Examples of manifolds

S. Allais, M. Joseph

Exercise 1. Show that a submanifold $M \subset \mathbb{R}^n$ of dimension m is a manifold of dimension m.

Exercise 2 (The sphere)**.**

1. Define a differential structure on $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1\}$ by using stereographic projections.
2. The Euclidean sphere $\mathbb{S}^n$ is also a manifold as a submanifold of $\mathbb{R}^{n+1}$. Show that the differentiable structure built in question 1 is the same as the one induced by $\mathbb{R}^{n+1}$.
3. *(bonus)* What happens if the Euclidean norm $\|\cdot\|_2$ is replaced by another norm $\|\cdot\|$?

Exercise 3 (Product manifolds)**.** Let M and N be smooth manifolds, show that $M \times N$ is a smooth manifold. What is its dimension?

Exercise 4 (The torus)**.**

1. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Show that $\mathbb{T}^n$ is compact and Hausdorff, and that the canonical projection $p : \mathbb{R}^n \to \mathbb{T}^n$ is an open map.
2. Let $\mathbb{S}^1$ be the set $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that $\mathbb{T}^1$ is homeomorphic to $\mathbb{S}^1$. More generally, show that $\mathbb{T}^n$ is homeomorphic to $(\mathbb{S}^1)^n \subset \mathbb{C}^n$.
   hint : prove that a continuous bijection $K \to Y$ with K compact Hausdorff and Y Hausdorff is a homeomorphism.
3. Build a differentiable structure on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that p is a local diffeomorphism.
4. Prove that the homeomorphism $\mathbb{T}^n \to (\mathbb{S}^1)^n$ built in question 2 is in fact a diffeomorphism.

Exercise 5 (The projective space)**.** Let $\mathbb{RP}^n$ be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation "belonging to the same vector line". For $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we denote by $[x_0 : \cdots : x_n]$ its class in $\mathbb{RP}^n$.

1. Show that $\mathbb{RP}^n$ is compact Hausdorff and that the canonical projection $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open.
2. Let $i \in \{0, \dots, n\}$. Show that $U_i = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$ is open in $\mathbb{RP}^n$, and construct an homeomorphism $\varphi_i : U_i \to \mathbb{R}^n$.
3. Show that $\mathbb{RP}^n$ equipped with the atlas $(\varphi_i)_{i \in \{0,\dots,n\}}$ is a smooth manifold.
4. Show that p is smooth.
5. Show that the restriction of p to $\mathbb{S}^n$ is a local diffeomorphism.
6. Show that $\mathbb{RP}^1$ is diffeomorphic to $\mathbb{S}^1$.

Exercise 6 (Klein Bottle)**.** Let G be the group $\mathbb{Z}^2$ where the group law is defined by $(n, m) \star (p, q) = ((-1)^m p + n, m + q)$. The group $(\mathbb{Z}^2, \star)$ acts on $\mathbb{R}^2$ by the following formula: $(n, m) \cdot (x, y) = ((-1)^m x + n, y + m)$

1. Prove that this action is properly discontinuous.

2. Let K be the quotient $[0,1] \times [0,1]/\sim$, where $\sim$ is the equivalence relation defined by $(x,0) \sim (1-x,1)$ and $(0,y) \sim (1,y)$ for all $x, y \in [0,1]$. Show that K is homeomorphic to the quotient $\mathbb{R}^2/G$. Show that $\mathbb{R}^2/G$ is compact Hausdorff.

3. Build a differential structure on $\mathbb{R}^2/G$ such that the canonical projection $p : \mathbb{R}^2 \to \mathbb{R}^2/G$ is a local diffeomorphism.

Correction: Hausdorff property and groups acting properly discontinuously

S. Allais, M. Joseph

Exercise. Show that if G is a discrete group acting properly discontinuously on a locally compact Hausdorff space X, *i.e.* for all $g \in G, x \mapsto g \cdot x$ is a continuous map, and

$$\{g \in G \mid g \cdot K \cap K \neq \varnothing\} \text{ is finite}, \quad \forall K \subset X \text{ compact subset},$$

then the quotient X/G is Hausdorff.

Proof. Let x and y be two distinct points in X/G. Let z be an element of $p^{-1}(\{x\})$ and t be an element of $p^{-1}(\{y\})$. By local compactness of X, there exists two disjoint open sets of X U and V of compact closure with $z \in U$ and $t \in V$. Since G acts properly discontinuously on X, there exists a finite subset F of G such that for all $g \in G \setminus F$,

$$(\overline{U} \cup \overline{V}) \cap g \cdot (\overline{U} \cup \overline{V}) = \varnothing.$$

This implies that for all g in $G \setminus F$, the set $\overline{U} \cap g \cdot \overline{V}$ is empty.

Lemma. *For all $g \in F$, there exists two open sets $U_g \subset U$ and $V_g \subset V$ with $z \in U_g$ and $t \in V_g$ such that $U_g \cap g \cdot V_g$ is empty.*

Proof. Since X is Hausdorff, there exist two disjoint open sets U'_g and V'_g with $z \in U'_g$ and $g \cdot t \in V'_g$. Set $U_g = U'_g \cap U$ and $V_g = g^{-1}(V'_g) \cap V$. □

Finally, define

$$A = \bigcap_{g \in F} U_g, \quad B = \bigcap_{g \in F} V_g.$$

The sets A and B are nonempty, since $z \in A$ and $t \in B$. To conclude, the sets $p(G \cdot A)$ and $p(G \cdot B)$ are open disjoint neighborhood of x and y in X/G.

□