$, the poles of $\zeta_p(s)$ lie in the set $\left\{ \frac{a_j}{b_j} : 1 \leq j \leq k \right\}$; but not every $\frac{a_j}{b_j}$ is necessarily

a pole of $\zeta_p(s)$, since the numerator of the fraction on the RHS of Eqn. (6.6) might have a zero at $\frac{a_j}{b_j}$. Let $\Xi = \left\{ j : 1 \leq j \leq k, \frac{a_j}{b_j} \text{ is a pole of } \zeta_p(s) \text{ for some } p \in A \right\}$. Set $M_1 = \max \left\{ \frac{a_j}{b_j} : j \in \Xi \right\}$ (we take $M_1 = -\infty$ if Ξ is empty).

Given $s \in \mathbb{R}$ and $i \in \{1, \dots, t\}$, we say that s is *i-dominant* if $d_i - e_i s > d_l - e_l s$ for all $l \neq i$. If s is not *i-dominant* for any i then we say that s is *critical*. The set of critical points is finite (each critical point satisfies an equation of the form $d_l - e_l s = d_{l'} - e_{l'} s$ for some distinct l and l' , and we assume that $e_l \neq e_{l'}$).

Lemma 6.5 *Let the notation be as above. Then α is rational or $-\infty$.*

Proof If $t = 0$ in Eqn. (6.6) then $\zeta_p(s) = 1$ for all $p \in A$, so $\alpha = -\infty$ and we are done. Hence we can suppose that $t \geq 1$; in particular, $\zeta_p(s)$ is a strictly decreasing function of s for $s > M_1$. For any $s \in \mathbb{R}$, if $\zeta(s)$ converges then standard results on infinite products of Dirichlet series imply that each $\zeta_p(s)$ converges. Hence $\alpha \geq M_1$.

For $s \in \mathbb{R}$, set

$$\beta(s) = \max_{1 \leq l \leq t} (d_l - e_l s) - n - \sum_{g_m > 0} g_m + \sum_{a_j/b_j \geq s} (b_j s - a_j).$$

Then $\beta(s)$ is piecewise linear, so it is continuous. We show that $\beta(s)$ is a strictly decreasing function of s for $s > M_1$. To see this, let $s \in \mathbb{R}$. If s is not critical then s is *i-dominant* for some i , and it follows that there exists $E > 0$ such that

$$E^{-1} p^{d_i - e_i s} \leq \left| \sum_{l=1}^t q_l (p^{d_l} + f_l(p)) p^{-e_l s} \right| \leq E p^{d_i - e_i s}$$

for all sufficiently large $p \in A$. Moreover, there exists $D > 0$ such that for all j and all $p \in A$, if $s > \frac{a_j}{b_j}$ then $1 - p^{a_j - b_j s} > D$, while if $s < \frac{a_j}{b_j}$ then $|1 - p^{a_j - b_j s}| > D p^{a_j - b_j s}$. It follows from the above discussion, the definition of $\zeta_p(s)$ and the bounds on the f_l that for any $s \in \mathbb{R}$ such that $s > M_1$, s is not critical and $s \neq \frac{a_j}{b_j}$ for any $1 \leq j \leq k$, there exists $C \geq 1$ such that

$$C^{-1} p^{\beta(s)} \leq \zeta_p(s) - 1 \leq C p^{\beta(s)} \tag{6.7}$$

for all sufficiently large $p \in A$. Since each $\zeta_p(s)$ is strictly decreasing for $s > M_1$, $\beta(s)$ must therefore also be strictly decreasing for $s > M_1$, as claimed. Hence there

is at most one point $s_0 > M_1$ such that $\beta(s_0) = -1$. Set $M_2 = s_0$ if this exists; otherwise set $M_2 = -\infty$ (note that in the latter case, $\beta(s) < -1$ for all $s > M_1$, as $\lim_{s \rightarrow \infty} \beta(s) = -\infty$.) We show that $\alpha = \max(M_1, M_2)$.

Let $s > M_1$ such that s is not critical. Suppose $s < M_2$. Then $\beta(s) > -1$, so $\sum_{p \in A} C^{-1} p^{\beta(s)}$ diverges since A has positive analytic density, so $\prod_{p \in A} (1 + C^{-1} p^{\beta(s)})$ diverges by Fact I. Hence $\zeta(s) = \prod_{p \in A} \zeta_p(s)$ diverges by Eqn. (6.7) and the comparison test, and it follows that $s \leq \alpha$. If $s > M_2$ then $\beta(s) < -1$, and a similar argument shows that $s \geq \alpha$. We deduce that if $M_2 \leq M_1$ then $\alpha = M_1$, and also that if $M_2 > M_1$ then $\alpha = M_2$. This completes the proof. \square

Proof (of Theorems 6.3 and 1.4) By Theorem 6.1 and Remark 6.2, Eqn. (6.2) holds for every prime p and the definable sets X_l are varieties over \mathbb{Z} for all but finitely many p . Hence for each p , $\zeta_p(s)$ can be written as a rational function, where the numerator is a polynomial in p^{-s} and the denominator is of the form $\prod_{j=1}^{k'} (1 - p^{a_j - b_j s})$ with each $b_j > 0$. (Here we have ordered the factors in the denominator of Eqn. (6.2) so that $b_1, \dots, b_{k'} > 0$ and $b_{k'+1}, \dots, b_k = 0$ for some $0 \leq k' \leq k$.) This implies that the abscissa of convergence of each $\zeta_p(s)$ is rational.

It now follows that by Fact (II), we can disregard finitely many primes: that is, it is enough to prove that $\prod_{p > p_0} \zeta_p(s)$ has rational abscissa of convergence for some prime p_0 . We can assume that $\zeta_p(s)$ has constant term 1 for every $p > p_0$. The $\zeta_p(s)$ all have non-negative coefficients by construction. Let $S_p^*(t) = S_p(t) - 1$ and let $\zeta_p^*(s) = \zeta_p(s) - 1 = S_p^*(p^{-s})$. Then $S_p^*(t)$ is the power series that arises from counting the equivalence classes of a uniformly \emptyset -definable family (in $\mathfrak{L} := \bigcup_{p > p_0} \{\mathbb{Q}_p\}$) of equivalence relations—just take the family of equivalence relations corresponding to $S_p(t)$ and remove the definable piece coming from $l = 0$ —so the power series $(S_p^*(t))_p$ prime are uniformly rational for $p > p_0$ by Theorem 6.1 and Remark 6.2. Hence $S_p^*(t)$ is of the form given in Eqn. (6.2) for all $p > p_0$, with the sum in the numerator beginning at $l = 1$ rather than $l = 0$. Explicitly, we have

$$S_p^*(t) = \frac{\sum_{l=1}^d q_l |X_l(\mathbb{F}_p)| t^l}{p^n \prod_{j=1}^k (1 - p^{a_j} t^{b_j})}, \quad (6.8)$$

for all $p > p_0$, where the X_l , etc., are as in Theorem 6.1.

We now apply Lemma 6.4 to the varieties X_l . We can choose $r, A_1, \dots, A_r, c, d_l, \mu_l$ such that Eqn. (6.5) holds for each of the X_l (note that complements, finite unions and finite intersections of Artin sets are Artin sets). By increasing p_0 if necessary, we can assume that each A_i is infinite and contains no primes less than or equal to p_0 ; in particular, each A_i has positive analytic density. It is enough by Fact (II) to show that

$\prod_{p \in A_i} \zeta_p(s)$ has rational abscissa of convergence for each i . It follows from Eqns. (6.5) and (6.8) that the hypotheses of Lemma 6.5 are satisfied for $(\zeta_p(s))_{p \in A_i}$, so the desired result follows from Lemma 6.5. \square

Remark 6.6 One-variable zeta functions that arise from cone integrals can be meromorphically continued beyond the abscissa of convergence, so one can apply Tauberian theorems and obtain a more precise growth estimate than that provided by Eqn. (6.4): see [29, Thm. 1.1]. One should not expect the same behavior for zeta functions arising from an arbitrary uniformly \emptyset -definable equivalence relation (cf. [29, Eqn. (1.3)] and the discussion that surrounds it).

Let us conclude this section with a short aside on positive characteristic local fields by explaining how Theorem 6.1 also yields transfer results between positive characteristic and mixed characteristic:

Corollary 6.7 *Let $\phi(x, y, l)$ be an \mathcal{L}_G -formula where l is a tuple of variables from the value group. The following are equivalent:*

1. *For all $p \gg 0$, the formula ϕ defines a family of finite equivalence relations $E_{p,l}$ on some set $D_{p,l}$ in \mathbb{Q}_p ;*
2. *For all $p \gg 0$, the formula ϕ defines a family of finite equivalence relations $E'_{p,l}$ on some set $D'_{p,l}$ in $\mathbb{F}_p((t))$.*

Moreover, whenever the above statements hold, there exists a prime p_0 such that for all $p \geq p_0$, the series $S_p(t) := \sum_{l \in \mathbb{N}^r} |D_{p,l}/E_{p,l}| t^l$ and $S'_p(t) := \sum_{l \in \mathbb{N}^r} |D'_{p,l}/E'_{p,l}| t^l$ are uniformly rational and $S_p = S'_p$.

Proof This follows immediately from (the proof of) Theorem 6.1 and the fact that for all non-principal ultrafilters \mathcal{U} on the set of primes, $\prod_p \mathbb{Q}_p/\mathcal{U}$ and $\prod_p \mathbb{F}_p((t))/\mathcal{U}$ are elementarily equivalent. \square

7 Zeta functions of groups

We now consider some applications to some zeta functions that arise in group theory. From now until the final part of Section 8 we take \mathfrak{L}_p to be $\{\mathbb{Q}_p\}$ for all p . Most of the examples in this section come from the theory of subgroup growth of finitely generated nilpotent groups. In Section 8 we consider the representation zeta function

of finitely generated nilpotent groups. We use Theorem 6.1 to prove uniform rationality of these zeta functions and Theorem 6.3 to prove that the abscissa of convergence of the corresponding global zeta function is rational. In the subgroup case this gives alternative proofs of results of [35] and [29].

Throughout this section Γ is a finitely generated nilpotent group. For any $n \in \mathbb{N}$, the number b_n of index n subgroups of Γ is finite (for background on subgroup growth, see [50]). The (global) subgroup zeta function of Γ is defined by $\xi_\Gamma(s) := \sum_{n=1}^{\infty} b_n n^{-s}$ and the p -local subgroup zeta function by $\xi_{\Gamma,p}(s) := \sum_{n=0}^{\infty} b_{p^n} p^{-ns}$ (the symbol ζ is commonly used to denote the subgroup zeta function but we reserve this for the representation zeta function in Section 8). These expressions converge if $\operatorname{Re}(s)$ is large enough. Grunewald, Segal and Smith observed in [35] that Euler factorization holds: we have

$$\xi_\Gamma(s) = \prod_p \xi_{\Gamma,p}(s),$$

where p ranges over all primes. Theorem 7.2 below (and [35, Theorem 1]) says that $\xi_{\Gamma,p}(s)$ is a rational function of p^{-s} . Hence $\xi_\Gamma(s)$ enjoys many of the properties of the Riemann zeta function.

To understand the behavior of the global subgroup zeta function, one needs to study the behaviour of the rational function $\xi_{\Gamma,p}(s)$ as p varies (cf. [6]). Du Sautoy and Grunewald introduced a class of p -adic integrals they called *cone integrals*. They showed [29, Theorem 1.3] that if $\tau_p(s) := \sum_{n=0}^{\infty} b_{p,n} p^{-ns}$ is the zeta function arising from an Euler product of suitable cone integrals then $\tau_p(s)$ is uniformly rational for $p \gg 0$ (in the variable $t := p^{-s}$) in the sense of Section 6. In fact, they proved a considerably stronger result [29, Theorem 1.4] and deduced various analytic properties of $\tau(s)$ [29, Theorem 1.5]: for instance, they showed that $\tau(s)$ can be meromorphically continued a short distance to the left of its abscissa of convergence [29, Theorem 1.1]. It follows from these results on cone integrals that $\xi_{\Gamma,p}(s)$ is uniformly rational for $p \gg 0$ [29, Section 5]. For Γ a finitely generated free nilpotent group of class 2, a stronger uniformity result holds: there is a polynomial $W(X, Y) \in \mathbb{Q}[X, Y]$ such that $\xi_{\Gamma,p}(s) = W(p, p^{-s})$ for every prime p [35, Theorem 2]. Du Sautoy, however, has given an example showing that this stronger result does not hold for Γ of arbitrary nilpotency class [28].

