Imaginaries in valued fields with operators

Silvain Rideau

UC Berkeley

Mai 25 2016

Valued fields

Example

- Let k be a field. On $k(X)$, $\mathrm{v}_X(X^n P/Q) = n \in \mathbb{Z}$ when $X \wedge P = X \wedge Q = 1$. Its competion is $k((X)) = \{\sum_{i>i_0} c_i X^i : c_i \in k\}$.
- On $\mathbb{Q}$, $\mathrm{v}_p(p^n a/b) = n \in \mathbb{Z}$ when $p \wedge a = p \wedge b = 1$. Its compeltion is $\mathbb{Q}_p$ the field of p-adic numbers.
- Let k be a field and Γ be an ordered Abelian group:

  $$k((t^\Gamma)) = \{\sum_{\gamma\in\Gamma} c_\gamma t^\gamma : \{\gamma : c_\gamma \neq 0\} \text{ well-ordered}\}$$

  and $\mathrm{v}(\sum_\gamma c_\gamma t^\gamma) = \min\{\gamma : c_\gamma \neq 0\}$.
- Let k be a perfect characteristic $p > 0$ field.

  $$\mathrm{W}(k) = \{\sum_{i>i_0} c_i^{p^{-i}} p^i : c_i \in k\} \text{ and } \mathrm{v}(\sum_{i>i_0} c_i^{p^{-i}} p^i) = \min\{i : c_i \neq 0\}.$$

Operators

- Contractive derivations: an additive morphism $\partial : K \to K$ that verifies:
  - the Leibniz rule
    $$\partial(xy) = \partial(x)y + x\partial(y)$$
  - $\mathrm{v}(\partial(x)) \geq \mathrm{v}(x)$.
- (Iterative) Hasse derivations: a collection $(\partial_n)_{n\geq 0}$ of additive morphisms $K \to K$ that verify
  - $D_0(x) = x$;
  - The generalised Leibniz rule:
    $$\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y);$$
  - $D_n(D_m(x)) = \binom{m+n}{n}\partial_{m+n}(x)$.
- Automorphisms (of the valued field).

Examples

- Scanlon, 2000: Model completion of valued fields with a contractive derivation. Let k, ∂ be differentially closed:

$$k((t^{\mathbb{Q}})) \text{ and } \partial(\sum_\gamma c_\gamma t^\gamma) = \sum_\gamma \partial(c_\gamma) t^\gamma.$$

- Hils-Kamensky-R., 2015: Strict separably closed valued fields of finite imperfection degree e with e commuting Hasse derivations. Let K be a separably closed such that $K = K^p(b_1 \dots b_e)$. Take $(\partial_{i,j})_{1 \le i \le e, j \ge 0}$ such that $\partial_{i,1}(b_i) = 1$, $\partial_{i,0}(b_l) = b_l$ and $\partial_{i,j}(b_l) = 0$ otherwise.
- Bélair-Macinyre-Scanlon, 2007: $(\mathrm{W}(k), \mathrm{W}(\sigma))$ where k is a difference field.
  - $k \models \mathrm{ACF}_p$ with the Frobenius automorphism.
  - $k \models \mathrm{ACFA}_p$.
- Durhan-Onay, 2015: $k((t^\Gamma))$ where $k \models \mathrm{ACFA}_0$, Γ an ordered Abelian group with an automorphism and $\sigma(\sum_\gamma c_\gamma t^\gamma) = \sum_\gamma \sigma(c_\gamma) t^{\sigma(\gamma)}$.
  - Γ is divisible with an ω-increasing automorphism.
  - Γ a $\mathbb{Z}$-group with the identity.

Imaginaries

An imaginary is an equivalent class of an $\varnothing$-definable equivalence relation.

Example

- Let $(X_y)_{y \in Y}$ be an $\varnothing$-definable family of sets.
  - Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$.
  - The set $Y/\equiv$ is a moduli space for the family $(X_y)_{y \in Y}$.

Definition

A theory T eliminates imaginaries if for all $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f defined on D such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$

Theorem (Poizat, 1983)

Algebraically closed fields and characteristic zero differentially closed fields eliminate imaginaries in the (differential) ring language.

Imaginaries in valued fields

Let (K, v) be a valued field, we define:

- $\mathbf{S}_n := \mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathcal{O})$.
  - It is the moduli space of rank n free $\mathcal{O}$-submodules of K^n.
- $\mathbf{T}_n := \mathrm{GL}_n(K)/\,\mathrm{GL}_{n,n}(\mathcal{O})$
  - $\mathrm{GL}_{n,n}(\mathcal{O})$ consists of the matrices $M \in \mathrm{GL}_n(\mathcal{O})$ whose reduct modulo $\mathfrak{M}$ has only zeroes on the last column but for a 1 in the last entry.
  - It is the moduli space of $\bigcup_{s\in\mathbf{S}_n} s/\,\mathfrak{M}s = \{a + \mathfrak{M}s : s \in \mathbf{S}_n \text{ and } a \in s\}$.

Let $\mathcal{L}_{\mathcal{G}} := \{\mathbf{K}, (\mathbf{S}_n)_{n\in\mathbb{N}_{>0}}, (\mathbf{T}_n)_{n\in\mathbb{N}_{>0}}; \mathcal{L}_{\mathrm{div}}, \sigma_n : \mathbf{K}^{n^2} \to \mathbf{S}_n, \tau_n : \mathbf{K}^{n^2} \to \mathbf{T}_n\}$.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

The $\mathcal{L}_{\mathcal{G}}$-theory of algebraically closed valued fields eliminates imaginaries.

Imaginaries and definable/invariant types

Proposition (Hrushovski, 2014)

Let T be a theory such that, for all $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$:

1. Any $\mathcal{L}^{\mathrm{eq}}(A)$-definable set is consistent with an $\mathcal{L}^{\mathrm{eq}}(A)$-definable type.
2. Any $\mathcal{L}^{\mathrm{eq}}(A)$-definable type p is $\mathcal{L}(A \cap M)$-definable.
3. Finite sets have canonical parameters.

Then T eliminates imaginaries.

Remark

It suffices to prove hypothesis 1 in dimension 1.

Imaginaries and definable/invariant types

Proposition

Let T be a theory such that, for all $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$:

1. Any $\mathcal{L}^{\mathrm{eq}}(A)$-definable set is consistent with an $\mathrm{Aut}(M^{\mathrm{eq}}/A)$-invariant type.
2. Any $\mathrm{Aut}(M^{\mathrm{eq}}/A)$-invariant type p is $\mathrm{Aut}(M^{\mathrm{eq}}/A \cap M)$-invariant.
3. Finite sets have canonical parameters.

Then T eliminates imaginaries.

Remark

If T is NIP, it suffices to prove hypothesis 1 in dimension 1.

Results

Theorem (R., 2014)

The model completion of valued fields with a contractive derivation eliminates imaginaries in the geometric language (with a new symbol for the derivation).

Theorem (Hils-Kamensky-R., 2015)

Strict separably closed valued field of imperfection degree e with e commuting Hasse derivations eliminate imaginaries in the geometric language (with new symbols for the Hasse derivations).

Conjecture

All the other examples eliminate imaginaries in the geometric language (with a new symbol for the automorphism).