Differential forms

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Exercise 1. 1. Let $f : t \mapsto e^t$ from $\mathbb{R}$ to $\mathbb{R}_+^*$ and let $\alpha = \frac{\mathrm{d}x}{x}$, compute $f^*\alpha$.

2. Same question with $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ and $\alpha = \mathrm{d}x \wedge \mathrm{d}y$.

Exercise 2 (tangent and cotangent bundles)**.** We recall that a Riemannian metric on a manifold M is a map $x \mapsto g_x$ such that, for all $x \in M$, g_x is a scalar product on T_xM and such that for all chart (U, φ), the map $U \to \mathcal{M}_n(\mathbb{R})$:

$$x \mapsto \left(g_x\left(\mathrm{d}\varphi(x)^{-1} \cdot e_i, \mathrm{d}\varphi(x)^{-1} \cdot e_j\right)\right)_{1 \leqslant i,j \leqslant n}$$

is smooth, where (e_i) is the canonical basis of $\mathbb{R}^n$.

1. Recall why every manifold admits Riemannian metrics.
2. Show that TM is diffeomorphic to T^*M. More precisely, show that there exists a diffeomorphism $\Psi : TM \to T^*M$ such that $\Psi|_{T_xM}$ is a linear isomorphism $T_xM \to T_x^*M$, for all $x \in M$.

Let us recall the definition of volume form.

Definition (Volume form)**.** Let M be a n-dimensional manifold, a volume form on M is a n-form $\nu \in \Omega^n(M)$ such that $\forall x \in M$, $\nu_x \neq 0$.

Exercise 3 (Spheres and real projective spaces)**.** 1. Let $\nu = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$ denote the standard volume form on $\mathbb{R}^{n+1}$ (that is the $(n+1)$-form equal to the determinant at each point) and let X be the radial vector field : $x \mapsto \sum x_i \frac{\partial}{\partial x_i}$.
Compute $\omega := X \lrcorner \nu$ (recall that $(Y \lrcorner \alpha)(Y_1, \dots, Y_p) = \alpha(Y, Y_1, \dots, Y_p)$).

2. Show that ω is invariant under the action of $SO_{n+1}(\mathbb{R})$.
3. Let $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the canonical embedding, show that $\mu := i^*(\omega)$ is a volume form.
4. What is $f^*\mu$, where $f : x \mapsto -x$ denotes the antipodal map? Does $\mathbb{RP}^n$ admit a volume form?

Exercise 4 (Contact form)**.** Let M be a 3-dimensional manifold. Let $\alpha \in \Omega^1(M)$ be a 1-form. Given a vector field $X \in \mathcal{X}(M)$, by $X \in \ker\alpha$ we mean $X_x \in \ker\alpha_x$ for all $x \in M$. We say that the family of planes $(\ker\alpha_x)_{x \in M}$ is *involutive* if for all $x \in M$,

$$[X, Y] \in \ker\alpha, \quad \forall X, Y \in \ker\alpha.$$

1. Show that for all 1-form $\alpha \in \Omega^1(M)$ and all vector fields X and Y, one has

$$\mathrm{d}\alpha(X, Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]).$$

2. Show that the family of planes $(\ker\alpha_x)_{x \in M}$ is involutive if and only if $\alpha \wedge \mathrm{d}\alpha$ is a volume form. We say that α is then a contact form. If M is $(2n+1)$-dimensional, a *contact form* is a 1-form α such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is a volume form.

3. Let $\alpha \in \Omega^1(M)$ be a contact form. Show that there exists a unique vector field $R \in \mathcal{X}(M)$ such that
$$\begin{cases} R \lrcorner \alpha := \alpha(R) = 1, \\ R \lrcorner \, \mathrm{d}\alpha := \mathrm{d}\alpha(R, \cdot) = 0. \end{cases}$$
This vector field is called the *Reeb vector field* associated to α.

4. Let (x, y, z) be the coordinates functions of $\mathbb{R}^3$, show that $\alpha := \mathrm{d}z - y\,\mathrm{d}x$ is a contact form and give its Reeb vector field.

Exercise 5 (Liouville form)**.** Let M be a n-dimensional manifold. Recall that local coordinates $(q_1, \dots, q_n)$ on M induce local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ on T^*M by
$$\begin{cases} U \times \mathbb{R}^n & \to \quad T^*U \\ (q, p) & \mapsto \quad \sum_{i=1}^n p_i \, \mathrm{d}q_i|_q. \end{cases}$$
The *Liouville form* λ of T^*M is defined by
$$\lambda_{(q,p)} := p \circ \mathrm{d}\pi_q \in T^*_{(q,p)}(T^*M), \quad \forall (q, p) \in T^*M,$$
where $\pi : T^*M \to M$ is the canonical projection.

1. Show that λ is well defined and that in the local coordinates (q, p) of T^*M defined above one has
$$\lambda = \sum_{i=1}^n p_i \, \mathrm{d}q_i.$$
that is $\lambda = \sum_i f_i(q, p)\,\mathrm{d}q_i + g_i(q, p)\,\mathrm{d}p_i$ with $f_i(q, p) = p_i$ and $g_i(q, p) = 0$.

2. A 1-form $\alpha \in \Omega^1(M)$ can be seen in particular as a smooth map $\alpha : M \to T^*M$, so that we can pullback a form of T^*M by α. Show that λ is characterized by the property
$$\alpha^*\lambda = \alpha, \quad \forall \alpha \in \Omega^1(M).$$

3. Show that $\omega := \mathrm{d}\lambda$ is a *symplectic form* that is a 2-form such that $\mathrm{d}\omega = 0$ and ω_x is a non-degenerated skew-symmetric bilinear form for all $x \in T^*M$.

4. Let $X \in \mathcal{X}(T^*M)$ be the vector field defined in local coordinates by
$$X(q, p) := \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}.$$
Show that $X \lrcorner \omega = \lambda$ and that ω^n is a volume form of T^*M.

5. Let $\Sigma^{2n-1} \subset T^*M$ be a hypersurface of T^*M which is transversal to X, that is $X(x) \notin T_x\Sigma$, for all $x \in \Sigma$. Show that $i^*\lambda$ is a contact form of Σ where $i : \Sigma \hookrightarrow T^*M$ denotes the inclusion map (see Exercise 4).