Theorem 7.2 below deals with some variations on the subgroup zeta function. In order to formulate the problem in terms of definable equivalence relations, we need to recall some facts about nilpotent pro- p groups, including the notion of a good basis for a subgroup of a torsion-free nilpotent group [35, Section 2]; we will need these ideas in Section 8 as well. We write \widehat{G}_p for the pro- p completion of a group G . Let $j: \Gamma \rightarrow \widehat{\Gamma}_p$

be the canonical map. Then $\widehat{\Gamma}_p$ is finitely generated as a pro- p group, so every finite-index subgroup of $\widehat{\Gamma}_p$ is open (cf. [21, Theorem 1.17]) and has p -power index (cf. [21, Lemma 1.18]). Since Γ is finitely generated nilpotent, every subgroup of p -power index is open in the pro- p topology on Γ ; in particular, there is a bijection $H \mapsto \overline{j(H)}$ between index p^n subgroups of Γ and index p^n subgroups of $\widehat{\Gamma}_p$, and $\overline{j(H)} \cong \widehat{H}_p$ (see [35, Proposition 1.2]). For any $H \trianglelefteq \Gamma$ of index p^n , we have $\Gamma/H \cong \widehat{\Gamma}_p/\overline{j(H)}$.

Let Δ be a finitely generated torsion-free nilpotent group. A *Mal'cev basis* is a tuple a_1, \dots, a_R of elements of Δ such that any element of Δ can be written uniquely in the form $a_1^{\lambda_1} \cdots a_R^{\lambda_R}$, where the $\lambda_i \in \mathbb{Z}$. We call the λ_i *Mal'cev coordinates*. Moreover, we require that group multiplication and inversion in Δ are given by polynomials in the λ_i with coefficients in \mathbb{Q} , and likewise for the map $\Delta \times \mathbb{Z} \rightarrow \Delta$, $(g, \lambda) \mapsto g^\lambda$. We may regard the a_i as elements of the pro- p completion $\widehat{\Delta}_p$, and analogous statements hold, except that λ and the Mal'cev coordinates λ_i now belong to \mathbb{Z}_p (see [35, Section 2]). In particular, the map $j: \Delta \rightarrow \widehat{\Delta}_p$ is injective and we may identify $\widehat{\Delta}_p$ with \mathbb{Z}_p^R .

Now let H be a finite-index subgroup of $\widehat{\Delta}_p$, of index p^n , say. In [35], a *good basis for H* is defined as an R -tuple $h_1, \dots, h_R \in H$ such that every element of H can be written uniquely in the form $h_1^{\lambda_1} \cdots h_R^{\lambda_R}$ ($\lambda_i \in \mathbb{Z}_p$), and satisfying an extra property which does not concern us here. We say that $h_1, \dots, h_R \in \widehat{\Delta}_p$ is a *good basis* if it is a good basis for some finite-index subgroup H of $\widehat{\Delta}_p$. For each i , we can write

$$h_i = a_1^{\lambda_{i1}} \cdots a_R^{\lambda_{iR}} \quad (7.1)$$

and we recover $|\widehat{\Delta}_p: H| = p^n$ from the formula

$$|\lambda_{11}\lambda_{22} \cdots \lambda_{RR}| = p^{-n}. \quad (7.2)$$

Any finite-index subgroup of $\widehat{\Delta}_p$ admits a good basis. Often we will identify a good basis h_1, \dots, h_R with the R^2 -tuple of coordinates (λ_{ij}) .

Proposition 7.1 *Let $\mathcal{D}_p \subseteq \mathbb{Z}_p^{R^2}$ be the set of good bases (λ_{ij}) of $\widehat{\Delta}_p$. Then the sets $(\mathcal{D}_p)_p$ *prime* are uniformly \emptyset -definable in p .*

Proof This follows from the proof of [35, Lemma 2.3]. □

For each non-negative n consider the following:

- (a) the number of index p^n subgroups of Δ ;
- (b) the number of normal index p^n subgroups of Δ ;

- (c) the number of index p^n subgroups A of Δ such that $\widehat{A}_p \cong \widehat{\Delta}_p$;
- (d) the number of conjugacy classes of index p^n subgroups of Δ ;
- (e) the number of equivalence classes of index p^n subgroups of Δ , where we define $A \sim B$ if $\widehat{A}_p \cong \widehat{B}_p$.

The rationality of $\sum_{n=0}^{\infty} b_{p,n}t^n$ in (a)–(d) of the following result are due to Grunewald, Segal, and Smith [35, Theorem 1]; for uniformity statements and the rationality of the abscissa of convergence in (a)–(d), see [29, Section 1] and the start of this section. Here we give a different proof. Observe that Theorem 7.2 for case (e) is new; here the equivalence relation does not arise from any obvious group action, and Theorem 6.1 gives a genuinely new way of proving uniform rationality. This illustrates the robustness of our methods, which are not sensitive to the precise details of how the objects to be counted are interpreted.

Theorem 7.2 *Let $b_{p,n}$ be as described in any of (a)–(e) above. Set $S_p(t) = \sum_{n=0}^{\infty} b_{p,n}t^n$. Then the power series $(S_p(t))_{p \text{ prime}}$ are uniformly rational. Moreover, the zeta function $\xi(s) := \prod_p \sum_{n=0}^{\infty} b_{p,n}p^{-ns}$ has rational abscissa of convergence.*

Proof Clearly $b_{p,0} = 1$ for all p , so rationality of the abscissa of convergence of $\xi(s)$ will follow from Theorem 1.4 once we have proved the other assertions of Theorem 7.2. To prove the rest of the theorem, we show how to interpret the objects that we are counting in a uniformly \emptyset -definable way, then apply Theorem 6.1. Consider case (a). Let \mathcal{D}_p be as in Proposition 7.1. Define $f_p: \mathcal{D}_p \rightarrow \mathbb{Z}_p$ by $f_p(\lambda_{ij}) = \lambda_{11} \cdots \lambda_{RR}$; note that the functions $(f_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p . Define an equivalence relation \mathcal{E}_p on \mathcal{D}_p as follows: two R -tuples $(\lambda_{ij}), (\mu_{ij})$, representing good bases h_1, \dots, h_R and k_1, \dots, k_R for subgroups H, K respectively, are equivalent if and only if $H = K$.

Now the equivalence relations $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p : each \mathcal{E}_p is the subset of $\mathcal{D}_p \times \mathcal{D}_p$ given by the conjunction for $1 \leq i, j \leq R$ of the formulae

$$(\exists \sigma_1^{(i)}, \dots, \sigma_R^{(i)} \in \mathbb{Z}_p) k_i = h_1^{\sigma_1^{(i)}} \cdots h_R^{\sigma_R^{(i)}}$$

and

$$(\exists \tau_1^{(j)}, \dots, \tau_R^{(j)} \in \mathbb{Z}_p) h_j = k_1^{\tau_1^{(j)}} \cdots k_R^{\tau_R^{(j)}},$$

and these become polynomial equations independent of p over \mathbb{Q} in the λ_{ij} , the μ_{ij} , the σ_i and the τ_j when we write the h_i and k_j in terms of their Mal'cev coordinates (Eqn. (7.1)).

Construct D_p and E_p from \mathcal{E}_p , \mathcal{D}_p and f_p as in the paragraph after Eqn. (6.4). Using Eqn. (7.2), we see that for each $n \in \mathbb{N}$, $D_{p,n}/E_{p,n}$ consists of precisely $b_{p,n}$

equivalence classes. We now deduce the rationality and uniform rationality assertions from Theorem 6.1 (taking $\mathfrak{L}_p = \{\mathbb{Q}_p\}$ for all p) and Remark 6.2.

The proofs in cases (b)–(e) are similar, modifying the definitions of \mathcal{D}_p and \mathcal{E}_p appropriately. For example, in (b) we replace \mathcal{D}_p by the set $\mathcal{D}_p^{\triangleleft}$ of tuples (λ_{ij}) that define a normal finite-index subgroup H ; a tuple (λ_{ij}) corresponding to a finite-index subgroup H belongs to $\mathcal{D}_p^{\triangleleft}$ if and only if it satisfies the formula

$$(\forall g \in \widehat{\Delta}_p)(\forall h \in H)(\exists \nu_1, \dots, \nu_R \in \mathbb{Z}_p) \quad ghg^{-1} = h_1^{\nu_1} \cdots h_R^{\nu_R},$$

which is made up of polynomial equations independent of p over \mathbb{Q} in the ν_i , the λ_{ij} and the Mal'cev coordinates of g and h . In case (d), the equivalence relation is the subset of $\mathcal{D}_p \times \mathcal{D}_p$ given by the formula:

there exists $g \in \widehat{\Delta}_p$, there exist $\sigma_i^{(j)}, \tau_i^{(j)} \in \mathbb{Z}_p$ for $1 \leq j \leq R$ such that $gh_jg^{-1} = k_1^{\sigma_1^{(j)}} \cdots k_R^{\sigma_R^{(j)}}$ and $g^{-1}k_jg = h_1^{\tau_1^{(j)}} \cdots h_R^{\tau_R^{(j)}}$ for $1 \leq j \leq R$.

This is made up of polynomial equations independent of p over \mathbb{Q} in the Mal'cev coordinates of g and of the h_i and the k_i . In cases (c) and (e), we can express the isomorphism condition in terms of polynomials in the Mal'cev coordinates; compare the proof of Proposition 7.4 below. \square

Remark 7.3 Du Sautoy and Grunewald prove that Theorem 7.2 (a) and (b) actually hold for an arbitrary finitely generated nilpotent group Γ , possibly with torsion. To prove this in our setting, write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ . Theorem 7.2 now follow for cases (a)–(e) from our arguments above with suitable modifications: for example, for case (a), we count not all index p^n subgroups of Δ , but only the ones that contain Θ . For details, compare the argument of the last two paragraphs of Lemma 8.6.

The proof for case (d) of Theorem 7.2 is not given explicitly in [35], but the appropriate definable integral can be constructed using the methods in the proof of [25, Theorem 1.2]; what makes this work is that the equivalence classes are the orbits of a group action. The language of [25] contains symbols for analytic functions, but our methods still apply there because we can use the results of Cluckers from the Appendix, which do hold in the analytic setting.

Here is another application, to the problem of counting finite p -groups.

Proposition 7.4 *Fix positive integers c, d . Let $c_{p,n}$ be the number of finite p -groups of order p^n and nilpotency class at most c , generated by at most d elements. Set $S_p(t) = \sum_{n=0}^{\infty} c_{p,n} t^n$. Then the power series $(S_p(t))_{p \text{ prime}}$ are uniformly rational. Moreover, the zeta function $\chi(s) := \prod_p \sum_{n=0}^{\infty} c_{p,n} p^{-ns}$ has rational abscissa of convergence.*

Proof As in Theorem 7.2, the rationality of the abscissa of convergence will follow from Theorem 1.4, and to prove the rest it is enough to interpret the objects we are counting in a uniformly \emptyset -definable way. Let Δ be the free nilpotent group of class c on d generators (note that Δ is torsion-free). Any finite p -group of order p^n and nilpotency class at most c and generated by at most d elements is a quotient of $\widehat{\Delta}_p$ by some normal subgroup of index p^n . Let $\mathcal{D}_p^{\leq c}$ and f_p be as in the proof of Theorem 7.2. Define an equivalence relation \mathcal{E}_p on $\mathcal{D}_p^{\leq c}$ as follows: two R -tuples $(\lambda_{ij}), (\mu_{ij})$, representing good bases h_1, \dots, h_R and k_1, \dots, k_R for subgroups H, K respectively, are equivalent if and only if $\widehat{\Delta}_p/H \cong \widehat{\Delta}_p/K$.

