A model theoretic theoretic account of the tilting equivalence

ongoing work with Tom Scanlon and Pierre Simon

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Tilting

Let K be an ultrametric normed field, with norm $|.|$ and unit ball $\mathcal{O}$.

- Assume $|p| < 1$.
- Let $\phi(x) = x^p$ be the Frobenius morphism, in characteristic p.

Definition

- Let $\mathcal{O}^\flat := \varprojlim_\phi \mathcal{O}/p\mathcal{O} := \{(x_i)_{i\in\omega} \in \mathcal{O}/p\mathcal{O} : x_{i+1}^p = x_i\}$;
- for every $x \in \mathcal{O}^\flat \setminus \{0\}$, let $|x| = p^i|\widetilde{x}_i|$ where $\widetilde{x}_i \in \mathcal{O} \setminus p\mathcal{O}$ is a lift of x_i — and let $|0| = 0$.

Lemma

$\mathcal{O}^\flat$ is a perfect (integral) complete ultrametric normed ring of characteristic p.

- We denote by $K^\flat$ its fraction field. Its unit ball is $\mathcal{O}^\flat$.

Perfectoid fields

Definition

We say that K is perfectoid (of residue characteristic p) if:

- it is complete;
- there exists $\varpi \in K$ with $|p| < |\varpi| < 1$;
- $\phi : \mathcal{O}/p\mathcal{O} \to \mathcal{O}/p\mathcal{O}$ sending x to x^p is surjective.

Examples

- Any characteristic p complete non trivially normed field;
- $\mathbb{Q}_p(p^{-\infty}) := \widehat{\bigcup_{n>0} \mathbb{Q}_p(p^{-n})}$ with the p-adic norm;
- $\mathbb{Q}(\mu_{p^\infty}) := \widehat{\bigcup_{n>0} \mathbb{Q}_p(\xi_{p^n})}$, where ξ_{p^n} is a primitive p^n-th root of 1, with the p-adic norm;
- $\mathbb{C}_p := \widehat{\mathbb{Q}_p^{\mathrm{alg}}}$ with the p-adic norm;
- Any mixed characteristic algebraically closed normed field.

Untilting I

Assume K is perfectoid.

Lemma

We have:

$$\mathcal{O}^\flat \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}$$

as multiplicative groups.

Definition

Let $\sharp$ be the (multiplicative) homomorphism:

$$\mathcal{O}^\flat \to \varprojlim_{x \mapsto x^p} \mathcal{O} \to \mathcal{O}.$$

For every $x \in \mathcal{O}^\flat$, we have $x^\sharp = \lim_i \widetilde{x}_i^{p^i}$ where $\widetilde{x}_i \in \mathcal{O}$ is a lift of x_i.

Untilting II, Witt vectors

Let R be a perfect characteristic p ring.

- There are polynomials $P_n, Q_n \in \mathbb{Z}[X]$, independent of R, such that $W(R) = (R^\omega, P, Q)$ is a characteristic zero ring.
- $W(R) \to W(R)/(p) \simeq R$ admits a section $[.] : R \to W(R)$.
- Element of $W(R)$ can be uniquely written as $\sum_{i \geq 0} [a_i] p^i$.
- $W(\mathbb{F}_p) \simeq \mathbb{Q}_p$ and $W(\mathbb{F}_p^{\mathrm{alg}}) \simeq \widehat{\mathbb{Q}_p(\mu_{\overline{p}})} = \mathbb{Q}_p(\xi_n : n \wedge p = 1)$.

Proposition

Assume K has characteristic zero. The map $\theta : W(\mathcal{O}^\flat) \to \mathcal{O}$

$$\sum_{i \geq 0} [a_i] p^i \mapsto \sum_{i \geq 0} a_i^\sharp p^i$$

is a surjective ring homomorphism. Its kernel is generated by any element $[\varpi] + pb$ where $|\varpi| = |p|$ and $\theta(b) = \varpi^\sharp p^{-1}$.

(Bounded) Continuous logic

- Structures are complete metric spaces of radius 1.
- Formulas are interpreted as uniformly continuous functions with values in $[0, 1]$. They are closed under composition with continuous functions $[0, 1]^n \to [0, 1]$, inf and uniform limits.
- Definable sets are closed sets such that the distance to these sets is given by a formula (with parameters).
- The setting naturally allows for definable sets in ω coordinates.

Definition

Let M be an $\mathfrak{L}$-structure and N be an $\mathfrak{F}$-structure.

- An interpretation of N in M is a definable set X in some power of M and a map $X \to N$ such that the pre-image of any $\mathfrak{F}$-formula is given by an $\mathfrak{L}$-formula.
- A bi-interpretation is a pair of interpretations $f : X \to N$ and $g : Y \to M$ whose compositions are definable maps.

Metric valued fields

- To any complete ultrametric normed field K, we associate the $\mathfrak{L}_{\mathrm{rg}}$ structure $(\mathcal{O}, |.|, 0, 1, +, -, \cdot)$.
- These do not form an elementary class: this structure has an elementary extension where $|x| = 1 \not\Leftrightarrow x$ invertible.
- Let (K, v) be a field with a microbial (Krull) valuation : there is a non trivial ordered group morphism $\mathrm{v}(K^{\times}) \to \mathbb{R}_{>0}$.
- Let $|.|_0 : K \to \mathrm{v}(K) \to \mathbb{R}_{\geq 0}$ be the associated ultrametric norm.
- Now consider the $\mathfrak{L}_{\mathrm{rg}}$-structure $(\mathcal{O}, |.|_0, 0, 1, +, -, \cdot)$, where $\mathcal{O} := \{x \in K : v(x) \leq 1\}$.
- These form an elementary class MVF. So does the class PERF of valuation rings of perfectoids fields.
- This continuous structure is continuously interpretable in the (classical) first order valued field structure of K.
- In fact, for every $a \in \mathcal{O} \setminus \{0\}$, the continuous structure induced on $\mathcal{O}/(a)$ is a (classical) first order structure.

