Cellularity and beyond

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Overview

PICTURES

CELLULARITY

Definition (Schmerl, 1990 [13])

M is *cellular* if whenever we choose some subset the components of one type, and fix everything else pointwise, $Aut(M)$ induces still the full symmetric group on the chosen components.

Definition

T is stable if it does not encode an infinite linear order.

Theorem

If M is cellular, then it is ω-categorical and ω-stable.

- Key intuition: M is cellular if and only if it encodes neither a linear order nor an infinite equivalence relation.

A collection of theorems

- (Macpherson-Pouzet-Woodrow, 1992 [12]) Given an age $\mathcal{A}$, let $Mod(\mathcal{A})$ be the countable structures of age $\mathcal{A}$. Then $|Mod(\mathcal{A})| \in \{1, \aleph_0, 2^{\aleph_0}\}$, and is $\leq \aleph_0 \iff M$ is cellular.
- (Laskowski-Mayer, 1996 [9]) Let M be (atomically) stable and countable. If $Sub(M)$ is the set of substructures, up to isomorphism, then $|Sub(M)| < 2^{\aleph_0} \iff |Sub(M)| \leq \aleph_0 \iff M$ is cellular.
- (Falque-Thiéry, 2020 [6]) If M is homogeneous and the unlabeled growth rate of M is at most a polynomial, then M is (essentially) cellular.
- Cellularity similarly corresponds to an initial interval for the labeled growth rate (Bodirsky-Bodor, 2018 [2]), even for arbitrary hereditary classes (Laskowski-Terry, 2018 [10]).
- (B.-Laskowski, 2019 [4]) Counting structures bi-embeddable with a given countable structure. (To be elaborated on.)

UNARY EXPANSIONS

- Given a property P, a structure/theory is *monadically P* if any expansion by (finitely many) unary relations still has P.
- Cellular structure are monadically cellular.

Theorem (B.-Laskowski [5])

M is monadically ω-categorical $\iff$ M is cellular.

MA-PRESENTATIONS

Definition

Given a set A, a relation $R \subset A^k$ is *mutually algebraic* if there is some N such that for any proper 2-partition of k, we have $\forall \overline{y} \exists^{\leq N} \overline{x}$ such that $R(\overline{x}, \overline{y})$.

Example

The edge relation in a bounded-degree graph is mutually algebraic. So is any unary relation.

Definition

M is *MA-presented* if every atomic relation is mutually algebraic.

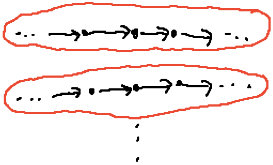
Decomposing MA-presented structures

Theorem (B.-Laskowski [5])

An MA-presented structure admits a decomposition like cellular structures, but without the finiteness conditions.

Example

Consider a model of $(\mathbb{Z}, succ)$.



- Components are connected components, which agree with algebraic closure.

Mutual algebraicity

Definition

A theory is *mutually algebraic* if, after expanding by constants, every model is q.f.-interdefinable with an MA-presented structure.

Example

Consider the theory of an equivalence relation with n infinite classes. After naming a point in each class, this is quantifier-free interdefinable with n unary relations.

Theorem (B.-Laskowski [5])

Given a mutually algebraic M, the cellular-like decomposition of any MA-presentation of M induces a corresponding decomposition of M. The decomposition of M is largely independent of the choice of MA-presentation.

Mutual algebraicity and cellularity

Theorem (B.-Laskowski [5])

M is cellular $\iff$ M is mutually algebraic and ω-categorical.

- Recall the components correspond to the algebraic closures of their elements, and ω-categoricity forces these to be finite.

Theorem (B.-Laskowski [5])

If M is mutually algebraic but not cellular, then some elementary extension contains infinitely many new pairwise-isomorphic infinite components.

- So if M is mutually algebraic but not cellular, an elementary extension encodes an infinite equivalence relation.

Supporting arrays

Definition

Given a structure M, a quantifier-free type p over M *supports an infinite array* if there is some $N \succ M$ with infinitely many disjoint realizations of p.

Lemma

$p(\bar{x})$ supports an infinite array $\iff$ $p \vdash x_i \neq m$ for every $x_i \in \bar{x}, m \in M$.