The result will follow as in Theorem 7.2 if we can show that the equivalence relations $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p . Let a_1, \dots, a_R be the Mal'cev basis of $\widehat{\Delta}_p$, as before. We claim that $\mathcal{E}_p \subseteq \mathcal{D}_p^{\leq c} \times \mathcal{D}_p^{\leq c}$ is given by the following conditions:

$$|f_p(\lambda_{ij})| = |f_p(\mu_{ij})|, \quad (7.3)$$

$$(\exists b_1, \dots, b_R \in \widehat{\Delta}_p)(\forall \nu_1, \dots, \nu_r \in \mathbb{Z}_p) a_1^{\nu_1} \cdots a_R^{\nu_R} \in H \iff b_1^{\nu_1} \cdots b_R^{\nu_R} \in K \quad (7.4)$$

and

$$(\forall \sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_r \in \mathbb{Z}_p)(\exists \nu_1, \dots, \nu_r \in \mathbb{Z}_p) (a_1^{\sigma_1} \cdots a_R^{\sigma_R} a_1^{\tau_1} \cdots a_R^{\tau_R} = a_1^{\nu_1} \cdots a_R^{\nu_R}) \wedge (b_1^{\sigma_1} \cdots b_R^{\sigma_R} b_1^{\tau_1} \cdots b_R^{\tau_R} \in b_1^{\nu_1} \cdots b_R^{\nu_R} K). \quad (7.5)$$

To prove this, suppose Eqns. (7.3), (7.4) and (7.5) hold. Then $|\widehat{\Delta}_p : H| = |\widehat{\Delta}_p : K|$ and the map $a_i H \mapsto b_i K$ defines an isomorphism from $\widehat{\Delta}_p/H$ onto $\widehat{\Delta}_p/K$. Conversely, if g is an isomorphism from $\widehat{\Delta}_p/H$ onto $\widehat{\Delta}_p/K$ then $|\widehat{\Delta}_p : H| = |\widehat{\Delta}_p : K|$, so $|f_p(\lambda_{ij})| = |f_p(\mu_{ij})|$. Moreover, we can choose $b_i \in \widehat{\Delta}_p$ such that $g(a_i H) = b_i K$ for $1 \leq i \leq R$. Then for all $\nu_1, \dots, \nu_r \in \mathbb{Z}$ we have

$$(*) \quad a_1^{\nu_1} \cdots a_R^{\nu_R} \in H \iff b_1^{\nu_1} \cdots b_R^{\nu_R} \in K$$

and for all $\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_r \in \mathbb{Z}$ there exist $\nu_1, \dots, \nu_r \in \mathbb{Z}$ such that

$$(**) \quad (a_1^{\sigma_1} \cdots a_R^{\sigma_R} a_1^{\tau_1} \cdots a_R^{\tau_R} = a_1^{\nu_1} \cdots a_R^{\nu_R}) \wedge (b_1^{\sigma_1} \cdots b_R^{\sigma_R} b_1^{\tau_1} \cdots b_R^{\tau_R} \in b_1^{\nu_1} \cdots b_R^{\nu_R} K);$$

since H, K are closed and the group operations are continuous, $(*)$ and $(**)$ hold with \mathbb{Z} replaced by \mathbb{Z}_p . This proves the claim. The formulae above involve only the functions $(f_p)_{p \text{ prime}}$ —which are uniformly \emptyset -definable in p —and polynomials independent of p over \mathbb{Q} in the Mal'cev coordinates, so the equivalence relations $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p , as required. \square

Du Sautoy's proof [27, Theorem 2.2], [26, Theorems 1.6 and 1.8] uses the fact that an isomorphism $\widehat{\Delta}_p/H \rightarrow \widehat{\Delta}_p/K$ lifts to an automorphism of $\widehat{\Delta}_p$, which implies that the equivalence relation \mathcal{E}_p arises from the action of the group $\text{Aut}(\widehat{\Delta}_p)$, a compact p -adic analytic group. This allows one to express the power series $\sum_{n=0}^{\infty} c_{p,n}t^n$ as a cone integral, from which uniform rationality follows (see the start of this section). Our proof is simpler in its algebraic input, as elimination of imaginaries allows us to use less information about \mathcal{E}_p .

Remark 7.5 Let Γ be a finitely generated nilpotent group and let $c_{p,n}$ be the number of isomorphism classes of quotients of Γ of order p^n . Then the power series $(\sum_{n=0}^{\infty} c_{p,n}t^n)_p$ prime are uniformly rational. If Γ is torsion-free then this follows immediately from the proof of Proposition 7.4. If Γ has torsion then we write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ and modify the proof of Proposition 7.4 accordingly (cf. Remark 7.3).

8 Twist isoclasses of characters of nilpotent groups

By a *representation* of a group G we shall mean a finite-dimensional complex representation, and by a *character* of G we shall mean the character of such a representation. A character is said to be *linear* if its degree is one. We write $\langle \cdot, \cdot \rangle_G$ for the usual inner product of characters of G . If χ is linear then we have

$$\langle \chi\sigma_1, \chi\sigma_2 \rangle_G = \langle \sigma_1, \sigma_2 \rangle_G \quad (8.1)$$

for all characters σ_1 and σ_2 . If $G' \leq G$ has finite index then we write $\text{Ind}_{G'}^G \cdot$ and $\text{Res}_{G'}^G \cdot$ for the induced character and restriction of a character respectively. For background on representation theory, see [14]. Below when we apply results from the representation theory of finite groups to representations of an infinite group, the representations concerned always factor through finite quotients.

We denote the set of irreducible n -dimensional characters of G by $R_n(G)$. If $N \trianglelefteq G$ then we say the character χ of an irreducible representation ρ factors through G/N if ρ factors through G/N (this depends only on χ , not on ρ).

Notation 8.1 We say a character σ of G is *admissible* if σ factors through a finite quotient of G . If p is prime then we say σ is *p -admissible* if σ factors through a finite p -group quotient of G . We write $R_n^{\text{ad}}(G)$ ($R_n^{(p)}(G)$) for the set of admissible (p -admissible) characters in $R_n(G)$. Note that $R_n^{(p)}(G)$ is empty if n is not a p -power [14, (9.3.2) Proposition].

Given $\sigma_1, \sigma_2 \in R_n(G)$, we follow [48] and say that σ_1 and σ_2 are *twist-equivalent* if $\sigma_1 = \chi\sigma_2$ for some linear character χ of G . Clearly this defines an equivalence relation on $R_n(G)$; we call the equivalence classes *twist isoclasses*.

Observation 8.2 Let σ_1, σ_2 be two irreducible degree n characters of G that are twist-equivalent: say $\sigma_2 = \chi\sigma_1$. If $N \trianglelefteq G$ such that σ_1, σ_2 both factor through G/N , then χ also factors through G/N .

If $N_1, N_2 \trianglelefteq G$ have finite (p -power) index then $N_1 \cap N_2$ also has finite (p -power) index. This implies that that when we are working with twist isoclasses in $R_n^{\text{ad}}(G)$ ($R_n^{(p)}(G)$), we need only consider twisting by admissible (p -admissible) linear characters.

Fix a finitely generated nilpotent group Γ . The set $R_n(\Gamma)$ can be given the structure of a quasi-affine complex algebraic variety. Lubotzky and Magid analyzed the geometry of this variety and proved the following result [48, Theorem 6.6].

Theorem 8.3 *There exists a finite quotient $\Gamma(n)$ of Γ such that every irreducible n -dimensional representation of Γ factors through $\Gamma(n)$ up to twisting. In particular, there are only finitely many twist isoclasses of irreducible n -dimensional characters.*

Thus the number of degree n twist isoclasses is a finite number a_n .

Definition 8.4 We define the (global) representation zeta function $\zeta_\Gamma(s)$ by $\zeta_\Gamma(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and the p -local representation zeta function $\zeta_{\Gamma,p}(s)$ by $\zeta_{\Gamma,p}(s) = \sum_{n=0}^{\infty} a_{p^n} p^{-ns}$.

It is shown in [63, Lemma 2.1] that $\zeta_\Gamma(s)$ converges on some right-half plane. Voll noted [66, Section 3.2.1] that $\zeta_\Gamma(s)$ has an Euler factorization

$$\zeta_\Gamma(s) = \prod_p \zeta_{\Gamma,p}(s)$$

for any finitely generated nilpotent group (cf. the proof of Lemma 8.5).

We now turn to the proof of Theorem 1.5. Clearly $a_1 = 1$ by construction, so the rationality of the abscissa of convergence of $\zeta_\Gamma(s)$ will follow as usual from Theorem 1.4. To prove the rest of Theorem 1.5, we show how to interpret twist isoclasses in a uniformly \emptyset -definable way. The equivalence relation in the parametrization is not simply the relation of twist-equivalence, which arises from the action of a group—the group of linear characters of Γ —but a more complicated equivalence relation.

The correspondence between index p^n subgroups of Γ and index p^n subgroups of $\widehat{\Gamma}_p$ gives a canonical bijection between $R_{p^n}^{(p)}(\Gamma)$ and $R_{p^n}^{(p)}(\widehat{\Gamma}_p)$, and it is clear that this respects twisting by p -admissible characters.

Lemma 8.5 *For every non-negative integer n , there is a bijective correspondence between the sets $R_{p^n}(\Gamma)/(\text{twisting})$ and $R_{p^n}^{(p)}(\widehat{\Gamma}_p)/(\text{twisting})$.*

Proof It suffices to show that given any $\sigma \in R_{p^n}(\Gamma)$, some twist of σ factors through a finite p -group quotient of Γ . By Theorem 8.3, we can assume that σ factors through some finite quotient F of Γ . Then F , being a finite nilpotent group, is the direct product of its Sylow l -subgroups F_l , where l ranges over all the primes dividing $|F|$. Moreover [14, Theorem 10.33], σ is a product of irreducible characters σ_l , where each σ_l is a character of F_l . Since the degree of an irreducible character of a finite group divides the order of the group [14, Proposition 9.3.2], all of the σ_l for $l \neq p$ are linear. We may therefore twist σ by a linear character of F to obtain a character that kills F_l for $l \neq p$, and this linear character is admissible by Observation 8.2. The new character factors through F_p , and we are done. \square

The key idea is that finite p -groups are monomial: that is, every irreducible character is induced from a linear character of some subgroup. We parametrize p -admissible irreducible characters of $\widehat{\Gamma}_p$ by certain pairs (H, χ) , where H is a finite-index subgroup of $\widehat{\Gamma}_p$ and χ is a p -admissible linear character of H : to a pair we associate the induced character $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$. We can parametrize these pairs using the theory of good bases for subgroups of $\widehat{\Gamma}_p$, and this description is well-behaved with respect to twisting. Two distinct pairs (H, χ) and (H', χ') may give the same induced character; this gives rise to a definable equivalence relation on the set of pairs.

If ψ is a character of H then we denote by $g.\psi$ the character of $g.H := gHg^{-1}$ defined by $(g.\psi)(ghg^{-1}) = \psi(h)$.

