Differential forms, Calculus of Lie-Cartan

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Exercise 1 (Divergence and curl of a vector field)**.** 1. Let $(E, \langle \cdot\,, \cdot \rangle)$ be an oriented Euclidean space of dimension 3. Let $\nu \in \Omega^3(E)$ be the volume form satisfying $\nu = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ in any oriented orthonormal basis (why does it exist?). Let X be a vector field on E, then $\langle X\,, \cdot \rangle$ defines a 1-form. We define $\mathrm{rot}(X)$ (also denoted $\mathrm{curl}(X)$) as the only vector field such that $\mathrm{rot}(X) \lrcorner \nu = \mathrm{d}(\langle X\,, \cdot \rangle) \in \Omega^2(E)$. Compute the expression of $\mathrm{rot}(X)$ in a direct orthonormal basis of E.

2. Let M be a smooth manifold equipped with a volume form ω. Let X be a vector field on M, we define $\mathrm{div}(X)$ as the only function such that $\mathrm{d}(X \lrcorner \omega) = \mathrm{div}(X)\omega$. Compute the expression of $\mathrm{div}(X)$ in local coordinates such that $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

3. Prove that $\mathrm{div}(X) \equiv 0$ if and only if the flow of X is volume preserving.

Exercise 2 (Calculus of Lie-Cartan)**.** Let M be a smooth manifold of dimension n. A smooth family of k-form $[-1, 1] \to \Omega^k(M)$, $t \mapsto \omega_t$ is a map such that in local coordinates $(x_1, \dots, x_n)$,

$$(\omega_t)_p = \sum_I \omega_{t,I}(p)\, \mathrm{d}x^I,$$

where $(t, p) \to \omega_{t,I}(p)$ are smooth maps $[-1, 1] \times U \to \mathbb{R}$. We denote by $\dot{\omega}_t$, $t \in [-1, 1]$, the k-form obtained by taking $\dot{\omega}_{t,I} = \partial_t \omega_{t,I}$ in local coordinates, equivalently $(\dot{\omega}_t)_p = \partial_t(\omega_t)_p$ for all $p \in M$ (the derivative is defined in the usual way on the finite dimensional vector spaces $\bigwedge^k T_p^* M$).

1. Let (ϕ_t) be the flow of the vector field X, prove that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} \phi_s^* \omega_s \right|_{s=t} = \phi_t^*(\mathcal{L}_X \omega_t + \dot{\omega}_t)$$

wherever ϕ_t and ω_t are well-defined.

2. Let (φ_t) be an isotopy, that is a family of diffeomorphisms of M such that $(t, p) \mapsto \varphi_t(p)$ is a smooth map. Let (X_t) be the time depend vector field associated to the isotopy, defined by

$$\left.\frac{\mathrm{d}\varphi_s}{\mathrm{d}s}\right|_{s=t} = X_t \circ \varphi_t.$$

Prove that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} \varphi_s^* \omega_s \right|_{s=t} = \varphi_t^*(\mathcal{L}_{X_t} \omega_t + \dot{\omega}_t),$$

wherever φ_t and ω_t are well-defined.

Exercise 3 (Hamiltonian dynamics)**.** Let $(q_1, \dots, q_n, p_1, \dots, p_n)$ be the canonical coordinates on $\mathbb{R}^{2n}$. Let $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function $(t, x) \mapsto H_t(x)$. The Hamilton equations of H refer the first order differential system

$$\begin{cases} q_i'(t) = \frac{\partial H_t}{\partial p_i}(q(t), p(t)), \\ p_i'(t) = -\frac{\partial H_t}{\partial q_i}(q(t), p(t)), \end{cases} \qquad 1 \leqslant i \leqslant n. \tag{1}$$

We assume that the associated flow (ϕ_t) satisfying $\phi_t(q(0), p(0)) = (q(t), p(t))$ for any solution (q, p) of (1) is well defined for $t \in \mathbb{R}$. Let $X_t \in \mathcal{X}(\mathbb{R}^{2n})$ be the 1-parameter family of vector fields associated to the isotopy (ϕ_t).

1. Let $\omega := \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$, show that (ϕ_t) being the flow of (1) is equivalent to

$$X_t \lrcorner \omega = -\,\mathrm{d}H_t, \quad \forall t \in \mathbb{R}.$$

2. Show that $\phi_t^* \omega = \omega$ for all $t \in \mathbb{R}$ and deduce that ϕ_t is volume preserving for all $t \in \mathbb{R}$ (Liouville theorem).

3. Let $\lambda := \sum_i p_i \,\mathrm{d}q_i$ be the Liouville form, find an explicit smooth map $a : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$\phi_1^* \lambda - \lambda = \mathrm{d}a.$$

Exercise 4 ($H^0(M)$)**.** Let M be a smooth manifold with N connected components, show that $H^0(M) \simeq \mathbb{R}^N$.

Exercise 5 ($H^*(\mathbb{T}^n)$)**.** For $k \in \mathbb{N}$, let $\Omega^k_{\mathbb{Z}^n}(\mathbb{R}^n)$ be the subspace of $\mathbb{Z}^n$-periodic k-form of $\mathbb{R}^n$, that is the set of $\alpha \in \Omega^k(\mathbb{R}^n)$ such that

$$\alpha_p = \sum_I \alpha_I(p)\,\mathrm{d}x^I, \quad \forall p \in \mathbb{R}^n,$$

where the $\alpha_I : \mathbb{R}^n \to \mathbb{R}$ are smooth and such that $\alpha_I(p+\ell) = \alpha_I(p)$ for all $p \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}^n$.

1. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map. Show that $(\Omega^*_{\mathbb{Z}^n}(\mathbb{R}^n), d)$ is a differential graded algebra (where d denotes the restriction of the exterior derivative to the subspace of $\mathbb{Z}^n$-periodic forms) and that π^* defines an isomorphism of differential graded algebras $\Omega^*(\mathbb{T}^n) \to \Omega^*_{\mathbb{Z}^n}(\mathbb{R}^n)$.

2. Given $t \in \mathbb{R}^n$, we denote by $\tau_t : \mathbb{R}^n \to \mathbb{R}^n$ the translation $\tau_t(p) := p + t$. We will admit that given a closed $\alpha \in \Omega^k(\mathbb{R}^n)$, there exists a smooth family $t \mapsto \beta_t$ of $(k-1)$-form (meaning the $(t,p) \mapsto (\beta_t)_I(p)$ are smooth) such that $\tau_t^*\alpha - \alpha = \mathrm{d}\beta_t$. Given $\alpha \in \Omega^k_{\mathbb{Z}^n}(\mathbb{R}^n)$, we define a form with constant coefficients $\overline{\alpha} \in \Omega^k_{\mathbb{Z}^n}(\mathbb{R}^n)$ by

$$\overline{\alpha} := \sum_I \left(\int_{t \in [0,1]^n} \alpha_I(t)\,\mathrm{d}t \right) \mathrm{d}x^I.$$

   Show that α and $\overline{\alpha}$ are cohomologous in $\Omega^*_{\mathbb{Z}^n}(\mathbb{R}^n)$.

3. Show that the de Rham cohomology of $\mathbb{T}^n$ is isomorphic to $\bigwedge^* \mathbb{R}^n$.