Theorem (Laskowski-Terry [11])

M is not mutually algebraic $\iff$ there is some $N \succ M$ and some $k \in \omega$ such that infinitely many k-types over N support infinite arrays.

- Arrays over $(\mathbb{Q}, <)$ and an infinite equivalence relation.

Unary expansions

Theorem (Laskowski [8])

- *Mutually algebraicity is preserved under expansions by unary (in fact mutually algebraic) relations.*
- *T is mutually algebraic $\iff$ T is monadically NFCP.*

Siblings

Definition

Two structures are *siblings* if they are bi-embeddable.
Given a structure M, $Sib(M)$ counts the number of siblings, up to isomorphism (including M itself).

Conjecture (Thomassé)

Given a countable relational structure M, $Sib(M) \in \{1, \aleph_0, 2^{\aleph_0}\}$.

- Note $(\mathbb{N}, +, \times, 0, 1)$ has only one sibling, so it doesn't seem like $Sib(M)$ measures model-theoretic complexity

Siblings and cellularity

Theorem (B.-Laskowski [4])

Given a countable structure M in a finite relational language, either

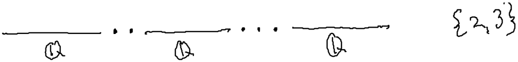
1. *M is cellular and has either 1 or $\aleph_0$ siblings.*
2. *M is not cellular, and there is some age-preserving $N \supset M$ such that N has $2^{\aleph_0}$ siblings.*

Corollary

- *Thomassé's conjecture is true for ω-categorical or countable universal structures (in a finite relational language).*
- *Thomassé's conjecture is true when coarsened to ages (in a finite relational language).*

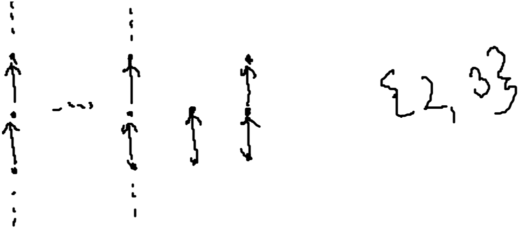
THE PARADIGMATIC CASES

1. $M = (\mathbb{Q}, <)$



2. M is an infinite equivalence relation

3. $M = (\mathbb{Z}, succ)$



More on the proof

- The proof follows the general strategy proposed.

1. The unstable case is handled similarly to $(\mathbb{Q}, <)$
2. The stable non-mutually algebraic case is handled similarly to the infinite equivalence relation, using the infinite arrays to mimic equivalence classes.
3. The mutually algebraic non-cellular case is handled similarly to $(\mathbb{Z}, succ)$ by adding infinitely many new infinite components.

- A significant technical hurdle is that these arguments take place on tuples, but "being in the same tuple" might not be definable.
- A lot of work is spent showing that we can treat tuples like singletons.

Monadic stability

Example

The theory of an infinite equivalence relation is monadically stable, but not mutually algebraic.

Theorem (Baldwin-Shelah [1])

The following are equivalent.

1. *T is monadically stable.*
2. *T is stable and monadically NIP.*
3. *Models of T admit a nice decomposition into trees of countable models.*
4. *There is no unary expansion with a definable infinite linear order on singletons.*
5. *T is stable and if $B \downarrow_D C$, then for any a, $aB \downarrow_D C$ or $B \downarrow_D aC$ (equiv., dependence is trivial and is transitive on singletons).*

Monadic stability and mutual algebraicity

- Since mutual algebraicity is the same as monadic NFCP, monadic stability is a generalization.

Theorem (B.-Laskowski)

T is mutually algebraic $\iff$ its models admit a nice tree decomposition of depth 1.

Theorem (B.-Laskowski)

If T is monadically stable but not mutually algebraic, then

1. *Some model admits a unary expansion with a definable infinite equivalence relation on singletons.*
2. *Some model admits a mutually algebraic expansion that codes graphs.*

Uses?

- It seems like monadic stability could be another stepping stone in proofs, similar to mutual algebraicity.
- The results about encoding configurations *on singletons* in unary expansions is very appealing, if a problem can be shown to be "blind" to unary expansions.