Lemma 8.6 (a) *Let $\sigma \in R_{p^n}^{(p)}(\widehat{\Gamma}_p)$. Then there exists $H \leq \widehat{\Gamma}_p$ such that $|\widehat{\Gamma}_p : H| = p^n$, together with a p -admissible linear character χ of H such that $\sigma = \text{Ind}_H^{\widehat{\Gamma}_p} \chi$.*

(b) *Let H be a p -power index subgroup of $\widehat{\Gamma}_p$ and let χ be a p -admissible linear character of H . Then $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is a p -admissible character of $\widehat{\Gamma}_p$, and $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is irreducible if and only if for all $g \in \widehat{\Gamma}_p - H$, $\text{Res}_{g.H \cap H}^{g.H} g.\chi \neq \text{Res}_{g.H \cap H}^H \chi$. Moreover, if ψ is a p -admissible linear character of $\widehat{\Gamma}_p$ and $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is irreducible then $\text{Ind}_H^{\widehat{\Gamma}_p} \left(\left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi \right) = \psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi$.*

(c) *Let $H, H' \leq \widehat{\Gamma}_p$ have index p^n , and let χ, χ' be p -admissible linear characters of H, H' respectively such that $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ and $\text{Ind}_{H'}^{\widehat{\Gamma}_p} \chi'$ are irreducible. Then $\text{Ind}_H^{\widehat{\Gamma}_p} \chi = \text{Ind}_{H'}^{\widehat{\Gamma}_p} \chi'$ if and only if there exists $g \in \widehat{\Gamma}_p$ such that $\text{Res}_{g.H \cap H'}^{g.H} g.\chi = \text{Res}_{g.H \cap H'}^{H'} \chi'$.*

Proof (a) Since σ is p -admissible, it factors through some finite p -group F . Since finite p -groups are monomial [14, Theorem 11.3], there exist $L \leq F$ of index p^n and a

linear character χ of L such that σ —regarded as a character of F —equals $\text{Ind}_L^F \chi$. Let H be the pre-image of L under the canonical projection $\widehat{\Gamma}_p \rightarrow F$. Regarding χ as a character of H , it is easily checked that $|\widehat{\Gamma}_p : H| = p^n$ and $\sigma = \text{Ind}_H^{\widehat{\Gamma}_p} \chi$.

(b) Since χ is p -admissible, the kernel K of χ has p -power index in $\widehat{\Gamma}_p$, so K contains a p -power index subgroup N such that $N \trianglelefteq \widehat{\Gamma}_p$. Clearly $N \leq \ker(\text{Ind}_H^{\widehat{\Gamma}_p} \chi)$, so $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is p -admissible. The irreducibility criterion follows immediately from [14, Theorem 10.25]. By Frobenius reciprocity,

$$\begin{aligned}
& \left\langle \text{Ind}_H^{\widehat{\Gamma}_p} \left(\left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi \right), \psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi \right\rangle_{\widehat{\Gamma}_p} \\
&= \left\langle \left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi, \text{Res}_H^{\widehat{\Gamma}_p} \left(\psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi \right) \right\rangle_H \\
&= \left\langle \left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi, \left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \text{Res}_H^{\widehat{\Gamma}_p} \left(\text{Ind}_H^{\widehat{\Gamma}_p} \chi \right) \right\rangle_H \\
&= \left\langle \chi, \text{Res}_H^{\widehat{\Gamma}_p} \left(\text{Ind}_H^{\widehat{\Gamma}_p} \chi \right) \right\rangle_H \text{ by Eqn. (8.1)} \\
&= \left\langle \text{Ind}_H^{\widehat{\Gamma}_p} \chi, \text{Ind}_H^{\widehat{\Gamma}_p} \chi \right\rangle_{\widehat{\Gamma}_p} \\
&= 1.
\end{aligned}$$

Now $\psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is irreducible, because $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ is, and the degrees of $\text{Ind}_H^{\widehat{\Gamma}_p} \left(\left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi \right)$ and $\psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi$ are equal. We deduce that $\psi \text{Ind}_H^{\widehat{\Gamma}_p} \chi = \text{Ind}_H^{\widehat{\Gamma}_p} \left(\left(\text{Res}_H^{\widehat{\Gamma}_p} \psi \right) \chi \right)$.

(c) The Mackey Subgroup Theorem [14, Theorem 10.13] gives

$$\text{Res}_{H'}^{\widehat{\Gamma}_p} \left(\text{Ind}_H^{\widehat{\Gamma}_p} \chi \right) = \sum_{\bar{g} \in H' \backslash \widehat{\Gamma}_p / H} \text{Ind}_{g.H \cap H'}^{H'} \left(\text{Res}_{g.H \cap H'}^{g.H} g \cdot \chi \right). \quad (8.2)$$

Here the sum is over a set of double coset representatives g for $H' \backslash \widehat{\Gamma}_p / H$ (the characters on the RHS of the formula are independent of choice of representative). Since $\text{Ind}_H^{\widehat{\Gamma}_p} \chi$ and $\text{Ind}_{H'}^{\widehat{\Gamma}_p} \chi'$ are irreducible, they are distinct if and only if their inner product is zero.

We have

$$\begin{aligned}
& \left\langle \text{Ind}_H^{\widehat{\Gamma}_p} \chi, \text{Ind}_{H'}^{\widehat{\Gamma}_p} \chi' \right\rangle_{\widehat{\Gamma}_p} \\
&= \left\langle \text{Res}_{H'}^{\widehat{\Gamma}_p} (\text{Ind}_H^{\widehat{\Gamma}_p} \chi), \chi' \right\rangle_{H'} \text{ by Frobenius reciprocity} \\
&= \sum_{\bar{g} \in H' \backslash \widehat{\Gamma}_p / H} \left\langle \text{Ind}_{g \cdot H \cap H'}^{H'} \left(\text{Res}_{g \cdot H \cap H'}^{g \cdot H} g \cdot \chi \right), \chi' \right\rangle_{H'} \text{ by the Mackey Subgroup Theorem} \\
&= \sum_{\bar{g} \in H' \backslash \widehat{\Gamma}_p / H} \left\langle \text{Res}_{g \cdot H \cap H'}^{g \cdot H} g \cdot \chi, \text{Res}_{g \cdot H \cap H'}^{H'} \chi' \right\rangle_{g \cdot H \cap H'} \text{ by Frobenius reciprocity.}
\end{aligned}$$

This vanishes if and only if each of the summands vanishes, which happens if and only if $\text{Res}_{g \cdot H \cap H'}^{g \cdot H} g \cdot \chi \neq \text{Res}_{g \cdot H \cap H'}^{H'} \chi'$ for every g , since the characters concerned are linear. The result follows. \square

Write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ : for example, we may take Δ to be the free class c nilpotent group on N generators for appropriate N and c . Let $\pi: \Delta \rightarrow \Gamma$ be the canonical projection, and let $i: \Theta \rightarrow \Delta$ be inclusion. Let $\widehat{\Delta}_p, \widehat{\Theta}_p$ be the pro- p completions of Δ, Θ respectively. Then π (respectively i) extends to a continuous homomorphism $\widehat{\pi}_p: \widehat{\Delta}_p \rightarrow \widehat{\Gamma}_p$ (respectively $\widehat{i}_p: \widehat{\Theta}_p \rightarrow \widehat{\Delta}_p$), and the three groups $\widehat{i}_p(\widehat{\Theta}_p)$, $\ker \widehat{\pi}_p$, and the closure of Θ in $\widehat{\Delta}_p$ all coincide (compare [21, Chapter 1, Ex. 21]; because Δ is finitely generated nilpotent, it can in fact be shown that \widehat{i}_p is injective, and hence an isomorphism onto its image). Clearly p -admissible representations of $\widehat{\Gamma}_p$ correspond bijectively to p -admissible representations of $\widehat{\Delta}_p$ that kill $\ker \widehat{\pi}_p$. Now Θ is finitely generated (see, e.g., [67, Lemma 1.2.2]), so we can choose a Mal'cev basis $\theta_1, \dots, \theta_s$ for Θ . We identify the θ_i with their images in $\widehat{\Delta}_p$.

Let μ_{p^n} be the group of all complex p^n th roots of unity, and let μ_{p^∞} be the group of all complex p -power roots of unity.

Lemma 8.7 *The groups μ_{p^∞} and $\mathbb{Q}_p/\mathbb{Z}_p$ are isomorphic.*

Proof Let $p^{-\infty}\mathbb{Z} \leq \mathbb{Q}$ be the group of rational numbers of the form np^{-r} for $n \in \mathbb{Z}$ and r a non-negative integer. Then $p^{-\infty}\mathbb{Z} \cap \mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Z}_p p^{-\infty}\mathbb{Z} = \mathbb{Q}_p$, so $\mathbb{Q}_p/\mathbb{Z}_p \cong p^{-\infty}\mathbb{Z}/\mathbb{Z}$, by one of the standard group isomorphism theorems. The map $q \mapsto e^{2\pi i q}$ gives an isomorphism from $p^{-\infty}\mathbb{Z}/\mathbb{Z}$ to μ_{p^∞} . \square

Let $\Phi: \mu_{p^\infty} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ be the isomorphism described above. Any p -admissible linear character of a pro- p group takes its values in μ_{p^∞} , so we use Φ to identify p -admissible linear characters with p -admissible homomorphisms to $\mathbb{Q}_p/\mathbb{Z}_p$.

Lemma 8.8 Let $\mathcal{D}_p \subseteq \mathbb{Z}_p^{R^2} \times \mathbb{Q}_p^R$ be the set of tuples (λ_{ij}, y_k) , where $1 \leq i, j \leq R$ and $1 \leq k \leq R$, satisfying the following conditions:

(a) the λ_{ij} form a good basis h_1, \dots, h_R for some finite-index subgroup H of $\widehat{\Delta}_p$ such that $\ker \widehat{\pi}_p \leq H$;

(b) the prescription $h_i \mapsto y_i \bmod \mathbb{Z}_p$ gives a well-defined p -admissible homomorphism $\chi: H \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ that kills $\ker \widehat{\pi}_p$;

(c) the induced character $\text{Ind}_H^{\widehat{\Delta}_p} \chi$ is irreducible.

Then the sets $(\mathcal{D}_p)_p$ prime are uniformly \emptyset -definable in p . Moreover, $\text{Ind}_H^{\widehat{\Delta}_p} \chi$ is a p -admissible character of $\widehat{\Delta}_p$ that kills $\ker \widehat{\pi}_p$ and hence induces a p -admissible character of $\widehat{\Gamma}_p$, and every p -admissible irreducible character of $\widehat{\Gamma}_p$ arises in this way.

Notation 8.9 Given $(\lambda_{ij}, y_k) \in \mathcal{D}_p$, we write $\Psi(\lambda_{ij}, y_k)$ for the pair (H, χ) . Since the h_i generate H topologically, the p -admissible homomorphism χ defined by the y_i is unique.

Proof Condition (a) is uniformly \emptyset -definable in p , by Proposition 7.1 (to the formulae that define the set of good bases we add the formulae $(\exists \nu_{1j}, \dots, \nu_{rj} \in \mathbb{Z}_p) \theta_j = h_1^{\nu_{1j}} \dots h_R^{\nu_{Rj}}$ for $1 \leq j \leq s$). Given that (a) holds, we claim that (b) holds if and only if there exists an R^2 -tuple (μ_{ij}) such that:

(i) (μ_{ij}) defines a good basis k_1, \dots, k_R for a finite-index subgroup K of $\widehat{\Delta}_p$;

(ii) $K \trianglelefteq H$;

(iii) $\ker \widehat{\pi}_p \subseteq K$;

(iv) there exist $\overline{y} \in \mathbb{Q}_p$, $r_1, \dots, r_R \in \mathbb{Z}_p$, $h \in H$ such that $|y| = |H/K|$ and for every i we have $\overline{h^{r_i}} = \overline{h_i}$ and $r_i y = y_i \bmod \mathbb{Z}_p$. (Here \overline{x} denotes the image of $x \in H$ under the canonical projection $H \rightarrow H/K$.)

