Partitions of Unity and Vector Fields

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Exercise 1. A vector field X on M is *singular* if $X(p) = 0$ for some $p \in M$, otherwise X is *non-singular*.

1. Find non-singular smooth vector fields on $\mathbb{T}^n$, $\mathbb{S}^{2n-1}$ and $\mathbb{RP}^{2n-1}$.
2. Is there such a vector field on the Möbius band?
3. Find a smooth vector field in $\mathbb{S}^{2n}$ with one singularity.

Exercise 2 (Extension of a vector field)**.** Let M be a compact submanifold of $\mathbb{R}^n$ and X a smooth vector field on M. Show that there exist a smooth vector field Y on $\mathbb{R}$ such that $X = Y|_M$.

Exercise 3 (Lie bracket)**.**

1. Let $(x_1, \dots, x_n)$ be a local system of coordinates on a smooth manifold M, prove that
$$\left[\sum_{i=1}^n X_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n Y_j \frac{\partial}{\partial x_j}\right] = \sum_{i=1}^n \left(\sum_{j=1}^n X_j \frac{\partial Y_j}{\partial x_j} - Y_j \frac{\partial X_j}{\partial x_j}\right) \frac{\partial}{\partial x_i},$$
where X_i's and Y_i's are smooth real maps.
2. Let $\varphi : M \to N$ be a diffeomorphism. Given a vector field X on M, we denote by $\varphi_* X$ the vector field on N defined by $T\varphi \circ X$ (why is it well defined?). Prove that $\varphi_*([X, Y]) = [\varphi_* X, \varphi_* Y]$ for all vector fields X and Y on M.
3. Let $f, g \in \mathcal{C}^\infty(M, \mathbb{R})$, prove that for all vector fields X and Y on M,
$$[fX, gY] = f(X \cdot g)Y - g(Y \cdot f)X + fg[X, Y].$$