Conjecture (Pouzet-Sauer-Thomassé)

Given an age $\mathcal{A}$, let $|Mod(\mathcal{A})/{\equiv}|$ count the bi-embeddability classes of countable structures of age $\mathcal{A}$. Then $|Mod(\mathcal{A})/{\equiv}| \in \{1, \aleph_0, \aleph_1, 2^{\aleph_0}\}$. Furthermore, it is 1 iff $\mathcal{A}$ is cellular.

- An example for $\aleph_0$ is an infinite equivalence relation; for $\aleph_1$ is $(\mathbb{Q}, <)$
- A guess: if $\mathcal{A}$ is not monadically NIP, then there are $2^{\aleph_0}$ classes; if $\mathcal{A}$ is not monadically stable, there are $\geq \aleph_1$ classes.
- Want to show unary expansions of $\mathcal{A}$ don't affect the outcome.

The ω-categorical case

Definition

M is hereditarily cellular of depth $\leq n$ if it admits a decomposition like cellular structures, except the non-exceptional components are allowed to be hereditarily cellular of depth $\leq n-1$.

Example

Infinite equivalence relations are hereditarily cellular of depth 2.

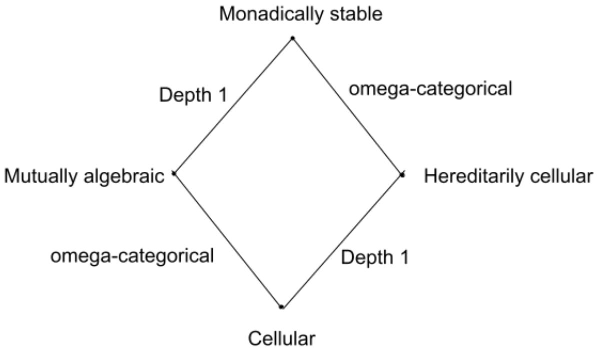
Theorem (Lachlan [7])

M is monadically stable and ω-categorical $\iff$ M is hereditarily cellular of depth n for some $n \in \omega$.

Theorem (B. [3])

A homogeneous M has subexponential unlabeled growth rate iff M is (essentially) hereditarily cellular.

THE MONADICALLY STABLE LANDSCAPE



Monadic NIP

Definition

T is NIP if it does not encode all graphs.

- These structures should be "tree-like" (or "order-like").
- In the following characterization, finite satisfiability mimics the behavior of forking in monadically stable theories.

Theorem (Shelah [14]; B.-Laskowski-Simon)

Write $B \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.5ex\hbox{$|$}}^{fs}_M C$ for $tp(B/MC)$ is finitely satisfiable in M. TFAE:

1. *T is monadically NIP.*
2. *If $B \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.5ex\hbox{$|$}}^{fs}_M C$, then for any a either $aB \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.5ex\hbox{$|$}}^{fs}_M C$ or $B \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.5ex\hbox{$|$}}^{fs}_M aC$.*
3. *T is dp-minimal and indiscernible-trivial.*

- This gives a linear decomposition, akin to one step of the tree-decomposition for monadic stability.

THE ω-CATEGORICAL CASE

Conjecture

Let M be homogeneous. The following are equivalent.

1. *M is monadically NIP.*
2. *The unlabeled growth rate of M is at most c^n for some c.*
3. *$Age(M)$ has no infinite antichains (under embeddability).*

- From a witness to failure of the previous theorem, we can code graphs in an (explicit) unary expansion of M.

Theorem (B.-Laskowski-Simon)

If M (not necessarily ω-categorical) has quantifier elimination and is not monadically NIP then

1. *The unlabeled growth rate of M is faster than any exponential.*
2. *For some expansion M^* of M by finitely many unary predicates, $Age(M^*)$ has an infinite antichain (under embeddability).*

Questions

Conjecture

Given an non-cellular age $\mathcal{A}$, $|Mod(\mathcal{A})/\equiv|$ is infinite.

Question

Is there a good notion of "codes an infinite equivalence relation" such that a universal theory T in a finite relational language is cellular $\iff$ T is stable and no model codes an infinite equivalence relation?

Question

What is a plausible refinement of Thomassé's conjecture describing how the cases split? If M is in the $2^{\aleph_0}$-case, must M "encode an infinite equivalence relation" or encode a linear order with $2^{\aleph_0}$ siblings?

Question

When/why are the monadic versions of properties relevant?

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