To see this, note that if (b) holds then $\ker \chi$ is a finite-index subgroup of H which satisfies (ii) and (iii). Take (μ_{ij}) to be any tuple defining a good basis for K . Then H/K , being isomorphic to a finite subgroup of $\mathbb{Q}_p/\mathbb{Z}_p$, is cyclic, so choose $h \in H$ that generates H/K and choose $y \in \mathbb{Q}_p$ such that $\chi(h) = y \bmod \mathbb{Z}_p$. We can choose $r_1, \dots, r_R \in \mathbb{Z}$ such that $\overline{h_i} = \overline{h^{r_i}}$ for each i , and it is easily checked that (iv) holds.

Conversely, suppose there exists a tuple (μ_{ij}) satisfying (i)–(iv). The map $\mathbb{Z}_p \rightarrow H$, $\lambda \mapsto h^\lambda$ is continuous because it is polynomial with respect to the Mal'cev coordinates, so there exists an open neighborhood U of 0 in \mathbb{Z}_p such that $h^\lambda \in K$ for all $\lambda \in U$. Since \mathbb{Z} is dense in \mathbb{Z}_p , we may therefore find $n_1, \dots, n_R \in \mathbb{Z}$ such that $\overline{h_i} = \overline{h^{n_i}}$ for each i . Hence H/K is cyclic with generator \overline{h} .

We have a monomorphism $\beta: H/K \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ given by $\beta(\overline{h^n}) = ny \bmod \mathbb{Z}_p$. Let χ be the composition $H \rightarrow H/K \xrightarrow{\beta} \mathbb{Q}_p/\mathbb{Z}_p$. The canonical projection $H \rightarrow H/K$ is

continuous [21, 1.2 Proposition], so we have $\chi(h^\lambda) = \lambda y \bmod \mathbb{Z}_p$ for every $\lambda \in \mathbb{Z}_p$. Condition (iv) implies that $\chi(h_i) = y_i \bmod \mathbb{Z}_p$ for every i , as required. This proves the claim.

Now condition (i) is uniformly \emptyset -definable in p , by Proposition 7.1. Condition (iii) can be expressed as

$$\begin{aligned} (\forall \nu_1, \dots, \nu_s \in \mathbb{Z}_p) [(\exists \sigma_1, \dots, \sigma_R \in \mathbb{Z}_p) \theta_1^{\nu_1} \cdots \theta_s^{\nu_s} = h_1^{\sigma_1} \cdots h_R^{\sigma_R}] \\ \Rightarrow (\exists \tau_1, \dots, \tau_R \in \mathbb{Z}_p) \theta_1^{\nu_1} \cdots \theta_s^{\nu_s} = k_1^{\tau_1} \cdots k_R^{\tau_R}. \end{aligned} \quad (8.3)$$

Eqn. (8.3) can be expressed in terms of polynomials independent of p over \mathbb{Q} in the λ_{ij} , the μ_{ij} , the ν_k , the σ_l and the τ_l , so condition (iii) is uniformly \emptyset -definable in p . (Note that the θ_k are fixed elements of Δ , so their Mal'cev coordinates are not just elements of \mathbb{Z}_p but elements of \mathbb{Z} .)

Similar arguments show that conditions (ii) and (iv) are also uniformly \emptyset -definable in p . In (iv), note that the conditions $\overline{h^{r_i}} = \overline{h_i}$ imply by the argument above that h is a generator for H/K , so the condition $|y| = |H/K|$ can be expressed as $(h^{y^{-1}} \in K) \wedge ((\forall z \in \mathbb{Q}_p) |z| < |y| \Rightarrow h^{z^{-1}} \notin K)$. This shows that (condition (a)) \wedge (condition (b)) is uniformly \emptyset -definable in p .

Condition (iii) implies that χ kills $\ker \widehat{\pi}_p$. Hence $\text{Ind}_{\widehat{H}}^{\widehat{\Delta}_p} \chi$ kills $\ker \widehat{\pi}_p$, so $\text{Ind}_{\widehat{H}}^{\widehat{\Delta}_p} \chi$ gives rise to an irreducible p -admissible character of $\widehat{\Gamma}_p$. By Lemma 8.6 (b), irreducibility of the induced character can be written as

$$(\forall g \in \widehat{\Delta}_p - H) (\exists h \in H) ghg^{-1} \in H \text{ and } \chi(ghg^{-1}) \neq \chi(h).$$

Writing this in terms of the Mal'cev coordinates, we see that condition (c) is uniformly \emptyset -definable in p .

By Lemma 8.6 (a), any p -admissible irreducible character σ of $\widehat{\Gamma}_p$ is of the form $\text{Ind}_L^{\widehat{\Gamma}_p} \chi$ for some finite-index subgroup L of $\widehat{\Gamma}_p$ and some p -admissible linear character χ of L . Let H be the pre-image of L under the canonical projection $\widehat{\Delta}_p \rightarrow \widehat{\Gamma}_p$. Regarding σ, χ as representations of $\widehat{\Delta}_p, H$ respectively, it is easily checked that $\sigma = \text{Ind}_H^{\widehat{\Delta}_p} \chi$. Choose (λ_{ij}) defining a good basis h_1, \dots, h_R for H , and choose y_k such that $\chi(h_k) = y_k \bmod \mathbb{Z}_p$ for all k . The above argument shows that $(\lambda_{ij}, y_k) \in \mathcal{D}_p$. This completes the proof. \square

Define $f_p: \mathcal{D}_p \rightarrow \mathbb{Z}_p$ by $f_p(\lambda_{ij}, y_k) = \lambda_{11} \cdots \lambda_{RR}$. Define an equivalence relation \mathcal{E}_p on \mathcal{D}_p by $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ if $\text{Ind}_{\widehat{H}}^{\widehat{\Delta}_p} \chi$ and $\text{Ind}_{\widehat{H}'}^{\widehat{\Delta}_p} \chi'$ are twist-equivalent, where $(H, \chi) =$

$\Psi(\lambda_{ij}, y_k)$ and $(H', \chi') = \Psi(\lambda'_{ij}, y'_k)$. The degree of $\text{Ind}_H^{\widehat{\Delta}_p} \chi$ equals $|f_p(\lambda_{ij}, y_k)|^{-1}$, and likewise for (λ'_{ij}, y'_k) , so if $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ then $f_p(\lambda_{ij}, y_k) = f_p(\lambda'_{ij}, y'_k)$.

Construct D_p and E_p from \mathcal{E}_p , \mathcal{D}_p and f_p as at the end of Section 6. It follows from Lemma 8.8 and the definition of \mathcal{E}_p that $D_{p,n}$ is the union of precisely a_{p^n} $E_{p,n}$ -equivalence classes (note that if one representation of $\widehat{\Gamma}_p$ is the twist of another by some linear character ψ of $\widehat{\Delta}_p$ then ψ is automatically a character of $\widehat{\Gamma}_p$, by Observation 8.2). To complete the proof of Theorem 1.5, it suffices by Theorem 6.1 and Remark 6.2 to show that $(D_p)_{p \text{ prime}}$ and $(E_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p . But the sets $(D_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p by Lemma 8.8, so it is enough to prove the following result.

Proposition 8.10 *The equivalence relations $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p .*

Proof Let a_1, \dots, a_R be a Mal'cev basis for Δ . Let $\mathcal{D}'_p \subseteq \mathbb{Q}_p^R$ be the set of R -tuples (z_1, \dots, z_R) such that the prescription $a_i \mapsto z_i \bmod \mathbb{Z}_p$ gives a well-defined p -admissible linear character of $\widehat{\Delta}_p$ that kills $\ker \widehat{\pi}_p$. We denote this character by $\Xi(z_1, \dots, z_R)$ (or just $\Xi(z_k)$). Similar arguments to those in the proof of Lemma 8.8 show that the sets $(\mathcal{D}'_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p . Let $(z_1, \dots, z_R) \in \mathcal{D}'_p$, let $(H, \chi) = \Psi(\lambda_{ij}, y_k)$ and let h_1, \dots, h_R be the corresponding good basis for H . Then $h_k = a_1^{\lambda_{k1}} \dots a_R^{\lambda_{kR}}$, so $\Xi(z_k) = \lambda_{k1} z_1 + \dots + \lambda_{kR} z_R$. Hence $\text{Res}_H^{\widehat{\Delta}_p} \Xi(z_k) \chi = \Psi(\lambda_{ij}, y_k + \lambda_{k1} z_1 + \dots + \lambda_{kR} z_R)$. Applying Lemma 8.6 (b) and (c), we see that if $(H', \chi') = \Psi(\lambda'_{ij}, y'_k)$ then $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ if and only if

$$\begin{aligned} (\exists (z_1, \dots, z_R) \in \mathcal{D}'_p) (\exists g \in \widehat{\Delta}_p) (\forall h \in H) ghg^{-1} \in H' \\ \Rightarrow \left(\text{Res}_H^{\widehat{\Delta}_p} \Xi(z_k) \chi \right) (h) = \chi'(ghg^{-1}). \end{aligned}$$

Writing this in terms of the Mal'cev coordinates, we obtain an equation independent of p involving \mathcal{D}'_p and absolute values of polynomials over \mathbb{Q} in the λ_{ij} , the y_k , the λ'_{ij} , the y'_k , the z_k , and the Mal'cev coordinates of g and h . We deduce that the equivalence relations $(\mathcal{E}_p)_{p \text{ prime}}$ are uniformly \emptyset -definable in p , as required. \square

Remark 8.11 Let M be a topologically finitely generated nilpotent pro- p group for some prime p . Note that since every finite-index subgroup of M is open and has p -power index, a representation $\rho: M \rightarrow \text{GL}_n(\mathbb{C})$ is p -admissible if and only if it is continuous (with respect to the discrete topology on $\text{GL}_n(\mathbb{C})$). Set $a_n = |\text{R}_{p^n}^{(p)}(M)/(\text{twisting})|$ and set $\zeta_H(s) = \sum_{n=0}^{\infty} a_n p^{-ns}$. Then $\zeta_H(s)$ is a rational function of p^{-s} with coefficients in \mathbb{Q} . We can see this as follows. Let Γ be a finitely generated dense subgroup of M ,

and choose an epimorphism π from a torsion-free finitely generated nilpotent group Δ onto Γ . Then π gives rise to a continuous epimorphism $\widehat{\pi}_p$ from the pro- p completion $\widehat{\Delta}_p$ to M . The kernel K of $\widehat{\pi}_p$ is a closed subgroup of $\widehat{\Delta}_p$, so K is also topologically finitely generated. Let $\Theta = \langle \theta_1, \dots, \theta_s \rangle$ be a finitely generated dense subgroup of K . The result now follows from the proof of Theorem 1.4 given above (cf. the paragraph after the proof of Lemma 8.6 and the proof of Lemma 8.8).

We finish the section by applying our approach to work of Stasinski and Voll on the representation zeta functions of nilpotent groups arising from smooth unipotent group schemes. We give a brief summary of the necessary background; see [63] for details. Let K be a number field with ring of integers \mathcal{O} and let $\mathbf{G} = \mathbf{G}_\Lambda$ be the smooth unipotent group scheme over \mathcal{O} corresponding to a nilpotent Lie lattice Λ over \mathcal{O} , in the sense of [63, 2.1]. If R is a ring extension of \mathcal{O} then we denote by $\mathbf{G}(R)$ the group of R -points of \mathbf{G} . Note that $\mathbf{G}(\mathcal{O})$ is a finitely generated torsion-free nilpotent group; moreover, for any finitely generated nilpotent group Γ , there exists a smooth unipotent group scheme \mathbf{H} over \mathbb{Z} such that $\zeta_{\Gamma,p}(s) = \zeta_{\mathbf{H}(\mathbb{Z}),p}(s)$ for all p sufficiently large.