A bi-interpretation

Proposition

Let $\mathcal{O} \models \text{PERF}$. The rings $\mathcal{O}$ and $\mathcal{O}^\flat$ are (uniformly) bi-interpretable.

- Let Ω be the definable set $\{x \in \mathcal{O}^\omega : x_{i+1}^p = x_i\}$.
- Let $f : \Omega \to \mathcal{O}^\flat$ be the identity.
- Let $g : (\mathcal{O}^\flat)^\omega = W(\mathcal{O}^\flat) \to \mathcal{O}$ be the quotient by the definable ideal $([\varpi] - pb)$.
- $f \circ g : W(\mathcal{O}^\flat) \to \mathcal{O}$ is the θ map.
- $g \circ f : (W(\mathcal{O}^\flat)/([\varpi] - pb))^\flat \to \mathcal{O}^\flat$ is given by:

$$\begin{aligned}(W(\mathcal{O}^\flat)/([\varpi] - pb))^\flat &\to (W(\mathcal{O}^\flat)/([\varpi] - pb, p))^\flat \\ &\to (\mathcal{O}^\flat/(\varpi))^\flat \to (\mathcal{O}^\flat)^\flat \to \mathcal{O}^\flat.\end{aligned}$$

The tilting equivalence

Let $k \leq \mathcal{O} \models \text{PERF}$ and $X \subseteq \Omega^n$ be ∞-k-definable in $\mathcal{O}$.

- $X \subseteq \Omega^n = (\mathcal{O}^\flat)^n$ is ∞-$k^\flat$-definable in $\mathcal{O}^\flat$. We denote it by $X^\flat$.

Corollary

We have:

$$\mathcal{S}_X(k) \simeq \mathcal{S}_{X^\flat}(k^\flat)$$

and the homeomorphism is functorial. In particular, we have an equivalence of categories:

$$\left\{ \begin{array}{c} \infty\text{-}k\text{-definable sets} \\ \text{in powers of } \Omega \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \infty\text{-}k^\flat\text{-definable sets} \\ \text{in powers of } \mathcal{O}^\flat \end{array} \right\}$$

This equivalence restricts to an equivalence:

$$\left\{ \begin{array}{c} k\text{-definable sets} \\ \text{in powers of } \Omega \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} k^\flat\text{-definable sets} \\ \text{in powers of } \mathcal{O}^\flat \end{array} \right\}$$

The adic spectrum

Let A be a topological k-algebra, e,g. $k[X]$ with the Gauss norm.

- A semi-valuation $v : A \to \Gamma$ is a map verifying:
  - $v(xy) = v(x) \cdot v(y)$;
  - $v(x+y) \leq \max\{v(x), v(y)\}$;
  - $v|_k = \mathrm{v}$;
- It is continuous if, for every $\gamma \in \Gamma$, $\{x : v(x) < \gamma\}$ is open.
- We define the adic spectrum

$$\mathrm{Spa}(A) := \{v : A \to \Gamma \text{ continuous semi-valuation}\}/\sim .$$

- It is endowed the topology generated by the open sets

$$\{v \in \mathrm{Spa}(A) : v(f) \leq v(g) \neq 0\},$$

where $f, g \in A$.

Adic spaces as type spaces

Proposition

The class ACMVF of (valuations rings of) algebraically closed microbial valued fields is the model companion of MVF.

- The proof is essentially the same as Robinson's in the (classical) first order case.

We fix $(P_i)_{i \leq m} \in k[x]$, where $|x| = n$.

- Let $I := (P_i : i \leq m)$.
- Let $X \subseteq \mathcal{O}^n$ be the zero set of $\max_i \{P_i(x) : i \leq m\}$.

Lemma

We have a continuous bijection:

$$\mathcal{S}_X^{\mathrm{ACMVF}}(k) \to \mathrm{Spa}(k[x]/I)$$

- The topology on $\mathcal{S}_X^{\mathrm{ACMVF}}(k)$ is the constructible adic topology.

Perfectoid spaces

A perfectoid space over k is a space which is covered by adic spectra of (affinoid) perfectoid k-algebras.

Theorem

There is an equivalence of categories:

$$\{\text{Perfectoid spaces over } k\} \leftrightarrow \{\text{Perfectoid spaces over } k^\flat\}.$$

- Let $\overline{X} = \{x \in \Omega : x_0 \in X\}$.

We recover part of this equivalence:

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{ACMVF}}_{\overline{X}}(k) & \longleftrightarrow & \mathcal{S}^{\mathrm{ACMVF}}_{\overline{X}^\flat}(k^\flat) \\ \downarrow & & \downarrow \\ \mathrm{Spa}(k[x^{p^{-\infty}}]/I) & \longleftrightarrow & \mathrm{Spa}((k[x^{p^{-\infty}}]/I)^\flat) \end{array}$$