Let \mathfrak{p} be a nonzero prime ideal of \mathcal{O} . Let $K_{\mathfrak{p}}$ be the completion of K at \mathfrak{p} and let $\mathcal{O}_{\mathfrak{p}}$ be the valuation ring of $K_{\mathfrak{p}}$. Let $\zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s) := \sum_{i=0}^{\infty} \tilde{a}_{p^i}(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) p^{-is}$ be the zeta function that counts the twist isoclasses of continuous irreducible complex representations of the pro- p group $\mathbf{G}(\mathcal{O}_{\mathfrak{p}})$, where p is the characteristic of the residue field of $K_{\mathfrak{p}}$. It follows from Eqn. (8.5) below that for p sufficiently large, $\tilde{a}_{p^i}(\mathbf{G}(\mathcal{O}_{\mathfrak{p}})) = 0$ unless p^i is a power of q , where q is the cardinality of the residue field of $K_{\mathfrak{p}}$. There is a ‘‘refined Euler product’’

$$\zeta_{\mathbf{G}(\mathcal{O})}(s) = \prod_{\mathfrak{p}} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s) \tag{8.4}$$

and the p -local representation zeta function is given by the ‘‘mini Euler product’’

$$\zeta_{\mathbf{G}(\mathcal{O}),p}(s) = \prod_{\mathfrak{p}|p} \zeta_{\mathbf{G}(\mathcal{O}_{\mathfrak{p}})}(s).$$

Let L be a finite extension of K and let \mathcal{O}_L be the ring of integers of L . Let \mathfrak{p} be a nonzero prime ideal of \mathcal{O} and let \mathfrak{B} be a nonzero prime ideal of \mathcal{O}_L that divides \mathfrak{p} . Let $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ and let \mathfrak{D} be the valuation ring of the completion $L_{\mathfrak{B}}$. Let p be the residue field characteristic of \mathfrak{o} and let q, q^f be the cardinality of the residue field of $\mathfrak{o}, \mathfrak{D}$, respectively. Note that $\mathbf{G}(\mathfrak{D})$ is a topologically finitely generated nilpotent pro- p group. We will show that $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ comes (up to a change of variable) from counting the

equivalence classes of a family of equivalence relations that are uniformly K -definable over \mathfrak{L} for an appropriate choice of \mathfrak{L} .

Let d, k, r be as defined on [63, p516]. Let $\mathbf{Y} = (Y_1, \dots, Y_d)$ be a tuple of indeterminates and define the $\lfloor r/2 \rfloor \times \lfloor r/2 \rfloor$ commutator matrix $\mathcal{R}(\mathbf{Y})$ as in [63, p516] by choosing a basis for the \mathfrak{D} -Lie algebra \mathfrak{g} that is associated to \mathbf{G} . Then for any $\mathbf{y} \in \mathfrak{D}^d$, $\mathcal{R}(\mathbf{y})$ is a matrix with entries from \mathfrak{D} . As in the proof of [63, Thm. A], we may choose the data that define $\mathcal{R}(\mathbf{Y})$ in a global way and ensure that the quantities b_i that appear in [63, Eqn. (2.6)] are all zero, at the cost of discarding finitely many rational primes. In particular, for p sufficiently large—say, for $p > p_0$ —the linear forms that appear as entries of the matrix $\mathcal{R}(\mathbf{Y})$ have coefficients from \mathcal{O} , and these coefficients do not depend on $L, \mathfrak{B}, \mathfrak{p}$ or p . We define the submatrix $\mathcal{S}(\mathbf{Y})$ of $\mathcal{R}(\mathbf{Y})$ as in [63, p516].

Let $\nu, \tilde{\nu}$ be as defined on [63, p518]. Let D be the set of tuples $(\mathbf{y}, N, \mathbf{a}, \mathbf{c}) \in \mathfrak{D}^d \times \mathbb{N} \times \mathbb{N}^{\lfloor r/2 \rfloor} \times \mathbb{N}^k$ such that $\mathbf{y} \neq \mathbf{0} \pmod{\mathfrak{B}^N}$, $\nu(\pi_N(\mathcal{R}(\mathbf{y}))) = \mathbf{a}$ and $\tilde{\nu}(\pi_N(\mathcal{S}(\mathbf{y}))) = \mathbf{c}$, where π_N denotes reduction of the matrix entries mod \mathfrak{B}^N . Define $g(N, \mathbf{a}) = \sum_{i=1}^{\lfloor r/2 \rfloor} (N - a_i)$ and $h(N, \mathbf{c}) = \prod_{i=1}^k (N - c_i)$. It follows from the definition of ν and $\tilde{\nu}$ that if $(\mathbf{y}, N, \mathbf{a}, \mathbf{c}) \in D$ then $g(N, \mathbf{a}), h(N, \mathbf{c}) \in \mathbb{N}$, and it is not hard to show using the theory of elementary divisors that $h(N, \mathbf{c}) \leq 2g(N, \mathbf{a})$ (recall that $\mathcal{S}(\mathbf{y})$ is a submatrix of $\mathcal{R}(\mathbf{y})$). Now define an equivalence relation E on D by

$$(\mathbf{y}, N, \mathbf{a}, \mathbf{c})E(\mathbf{y}', N', \mathbf{a}', \mathbf{c}') \iff$$

$$N = N', \quad \mathbf{a} = \mathbf{a}', \quad \mathbf{c} = \mathbf{c}' \text{ and } \mathbf{y} = \mathbf{y}' \pmod{\mathfrak{B}^{N+2g(N, \mathbf{a})-h(N, \mathbf{c})}}.$$

It is easily seen that the functions $\nu, \tilde{\nu}, g$ and h are definable over K , so D and E are definable over K . Set $D_l = \{(\mathbf{y}, N, \mathbf{a}, \mathbf{c}) \in D : g(N, \mathbf{a}) = l\}$ for $l \in \mathbb{N}$ and let E_l be the restriction of E to D_l .

Let $e_{\mathfrak{D}, l}$ be the number of equivalence classes of E_l on D_l . It follows from [63, Eqn. (2.7)] that for $p > p_0$,

$$\zeta_{\mathbf{G}(\mathfrak{D})}(s-2) = \sum_{l \in \mathbb{N}} e_{\mathfrak{D}, l} q^{-fls}. \quad (8.5)$$

(Note that if $p > p_0$ then $e_{\mathfrak{D}, l} = \sum_{\{(N, \mathbf{a}, \mathbf{c}) : g(N, \mathbf{a}) = l\}} q^{f(2g(N, \mathbf{a}) - h(N, \mathbf{c}))} \mathcal{N}_{N, \mathbf{a}, \mathbf{c}}^0$, where $\mathcal{N}_{N, \mathbf{a}, \mathbf{c}}^0$ is as in [63, Eqn. (2.6)]. Moreover, although [63, Eqn. (2.7)] is stated only for $L = K$, the equation holds for arbitrary L because the coefficients of the linear forms that appear as entries of $\mathcal{R}(\mathbf{Y})$ and $\mathcal{S}(\mathbf{Y})$ do not change when one extends the field from K to L .)

Now define \mathcal{L}_p to be empty if $p \leq p_0$ and the set of localisations $L_{\mathfrak{B}}$ if $p > p_0$, where L runs over all the finite extensions of K and \mathfrak{B} runs over all the nonzero prime

ideals of L that divide p . Set $\mathfrak{L} = \bigcup_p \mathfrak{L}_p$. Then we see that $(D_{L_{\mathfrak{B}}})_{L_{\mathfrak{B}} \in \mathfrak{L}}$ and $(E_{L_{\mathfrak{B}}})_{L_{\mathfrak{B}} \in \mathfrak{L}}$ are uniformly K -definable in \mathfrak{L} ; again, the key point is that the entries of $\mathcal{R}(\mathbf{Y})$ and $\mathcal{S}(\mathbf{Y})$ are linear forms with coefficients from \mathcal{O} , and these coefficients do not depend on L , \mathfrak{B} , \mathfrak{p} or p . Applying Theorem 6.1, Remark 6.2.3 and Remark 8.11, we obtain the following result.

Theorem 8.12 *Let the notation be as above. Let $S_{L_{\mathfrak{B}}}(t) = \sum_{l \in \mathbb{N}} e_{\mathfrak{D}, l} t^l$. Then the power series $(S_{L_{\mathfrak{B}}}(t))_{L_{\mathfrak{B}}}$ are uniformly rational for $p \gg 0$. In particular, $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ has the form*

$$\zeta_{\mathbf{G}(\mathfrak{D})}(s) = \frac{\sum_{l=1}^d q_l |X_l(\mathbb{F}_{q^f})| q^{-fls}}{q^{fn} \prod_{j=1}^k (1 - q^{f(a_j - b_j s)})} \quad (8.6)$$

for all $p \gg 0$, where the X_l , etc., are as in Theorem 6.1.

Moreover, each $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ is a rational function of p^{-s} with coefficients in \mathbb{Q} .

Remark 8.13 1. Theorem 8.12 shows that $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ depends on \mathfrak{p} and \mathfrak{D} only by way of the residue field of \mathfrak{D} . A different expression for $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ is given in [63, Thm. A]; this expression implies very strong uniformity behavior when one varies L and \mathfrak{B} for fixed \mathfrak{p} . It is not stated explicitly in [63] that the power series $(S_{L_{\mathfrak{B}}}(t))_{L_{\mathfrak{B}}}$ are uniformly rational for $p \gg 0$, but this can be seen from the proof of [63, Thm. A]; cf. [5, Sec. 4]. To prove rationality of $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ for every \mathfrak{D} , however, we need Remark 8.11.

2. Stasinski and Voll show that $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ satisfies a functional equation [63, Thm. A]. Our methods—which apply to a very general class of problems—do not produce the functional equation that holds in this particular setting; there is no reason to expect the zeta function of an arbitrary definable equivalence relation to satisfy a functional equation.
3. Dung and Voll show that the abscissa of convergence α of $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ is rational and does not depend on \mathcal{O} , and they prove that $\zeta_{\mathbf{G}(\mathfrak{D})}(s)$ can be meromorphically continued a short distance to the left of the line $\operatorname{Re}(s) = \alpha$ [24, Thm. A]. For related results in the context of semisimple arithmetic groups, see [6], [1], [2] and [3].

Example 8.14 Let \mathbf{H} be the smooth unipotent group scheme over \mathbb{Z} corresponding to the Heisenberg group: so for every ring R , $\mathbf{H}(R)$ is the group of 3×3 upper unitriangular matrices with entries from R . Then for any number field L ,

$$\zeta_{\mathbf{H}(\mathcal{O}_L)}(s) = \frac{\zeta_L(s-1)}{\zeta_L(s)}, \quad (8.7)$$

where $\zeta_L(s)$ is the Dedekind zeta function of L . For $L = \mathbb{Q}$ this follows from results of Nunley and Magid [54], who explicitly calculated the twist isoclasses of $\mathbf{H}(\mathbb{Z})$. For L a quadratic extension of \mathbb{Q} , Eqn. (8.7) follows from work of Ezzat [31, Theorem 1.1], while for general L , it is a special case of results of Stasinski and Voll [63, Thm. B].

The expression for the subgroup zeta function of $\mathbf{H}(\mathbb{Z})$ is more complicated: it is given by

$$\xi_{\mathbf{H}(\mathbb{Z})}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)},$$

where $\zeta(s)$ is the Riemann zeta function [29, Section 1]. Expressions for $\xi_{\mathbf{H}(\mathcal{O}_L)}(s)$ were obtained by Grunewald, Segal and Smith for quadratic and cubic extensions of \mathbb{Q} , but no general formula is known (see [35, Sec. 8])¹. This suggests that the representation zeta function is better behaved than the subgroup zeta function. The same seems to be true also for semisimple arithmetic groups [49].

Theorem 8.12 (and Example 8.14) illustrate a significant difference between the subgroup zeta functions and representation zeta functions of groups of points of smooth unipotent group schemes: the former do not have the same double uniformity properties as the latter. For instance, let $K = \mathbb{Q}$ and let \mathbf{G} be the smooth unipotent \mathbb{Z} -scheme \mathbb{G}_a (the additive group). The p -local subgroup zeta function of $\mathbf{G}(\mathbb{Z}) = \mathbb{Z}$ is given by $\xi_{\mathbb{Z},p}(s) = \frac{1}{1-p^{-s}}$. Now let $L = \mathbb{Q}(i)$ and let p be any prime such that $p \equiv 3 \pmod{4}$. Let \mathfrak{B} be a prime ideal of $\mathbb{Z}[i]$ that divides p , and let \mathfrak{D} be the valuation ring of the completion of $\mathbb{Q}[i]$ at \mathfrak{B} ; note that \mathfrak{D} is isomorphic as an additive group to \mathbb{Z}_p^2 . The residue field of \mathfrak{D} has cardinality $q = p^2$. Recall from Section 7 that if Γ is a torsion-free finitely generated nilpotent group then subgroups of $\widehat{\Gamma}_p$ are parametrised by good bases, which for $\Gamma = \mathbb{Z}$ and $\widehat{\Gamma}_p = \mathbb{Z}_p$ are just 1-tuples of nonzero elements of \mathbb{Z}_p . But 1-tuples of nonzero elements of $\mathbb{Z}[i]$ parametrise not finite-index subgroups of $\mathbb{Z}[i]$ but finite-index subrings of $\mathbb{Z}[i]$ (cf. [35, Sec. 3]), and $\xi_{\mathfrak{D},p}(s)$ is equal not to $\frac{1}{1-q^{-s}} = \frac{1}{1-p^{-2s}}$ but to $\frac{1}{(1-p^{-s})(1-p^{-s+1})}$ (this formula follows from [50, Thm. 15.1]). In the language of Section 6, the definable sets and equivalence relations that we use to parametrise

¹Schein and Voll have obtained results on the structure of the normal subgroup zeta function of $\mathbf{H}(\mathcal{O})$ [58], [59].

finite-index subgroups via good bases are uniformly \emptyset -definable in p , but need not be uniformly \emptyset -definable in \mathfrak{L} if we take \mathfrak{L}_p to contain more than one extension of \mathbb{Q}_p .

The uniform definability established in Theorem 8.12 cannot be seen from our parametrisation of twist isoclasses, which involves good bases: to prove double uniformity we needed the Kirillov orbit formalism of [63]. Our results give a further illustration of the power of the machinery developed in [44], [64] and [63].

A Rationality results for p -adic subanalytic equivalence relations, by Raf Cluckers

Dedicated to Jan Denef, Lou van den Dries, Leonard Lipshitz and Angus Macintyre

A.1 Introduction

One way to understand Denef's rationality results of [15] for the generating power series $\sum_{n \geq 0} A_n T^n$ with coefficients

$$A_n := \#\{\underline{x} \in (\mathbb{Z}/(p^n\mathbb{Z}))^d : \varphi(x, n), \underline{x} = x \bmod (p^n)\},$$

for $n \geq 0$, where φ is a definable condition on n and on $x \in \mathbb{Z}_p^d$, goes by writing A_n as

$$\int_{\mathbb{Z}_p^d} p^{f(x,n)} |dx|$$

for some well-chosen definable function f and by studying the way such integrals may in general depend on the parameter n . This has started a vast study of the dependence of such integrals on more general parameters and on p , culminating in a way in the theory of motivic integration, see [52, 55, 18, 13, 41]. Most of this study works equally well in the semi-algebraic setting as in subanalytic settings, using model-theoretic results from the foundational [51], resp. [19].

In this appendix we show that this method also applies to generating power series $\sum_{n \geq 0} a_n T^n$ with coefficients

$$a_n := \#(X_n / \sim_n),$$

where \sim_n is a definable family of equivalence relations with finitely many equivalence classes, depending definably on an integer parameter $n \geq 0$. This result coincides with the result in Section 6 in the case that one uses the semi-algebraic language (based on Macintyre's language, [51]), but the present result differs in two important

ways. Firstly, our method is very robust in the choice of the language to define the equivalence relations. In particular, the subanalytic language of [19] can be used, or any intermediate structure between the semi-algebraic and this subanalytic language which is given by an analytic structure in the sense of [12]. Secondly, our method derives the rationality result, and more generally explains parameter dependence on arbitrary parameters, without using any form of elimination of imaginaries. Proving elimination of imaginaries is often not easy, and seems to be dependent on the language as is shown in [38]; in particular, in the subanalytic language on \mathbb{Q}_p the elimination of imaginaries is not yet completely understood. For simplicity of notation we will focus on those settings where elimination of imaginaries is not yet understood: the subanalytic setting on p -adic numbers and certain substructures coming from an analytic structure as in [12] (which in fact includes the semi-algebraic case). Our results also hold for many possible other languages allowing a typical kind of cell decomposition for the definable sets, but we leave this generality for the reader to discover. Our method can be adapted to obtain uniformity properties in p , both in the semi-algebraic and the subanalytic settings, but we leave this to future work.

Although our arguments go through for any finite field extension of \mathbb{Q}_p , we will work for simplicity with \mathbb{Q}_p itself.

A.1.1

Let us enrich the ring language on \mathbb{Q}_p with an analytic structure as in Section 4 of [12]. As an example of an analytic structure, one may work with the subanalytic language as in [19, 23], where one adds to the ring language a function symbol f for each power series $\sum_{i \in \mathbb{N}^n} a_i x^i$ in n variables over \mathbb{Z}_p for any $n \geq 0$, whose coefficients go to zero as $|i|$ grows, and interpret it as the restricted analytic function

$$\mathbb{Q}_p^n \rightarrow \mathbb{Q}_p : x \mapsto \begin{cases} \sum_{i \in \mathbb{N}^n} a_i x^i & \text{if } x \in \mathbb{Z}_p^n, \\ 0 & \text{otherwise.} \end{cases}$$

Let us further enrich this language by joining a sort for the value group \mathbb{Z} , enriched with $-\infty$ for the valuation of zero, the valuation map $\text{ord} : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{-\infty\}$, a sort for the residue field \mathbb{F}_p , and a map $\bar{ac} : \mathbb{Q}_p \rightarrow \mathbb{F}_p$ which sends 0 to 0 and nonzero x to $xp^{-\text{ord } x} \bmod (p)$. We denote this three-sorted language by \mathcal{L}_{an} , which refers to the analytic nature of the language.

The first theorem that we present in this introduction is a rather concrete form of Theorem A.10 below.

Theorem A.1 *Let \sim_y be an \mathcal{L}_{an} -definable family of equivalence relations on nonempty sets $X_y \subseteq \mathbb{Q}_p^d$ for some $d > 0$, where the family parameters y run over some \mathcal{L}_{an} -definable set Y . Suppose for each $y \in Y$ that each equivalence class of \sim_y has nonempty interior in \mathbb{Q}_p^d . Then there exist $N > 0$ and \mathcal{L}_{an} -definable families of functions $f_y : X_y \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\alpha_y : X_y \rightarrow [1, N] \cap \mathbb{Z}$, such that for each $y \in Y$ and each $a \in X_y$,*

$$\int_{x \sim_y a} \frac{p^{f_y(x)}}{\alpha_y(x)} |dx| = 1,$$

where $|dx|$ stands for the Haar measure on \mathbb{Q}_p^d normalized so that \mathbb{Z}_p^d has measure 1, and where $p^{-\infty}$ stands for 0.

By the theorem and with its notation, if moreover each quotient X_y / \sim_y is finite, say, of size a_y , it immediately follows for $y \in Y$ that

$$\int_{x \in X_y} \frac{p^{f_y(x)}}{\alpha_y(x)} |dx| = a_y, \tag{A.1}$$

which follows the philosophy mentioned above of relating finite counting to taking integrals (this philosophy is also followed in Section 6 in the semi-algebraic context, via elimination of imaginaries). The integral description Eqn. (A.1) and the more flexible variant Theorem A.10 of Theorem A.1 lead in a nowadays standard way to the following rationality result. Note that a multi-variate version of Theorem A.2 can be obtained by similar arguments.

Theorem A.2 *Let \sim_n be an \mathcal{L}_{an} -definable family of equivalence relations on nonempty sets $X_n \subseteq \mathbb{Q}_p^d$ for some $d > 0$, where n runs over non-negative integers. Suppose for each $n \geq 0$ that the quotient X_n / \sim_n is finite, say, of size a_n . Then*

$$\sum_{n \geq 0} a_n T^n$$

is a rational power series over \mathbb{Q} whose denominator is a product of factors of the form $(1 - p^i T^j)$ for some integers i and some $j > 0$.

A.2 Proofs

For integers $m > 0$ and $n > 0$, write $Q_{m,n}$ for the set of all p -adic numbers of the form $p^{na}(1 + p^m x)$ with $x \in \mathbb{Z}_p$ and $a \in \mathbb{Z}$.

The following lemma is a direct corollary of cell decomposition results in [11] and [12, Section 6].

Lemma A.3 For any \mathcal{L}_{an} -definable sets Y and $X \subseteq Y \times \mathbb{Q}_p$, one can write X as a finite disjoint union of \mathcal{L}_{an} -definable sets of the form

$$\{(y, x) \in Y \times \mathbb{Q}_p : \text{ord}(x - c(y)) \in G_y, (x - c(y)) \in \lambda Q_{m,n}\},$$

where $c : Y \rightarrow \mathbb{Q}_p$ is an \mathcal{L}_{an} -definable function, G_y is an \mathcal{L}_{an} -definable family of subsets of $\mathbb{Z} \cup \{-\infty\}$ with parameter $y \in Y$, and λ lies in \mathbb{Q}_p .

Note that any set $Q_{m,n}$ equals a finite disjoint union of sets of the form λP_ℓ for $\lambda \in \mathbb{Q}_p$, where P_ℓ stands for the nonzero ℓ th powers in \mathbb{Q}_p , and also the other way around: any set P_ℓ equals a finite disjoint union of sets of the form $\lambda Q_{m,n}$ for $\lambda \in \mathbb{Q}_p$.

The rest of this note is devoted to the proofs of Theorems A.1, A.2 and A.10. We first give some definitions and lemmas. By a ball we mean a subset $B \subseteq \mathbb{Q}_p$ of the form

$$\{x \in \mathbb{Q}_p : \text{ord}(x - c) > g\}$$

for some $g \in \mathbb{Z}$ and some $c \in \mathbb{Q}_p$.

Let Vol stand for the Haar measure on \mathbb{Q}_p , normalized so that \mathbb{Z}_p has measure 1.

Definition A.4 Let $n \geq 1$, $r_i \geq 0$ for $i = 1, \dots, n$, and let a nonempty set $Y \subseteq \mathbb{Z}_p^n$ be given.

If $n = 1$, then Y is called a *multi-ball of multi-volume* r_1 if $r_1 = \text{Vol}(Y)$ and either Y is a singleton (in which case $r_1 = 0$), or Y is a ball (in which case $r_1 > 0$).

If $n \geq 2$, then the set Y is called a *multi-ball of multi-volume* (r_1, \dots, r_n) if and only Y is of the form

$$\{(x_1, \dots, x_n) : (x_1, \dots, x_{n-1}) \in A, x_n \in B_{x_1, \dots, x_{n-1}}\},$$

where $A \subseteq \mathbb{Z}_p^{n-1}$ is a multi-ball of multi-volume (r_1, \dots, r_{n-1}) , $B_{x_1, \dots, x_{n-1}}$ is a subset of \mathbb{Z}_p which may depend on (x_1, \dots, x_{n-1}) , with $\text{Vol}(B_{x_1, \dots, x_{n-1}}) = r_n$, and such that $B_{x_1, \dots, x_{n-1}}$ is either a singleton or a ball. The multi-volume of a multi-ball Y is denoted by $\text{MultiVol}(Y)$.

An example of a multi-ball in \mathbb{Z}_p^3 of multi-volume $(1, 0, p^{-1})$ is the set

$$\{(x, y, z) : x \in \mathbb{Z}_p, y = x, z \in x + p\mathbb{Z}_p\}.$$

Let us put on \mathbb{R}^n the lexicographical ordering.

Definition A.5 Consider a set $X \subseteq \mathbb{Z}_p^n$. The multi-box of X , denoted by $\text{MB}(X)$, is the union of the multi-balls Y contained in X and with maximal multi-volume $\text{MultiVol}(Y)$ in \mathbb{R}^n (for the lexicographical ordering), where maximality is among all multi-balls contained in X . We write $\text{MultiVol}(X)$ for $\text{MultiVol}(Y)$ for any multi-ball Y contained in X with maximal multi-volume.

For a set $X \subseteq \mathbb{Z}_p^n$, we next define, by induction on n , an \mathbb{N} -valued function MultiNumber_X on X called the multinumber function of X .

Definition A.6 For a set $X \subseteq \mathbb{Z}_p^n$, let MultiNumber_X be the constant function on X taking as value the number of distinct multi-balls Y contained in X with maximal multi-volume if this is finite, and taking the value $+\infty$ otherwise.

For a set $X \subseteq \mathbb{Z}_p^n$ with $n > 1$ and with an ordering on the coordinates, let $p : \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^{n-1}$ be the projection on the first $n-1$ coordinates. We define $\text{MultiNumber}_X : X \rightarrow \mathbb{N}$, for this ordering, as the function sending $x = (p(x), x_n)$ to the product

$$\text{MultiNumber}_{p(X)}(p(x)) \cdot \text{MultiNumber}_{X_{p(x)}}(x_n),$$

where $X_{p(x)} \subseteq K$ is the fiber above $p(x)$ of the projection map $X \rightarrow p(X)$. Here, the product of $+\infty$ with any $a > 0$ is set to be $+\infty$.

The following two simple lemmas are key.

Lemma A.7 *Let X be a subset of \mathbb{Z}_p satisfying $X = \text{MB}(X)$ and let $N \geq 1$ be an integer. Suppose that X can be written as the disjoint union of N sets of the form*

$$A_j = \{x \in \mathbb{Z}_p : \text{ord}(x - c_j) \in G_j, (x - c_j) \in \lambda_j Q_{m_j, n_j}\}, \quad (\text{A.2})$$

for $j = 1, \dots, N$, where c_j and λ_j lie in \mathbb{Q}_p , G_j is a subset of $\mathbb{Z} \cup \{-\infty\}$, and $m_j, n_j \geq 1$. Then one has for $x \in X$ that

$$\text{MultiNumber}_X(x) \leq \frac{p}{p-1} N.$$

Proof If X is a finite set, then the A_j are of size at most 1, and then the bound is clear. Hence, we may and do suppose that X is infinite. Then one of the sets A_j is infinite, and since any infinite set of the form (A.2) contains a ball, it follows that X contains at least one ball of maximal size. Since \mathbb{Z}_p has finite measure and since $X = \text{MB}(X)$, X equals a finite union of balls of the same volume, and hence, $\text{MultiNumber}_X(x)$ is finite and nonzero, and moreover constant since $n = 1$. Write s for $\text{MultiNumber}_X(x)$.

The set X thus equals a disjoint union of balls B_i for $i = 1, \dots, s$ all of equal maximal volume (among the balls contained in X). Each of the sets A_j for $j = 1, \dots, N$ contains at most one ball of maximal volume, among all the balls included in A_j , by the simple form (A.2). Write B_{A_j} for this ball of maximal volume contained in A_j if it exists, and otherwise let B_{A_j} be the empty set. By the simple form (A.2), one has for each j that

$$\text{Vol}(A_j) \leq \frac{p}{p-1} \cdot \text{Vol}(B_{A_j}). \quad (\text{A.3})$$

Since B_1 has maximal volume among all balls contained in X , one has for each j that $\text{Vol}(B_{A_j}) \leq \text{Vol}(B_1)$. We calculate, by finite additivity of Vol ,

$$s \text{Vol}(B_1) = \sum_{i=1}^s \text{Vol}(B_i) = \text{Vol}(X) = \sum_{j=1}^N \text{Vol}(A_j). \quad (\text{A.4})$$

Combining (A.3) and (A.4), the lemma follows. \square

Lemma A.8 *Let \sim_y be an \mathcal{L}_{an} -definable family of equivalence relations on \mathbb{Z}_p^n for the family parameter y running over some \mathcal{L}_{an} -definable set Y . For $x \in \mathbb{Z}_p^n$, write x / \sim_y to denote the equivalence class of x modulo \sim_y . We regard x / \sim_y as a subset of \mathbb{Z}_p^n . Then the following properties hold.*

The union

$$\bigcup_{x \in \mathbb{Z}_p^n} \text{MB}(x / \sim_y)$$

is an \mathcal{L}_{an} -definable family of subsets of \mathbb{Z}_p^n with parameter $y \in Y$. There exists an \mathcal{L}_{an} -definable family of functions $g_y : \mathbb{Z}_p^n \rightarrow (\mathbb{Z} \cup \{-\infty\})^n$ such that $(p^{g_{y,i}(x)})_{i=1}^n$ equals $\text{MultiVol}(x / \sim_y)$ for each x in \mathbb{Z}_p^n . Finally, $x \mapsto \text{MultiNumber}_{x/\sim_y}(x)$ has uniformly bounded range (uniformly bounded in x and y), and depends definably on x and y .

Proof Clearly the condition on $x \in \mathbb{Z}_p^n$ to lie inside $\text{MB}(x / \sim_y)$ is an \mathcal{L}_{an} -definable condition, and also the existence of the \mathcal{L}_{an} -definable family of functions g_y is immediate.

We now show the finiteness of $\text{MultiNumber}_{x/\sim_y}$ and that it is uniformly bounded in x and y . It is enough, by the definitions of $\text{MB}(\cdot)$ with the lexicographic ordering on the multi-volumes and of MultiNumber with the corresponding order on the coordinates, to consider the case that $n = 1$. Let us thus assume that $n = 1$ and consider the \mathcal{L}_{an} -definable family of sets

$$R_y := \{(w, x) \in \mathbb{Z}_p^{1+1} : w \sim_y x\},$$

with family parameter $y \in Y$. By Lemma A.3, applied to the family $R_{y,x} := \{w \in \mathbb{Z}_p^1 : w \sim_y x\}$ with parameter (y, x) , there exists $N \geq 1$ such that any set $R_{y,x}$ equals a finite disjoint union of at most N definable sets of the form in Eqn. (A.2) of Lemma A.7. Applying that lemma, and noting that $R_{y,x}$ is nothing else than x / \sim_y , yields that

$$\text{MultiNumber}_{x/\sim_y}(x) \leq \frac{p}{p-1}N,$$

for all x and y . This proves that $\text{MultiNumber}_{x/\sim_y}$ has a uniformly bounded range, uniformly in x and y . Having such a uniformly bounded range, the definability of $\text{MultiNumber}_{x/\sim_y}$ on x and y becomes an exercise. \square

Let I be a subset of $\{1, 2, \dots, d\}$ for some $d \geq 1$. Let $\mu_{I,d}$ be the measure on \mathbb{Q}_p^d which is the product measure of the following measures on the d Cartesian factors of \mathbb{Q}_p^d : the normalized Haar measure on the i th factor \mathbb{Q}_p of \mathbb{Q}_p^d if $i \in I$, and the counting measure on the i th factor \mathbb{Q}_p of \mathbb{Q}_p^d if $i \notin I$.

The following proposition is a close variant of the well-known rationality result from [19].

Proposition A.9 *Let $f_n : \mathbb{Z}_p^d \rightarrow \mathbb{Z} \cup \{-\infty\}$ be an \mathcal{L}_{an} -definable family of functions, with an integer parameter $n \geq 0$. Suppose that, for each $n \geq 0$, the function $x \mapsto p^{f_n(x)}$ is integrable for the measure $\mu_{I,d}$. Then the generating power series*

$$\sum_{n \geq 0} X_n T^n$$

with

$$X_n = \int_{x \in \mathbb{Z}_p^d} p^{f_n(x)} \mu_{I,d}$$

is a rational power series over \mathbb{Q} , with denominator a product of factors of the form $(1 - p^i T^j)$ for some integers i and some $j > 0$.

Proof By Lemma A.3 and by finite additivity of the integral operator, one reduces to the case that the support of $p^{f_n(x)}$ is contained in the graph of an \mathcal{L}_{an} -definable function

$$\mathbb{Z}_p^a \rightarrow \mathbb{Z}_p^b$$

for some integers $a \geq 0$ and $b \geq 0$. But then one may suppose that $I = \{1, \dots, d\}$. Now the result is a standard variant from the rationality result for p -adic integrals from [19]. \square

Proposition A.9 has several generalizations. For example, parameter integrals of a more general type and with more general parameters of any of the sorts $\mathbb{Z}, \mathbb{Q}_p, \mathbb{F}_p$, as well as uniformity in p , are well understood, see, e.g., [55, 13]. We will not need more general results of this type here, and can come directly to the main result.

Theorem A.10 *Let \sim_y be an \mathcal{L}_{an} -definable family of equivalence relations on nonempty sets $X_y \subseteq \mathbb{Q}_p^d$ for some $d > 0$, where the family parameters y run over some \mathcal{L}_{an} -definable set Y . Then there exist $N > 0$ and \mathcal{L}_{an} -definable families of functions $f_{I,y} : X_y \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $\alpha_y : X_y \rightarrow [1, N] \cap \mathbb{Z}$, such that for each $y \in Y$ and each $a \in X_y$,*

$$\sum_I \int_{x \sim_y a} \frac{p^{f_{I,y}(x)}}{\alpha_y(x)} \mu_{I,d}(x) = 1, \quad (\text{A.5})$$

where the sum runs over the subsets I of $\{1, \dots, d\}$.

Proof Clearly we may suppose that the sets X_y are subsets of \mathbb{Z}_p^d , up to increasing d and mapping any unbounded coordinate $w \in \mathbb{Q}_p$ to $(w, 0) \in \mathbb{Z}_p^2$ if $|w| \leq 1$ and to $(0, w^{-1}) \in \mathbb{Z}_p^2$ if $|w| > 1$. Apply Lemma A.8 to the family \sim_y to find an \mathcal{L}_{an} -definable family of functions $g_y = (g_{y,i})_{i=1}^d$. Now one can take for $f_{I,y}$ the function that maps x to

$$\sum_{i, g_{y,i}(x) \neq -\infty} g_{y,i}(x)$$

if x lies in $\text{MB}(x/\sim_y)$ and $\text{MB}(x/\sim_y)$ has nonzero and finite $\mu_{I,d}$ -measure, and to $-\infty$ in all other cases. For $\alpha_y(x)$ one takes $\text{MultiNumber}_{x/\sim_y}(x)$. The α_y are an \mathcal{L}_{an} -definable family of functions with finite range by Lemma A.8. Now clearly Eqn. (A.5) holds for all $y \in Y$ and $a \in X_y$, as desired. \square

We can now prove the rationality result of Theorem A.2.

Proof [Theorem A.2] Consider $N, f_{I,n}$ and α_n as given by Theorem A.10, with Y the set of nonnegative integers n . For each integer i with $1 \leq i \leq N$, let $X_{n,i}$ be the subset of X_n on which α_n takes the value i . Let $a_{n,i}$ be number of equivalence classes of the restriction of \sim_n to $X_{n,i}$ if $X_{n,i}$ is nonempty, and let $a_{n,i}$ be zero otherwise. By Proposition A.9 and by the integral description of Theorem A.1, each of the generating series

$$\sum_{n \geq 0} a_{n,i} T^n$$

is rational and of the desired form. Since

$$\sum_{n \geq 0} a_n T^n = \sum_{i=1}^N \sum_{n \geq 0} a_{n,i} T^n,$$

we are done. □

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